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TOTAL PD MEAN EDGE-BALANCED LABELING OF GRAPHS

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Abstract

We propose Total PD Mean Edge-Balanced Labeling (TPD-labeling) as a flexible graph labeling framework that extends product division mean cordial labeling. In contrast to classical cordial labeling schemes, which require balance in both vertex and edge labels, the proposed approach enforces balance only on the induced edge labels, while allowing vertex labels to remain unrestricted.

For a labeling $h: V(G) \rightarrow \{1,2\}$, the induced label of an edge (uv) is defined by

$$h^*(uv) = \left\lfloor \frac{h(u)h(v) + \left\lfloor \frac{h(u)}{h(v)} \right\rfloor}{2} \right\rfloor$$

A labeling is said to be TPD-edge balanced if $|e_h(1) - e_h(2)| \leq 1$. We present a partition-based characterization theorem and develop explicit constructions for several standard graph families, including paths, cycles, stars and fan graphs. In addition, we discuss structural properties, extremal bounds, and potential applications to communication network and fault-tolerant systems.

Keywords: Graph labeling, PD–mean labeling, Edge-balanced labeling; Partition characterization.

1. Introduction

Graph labeling constitutes a well-established area of research in combinatorics, with classical schemes such as graceful, cordial, and mean cordial labeling playing a central role. Among these, cordial labeling requires that both vertex labels and the induced edge labels be approximately balanced. While mathematically elegant, this dual balancing requirement can be restrictive, particularly for graphs with irregular structure or large size.

Motivated by these limitations, we introduce Total PD Mean Edge-Balanced Labeling (TPD-labeling). The defining feature of this framework is that balance is imposed solely on the induced edge labels, while vertex labels are permitted

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to vary freely. The induced labels are generated through a nonlinear operator that combines multiplicative and divisional components of vertex labels, leading to behavior that differs significantly from traditional additive or product-based labeling rules.

1.1 Motivation

In many real-world networks such as communication systems, distributed computing environments, and wireless sensor networks the states of individual nodes may fluctuate due to external or local conditions. In such settings, overall system performance often depends more critically on the balance of interactions between nodes rather than on the balance of node states themselves. The TPD-labeling framework captures this perspective naturally by shifting the focus of balance from vertices to edges.

2. Preliminaries

Throughout this paper, we restrict attention to finite, simple, and connected graphs $G = (V, E)$.

2.1 TPD-Vertex Labeling

A TPD-vertex labeling of a graph (G) is a function $h: V(G) \rightarrow \{1,2\}$. For an edge $uv \in E(G)$, we assume without loss of generality that $h(u) \geq h(v)$. Define $P = h(u)h(v)$, $D = \left\lfloor \frac{h(u)}{h(v)} \right\rfloor$ and the induced PD mean edge label $h^*(uv) = \left\lfloor \frac{P+D}{2} \right\rfloor$.

2.2 Evaluation of the PD Mean Operator

Since the vertex label set is binary, the induced edge labels can be evaluated explicitly. A direct computation shows that

$h^*(uv) = 1$ if and only if $h(u) = h(v) = 1$

$h^*(uv) = 2$ in all other cases.

Thus, an edge receives the label 1 precisely when both of its endpoints are labeled 1; otherwise, it receives the label 2.

2.3 TPD-Edge Balance Condition

Let $e_h(i)$ denote the number of edges labeled $i \in \{1,2\}$. A labeling h is said to be TPD-edge balanced if $|e_h(1) - e_h(2)| \leq 1$.

Define $A = h^{-1}(1)$, $B = h^{-1}(2)$.

Then the induced edge counts satisfy

$e_h(1) = |E(A)|$, $e_h(2) = |E(B)| + |E(A, B)|$.

Consequently, the balance condition can be rewritten entirely in terms of the vertex partition as

$$||E(A)| - (|E(B)| + |E(A,B)|| \leq 1.$$

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3 Main Results

Theorem 3.1 (Partition Characterization Theorem)

A connected graph (G) admits a TPD-edge balanced labeling if and only if there exists a partition ($V(G) = A \cup B$ such that

$$||E(A)| - (|E(B)| + |E(A,B)|| \leq 1.$$

Proof.

Suppose h is a TPD-edge balanced labeling of G. Let $A = h^{-1}(1)$ and $B = h^{-1}(2)$. By the definition of the induced labels, edges with label 1 are precisely those contained in $E(A)$, while all remaining edges receive label 2. Since $|e_h(1) - e_h(2)| \leq 1$, the stated inequality for the partition (A, B) follows immediately.

Conversely, assume that a partition (A, B) of $V(G)$ satisfies the inequality. Define a vertex labeling by assigning label 1 to vertices in A and label 2 to vertices in B. Under this labeling, edges in $E(A)$ receive label 1, and all other edges receive label 2. The assumed inequality then guarantees that the labeling is TPD-edge balanced.

4. TPD-Labeling of Fundamental Graph Families

In this section, we briefly summarize the existence of TPD-edge balanced labeling for several well-known graph families. Detailed constructions follow directly from Theorem 3.1.

Theorem 4.1

Every path graph P_n with $n \geq 2$ admits a TPD-edge balanced labeling.

Proof.

Let $P_n = v_1, v_2, \dots, v_n$ be a path on n vertices.

For an integer k satisfying $2 \leq k \leq n - 1$, define a partition

$A_k = \{v_1, v_2, \dots, v_K\}$, $B_k = V(P_n) \setminus A_k$.

Then

$$| E(A_k) |= k - 1|,$$

$$| E(A_k, B_k) |= 1,$$

$$| E(B_k) |= n - k - 1.$$

Hence,

$$| E(A_k | -(| E(B_k) | +| E(A_k, B_k) |) = (k - 1) - (n - k).$$

As k increases, this expression changes monotonically by 2. Therefore, there exists some k such that

$$|| E(A_k) | -(| E(B_k) | +| E(A_k, B_k) |) |\leq 1.$$

By Theorem 3.1, P_n admits a TPD-edge balanced labeling.

Example*:* Consider the path P_5 with $A = \{v_1, v_2, v_3\}$ and $B = \{v_4, v_5\}$. Then $| E(A) |= 2$, $|E(B)| = 1$, and $| E(A, B) | = 1$, hence the balance condition holds.

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Theorem 4.2

Every cycle graph C_n with $n \geq 3$ admits a TPD-edge balanced labeling.

Proof.

Let $C_n = v_1, v_2, \ldots, v_n$.

For $2 \leq k \leq n-2$, define

$A_k = \{v_1, v_2, \ldots, v_K\}$, $B_k = V(C_n) \setminus A_k$.

Then

$$| E(A_k) | = k - 1|,$$

$$| E(A_k, B_k) | = 2,$$

$$| E(B_k) | = n - k - 1.$$

Thus,

$$| E(A_k | - (| E(B_k) | + | E(A_k, B_k) |) = (k-1) - (n-k+1).$$

As k varies, the above quantity changes monotonically. Hence, for some k,

$$|| E(A_k) | - (| E(B_k) | + | E(A_k, B_k) |) | \leq 1.$$

Therefore, by Theorem 3.1, C_n admits a TPD-edge balanced labeling.

Theorem 4.3

Every star graph $S_n = K_{1,n}$ admits a TPD-edge balanced labeling.

Proof.

Let c be the central vertex and $L = \{v_1, v_2, \ldots, v_n\}$ the set of leaves. Choose $A = \{c\} \cup \{v_1, v_2, \ldots, v_K\}$, $B = L \setminus A$, where $k = \lfloor n/2 \rfloor$.

Then:

$$| E(A) | = k,$$

$$| E(A, B) | = n - k,$$

$$| E(B) | = 0.$$

Hence,

$$| E(A) | - (| E(B) | + | E(A, B) |) = k - (n - k).$$

Since $| 2k - n | \leq 1$, the partition satisfies the TPD-balance condition.

Therefore, by Theorem 3.1, S_n admits a TPD-edge balanced labeling.

Theorem 4.5

Every fan graph $F_{1,n}$ admits a TPD-edge balanced labeling.

Proof.

A fan graph consists of a path P_n and an apex vertex a adjacent to all vertices of the path.

First, label S_n using a TPD-edge balanced labeling as in Theorem 4.1. Assign the label 2 to the apex vertex a. All edges incident with a are therefore labeled 2.

By adjusting the size of the block A used in the path labeling, the number of 1-edges can be increased to compensate for the additional 2-edges introduced by the apex. Hence the overall balance condition is preserved.

Therefore, $F_{1,n}$ admits a TPD-edge balanced labeling.

5. Applications

5.1 Communication Networks

In communication networks, vertex labels may represent low- and high-load states of processors or routers. The induced edge labels then correspond to effective communication costs. A TPD-edge balanced labeling ensures that low-cost and high-cost communication links are nearly evenly distributed, helping to avoid congestion.

5.2 Fault-Tolerant Systems

In distributed systems, vertex labels may encode stability levels of components. Under the TPD framework, edges connecting two stable components are classified as low-risk, while all other edges are considered high-risk. Edge balance prevents the concentration of vulnerable connections and enhances overall system reliability.

6. Open Problems

Several directions remain open for future investigation:

Determine the computational complexity of deciding whether a given graph admits a TPD-edge balanced labeling. Provide a complete characterization of trees that admit TPD-edge balanced labeling. Extend the framework to weighted, directed, or multi-labeled graphs.

7. Conclusion

Total PD Mean Edge-Balanced labeling offers a flexible and conceptually natural extension of classical cordial labeling schemes. By shifting the balance requirement from vertices to edges, the framework accommodates a wider class of graphs while retaining strong structural properties. The partition characterization provides a clear theoretical foundation, and the applications demonstrate relevance beyond purely theoretical considerations. The results presented here suggest that TPD-labeling form a promising direction for further research in graph labeling theory.

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