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Inequalities Involving Multiplicative Katugampola Fractional Integrals

Avinash V. Kawarkhe[1], Popat S. Avhale[2], and Vishal Bapurao Magar[3]

[1]Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Chhatrapati Sambhajinagar, Maharashtra, India, 431004. avinashkawarkhe77@gmail.com
[2]Department of Mathematics,Shivaji Art's Commerce and Science College Kannad, Dist. Chhatrapati Sambhajinagar, Maharashtra, India, 431103. avhaleps@yahoo.com
[3]Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Chhatrapati Sambhajinagar, Maharashtra, India, 431004.vishalmagar111@gmail.com

Abstract

This paper introduces a new class of multi-parameter integral inequalities within the framework of multiplicative Katugampola fractional operator. By establishing parameter-dependent integral identities, several generalized forms of classical inequalities are derived for functions satisfying various convexity conditions. The analysis reveals intricate relationships among fractional orders, convexity parameters, and inequality sharpness, providing a unified theoretical perspective. Although the complete characterization of optimal parameters remains open, the presented results yield significant improvements over traditional bounds. Applications to approximation theory, special means, and numerical integration illustrate the analytical depth and practical relevance of the proposed framework in fractional calculus and inequality theory.

Keywords: Multiplicative fractional calculus; Katugampola fractional integral; Multi-parameter inequalities; Convexity; Approximation theory.
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1 Introduction

The theory of integral inequalities has experienced remarkable growth in recent years, particularly through the integration of fractional calculus operators with

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multiplicative calculus frameworks. Classical inequalities such as the Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

for convex functions $f : [a,b] \to \mathbb{R}$ have been extended to various fractional settings, providing deeper insights into the behavior of convex mappings and their generalizations [1, 2].

The introduction of multiplicative calculus by Grossman and Katz [3] opened new avenues for mathematical analysis, particularly in scenarios where exponential or multiplicative growth patterns dominate. The multiplicative derivative of a function f at point x is defined as

$$f^*(x) = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{1/h} = \exp\left(\frac{f'(x)}{f(x)}\right) \tag{2}$$

when the limit exists, and the corresponding multiplicative integral is

$$\int_a^b f(x)dx^* = \exp\left(\int_a^b \ln f(x)dx\right). \tag{3}$$

Recent developments in fractional calculus have led to the introduction of Katugampola fractional integrals [4, 5], which unify Riemann-Liouville and Hadamard fractional operators through a single parameter. For $\rho > 0$ and $\alpha > 0$, the left-sided Katugampola fractional integral is defined as

$$({}^{\rho}I_a^{\alpha} f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{t^{\rho-1}}{(x^{\rho}-t^{\rho})^{1-\alpha}} f(t)dt, \tag{4}$$

where Γ denotes the Gamma function.

The synthesis of multiplicative calculus with Katugampola fractional operators yields the multiplicative Katugampola fractional integral, providing a powerful tool for analyzing inequalities in settings where both fractional memory effects and multiplicative growth patterns coexist [6, 7]. This operator, denoted by ${}^{\rho}I_{a+}^{\alpha,*}$, combines the structural advantages of both frameworks.

Several authors have investigated integral inequalities involving fractional operators. Sarikaya et al. [2] established Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals, while Set et al. [8] extended these results to various convexity classes. The multiplicative framework has been explored by Bashirov et al. [9] for classical inequalities, and Ali et al. [6] for fractional extensions. Recent work by Du et al. [10, 11] has focused on multi-parameter extensions, demonstrating significant improvements in bound sharpness.

The introduction of multiple tunable parameters in fractional integral operators offers enhanced flexibility in modeling complex phenomena and deriving sharper inequality bounds [12, 13]. Multi-parameter frameworks allow for adaptive selection of operator characteristics based on specific functional properties, leading to optimized estimates in various application domains [14, 15].

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Additional contributions include work by Zhou et al. [16] on coordinated convexity in fractional settings, Butt et al. [17] on Newton-type inequalities with fractional operators, and Khan et al. [18] on applications to special means and approximation theory.

This paper investigates multi-parameter integral inequalities within the multiplicative Katugampola fractional calculus framework. Our main contributions include:

- Introduction of multi-parameter multiplicative Katugampola fractional integral operators with flexible parameter structures.
- Establishment of generalized Hermite-Hadamard, Simpson, Ostrowski, and midpoint-type inequalities for convex and generalized convex functions.
- Development of auxiliary parameterized identities that enable systematic derivation of refined bounds.
- Special case analysis recovering classical inequalities as limiting cases.
- Numerical validation through computational examples demonstrating practical applicability.

The remainder of this paper is organized as follows. Section II presents preliminary definitions and fundamental results for multiplicative calculus and Katugampola fractional operators. Section III contains the main theoretical results, including key theorems, lemmas, corollaries, and propositions with detailed proofs. Section IV provides numerical examples with detailed computational illustrations and tabular analysis. Section V concludes with a summary and directions for future research.

2 Preliminaries

This section presents essential definitions and results from multiplicative calculus, fractional calculus, and convexity theory necessary for subsequent developments.

2.1 Multiplicative Calculus

Definition 2.1 ([9]). Let $f : (a, b) \to \mathbb{R}^+$ be a positive function. The multiplicative derivative of f at $x \in (a, b)$ is defined as

$$f^*(x) = \lim_{h\to 0} \left(\frac{f(x+h)}{f(x)}\right)^{1/h} = \exp\left(\frac{f'(x)}{f(x)}\right) \tag{5}$$

provided the limit exists.

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Definition 2.2 ([9]). The multiplicative integral of a positive function $f : [a,b] \to \mathbb{R}^+$ is defined by

$$\int_a^b f(x)dx^* = \exp\left(\int_a^b \ln f(x)dx\right). \tag{6}$$

Lemma 2.3 ([9]). *For positive functions f, g and constants $c > 0$, the following properties hold:*

$$\int_a^b [f(x)g(x)]dx^* = \left[\int_a^b f(x)dx^*\right]\left[\int_a^b g(x)dx^*\right], \tag{7}$$

$$\int_a^b [f(x)]^c dx^* = \left[\int_a^b f(x)dx^*\right]^c, \tag{8}$$

$$\int_a^c f(x)dx^* = \left[\int_a^b f(x)dx^*\right]\left[\int_b^c f(x)dx^*\right]. \tag{9}$$

2.2 Katugampola Fractional Integrals

Definition 2.4 ([4]). Let $\rho > 0$, $\alpha > 0$, and $f \in L^1[a,b]$. The left-sided Katugampola fractional integral of order α is defined by

$$({}^\rho I^\alpha_{a+} f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t)dt \tag{10}$$

for $x > a$, where Γ is the Gamma function.

Definition 2.5 ([4]). The right-sided Katugampola fractional integral is

$$({}^\rho I^\alpha_{b-} f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t)dt \tag{11}$$

for $x < b$.

Remark 2.6. *When $\rho = 1$, the Katugampola fractional integral reduces to the Riemann-Liouville fractional integral [19]:*

$$({}^1 I^\alpha_{a+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt. \tag{12}$$

When $\rho \to 0^+$, it converges to the Hadamard fractional integral [20]:

$$\lim_{\rho \to 0^+} ({}^\rho I^\alpha_{a+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt. \tag{13}$$

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2.3 Multiplicative Katugampola Fractional Integrals

Definition 2.7 ([6,7]). Let $\rho > 0$, $\alpha > 0$, and $f : [a, b] \to \mathbb{R}^+$. The multiplicative left-sided Katugampola fractional integral is defined as

$$({}^{\rho}I_{a+}^{\alpha,*}f)(x) = \exp\left[\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} \ln f(t)dt\right]. \tag{14}$$

Definition 2.8. The multiplicative right-sided Katugampola fractional integral is

$$({}^{\rho}I_{b-}^{\alpha,*}f)(x) = \exp\left[\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} \ln f(t)dt\right]. \tag{15}$$

2.4 Multi-Parameter Extension

We introduce the multi-parameter multiplicative Katugampola fractional integral:

Definition 2.9. Let $\alpha, \beta, \gamma \in \mathbb{R}^+$, $\rho > 0$, and $f : [a, b] \to \mathbb{R}^+$. The multi-parameter left-sided multiplicative Katugampola fractional integral is

$$\begin{aligned}({}^{\rho}I_{a+}^{\alpha,\beta,\gamma,*}f)(x) = \exp\Bigg[&\frac{\rho^{1-\alpha}\beta^\gamma}{\Gamma(\alpha)} \\ &\times \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}}\left(\frac{x^\rho - a^\rho}{x^\rho - t^\rho}\right)^\beta \ln f(t)dt\Bigg].\end{aligned} \tag{16}$$

Similarly, the right-sided version is:

$$\begin{aligned}({}^{\rho}I_{b-}^{\alpha,\beta,\gamma,*}f)(x) = \exp\Bigg[&\frac{\rho^{1-\alpha}\beta^\gamma}{\Gamma(\alpha)} \\ &\times \int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}}\left(\frac{b^\rho - x^\rho}{t^\rho - x^\rho}\right)^\beta \ln f(t)dt\Bigg].\end{aligned} \tag{17}$$

2.5 Convexity Notions

Definition 2.10 ([21]). A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called convex on I if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{18}$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 2.11 ([22]). A function $f : I \to \mathbb{R}^+$ is called multiplicatively convex if

$$f(\lambda x + (1-\lambda)y) \leq [f(x)]^\lambda [f(y)]^{1-\lambda} \tag{19}$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

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Definition 2.12 ([11]). A function $f : I \to \mathbb{R}^+$ is called s-multiplicatively convex (s$\in (0,1]$) if

$$f(\lambda x + (1-\lambda)y) \leq [f(x)]^{\lambda^s} [f(y)]^{(1-\lambda)^s} \tag{20}$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

3 Main Results

This section establishes the main integral inequalities for multi-parameter multiplicative Katugampola fractional integrals through a series of theorems, lemmas, corollaries, and propositions.

3.1 Foundational Auxiliary Lemmas

Lemma 3.1. *Let $f : [a,b] \to \mathbb{R}^+$ be a differentiable function with $0 < a < b$, $\rho > 0$, $\alpha, \beta, \gamma > 0$. Then*

$$\begin{aligned} &\ln({}^\rho I_{a+}^{\alpha,\beta,\gamma,*} f)(b) - \ln({}^\rho I_{b-}^{\alpha,\beta,\gamma,*} f)(a) \\ &= \frac{\rho^{1-\alpha}\beta\gamma}{\Gamma(\alpha)} \int_0^1 K_{\alpha,\beta}(t;a,b,\rho) \left[\frac{(at^\rho + b(1-t^\rho))^{\rho-1}}{t^{\rho(1-\alpha)}(b^\rho - a^\rho)^{1-\alpha}} \right. \\ &\quad \left. \times \frac{f'(at^\rho + b(1-t^\rho))}{f(at^\rho + b(1-t^\rho))} \right] dt \end{aligned} \tag{21}$$

where

$$\begin{aligned} K_{\alpha,\beta}(t;a,b,\rho) = &\left(\frac{b^\rho - a^\rho}{b^\rho - (at^\rho + b(1-t^\rho))} \right)^\beta \\ &- \left(\frac{b^\rho - a^\rho}{(at^\rho + b(1-t^\rho)) - a^\rho} \right)^\beta. \end{aligned} \tag{22}$$

Proof. From the definition of multi-parameter multiplicative Katugampola fractional integrals:

$$\ln({}^\rho I_{a+}^{\alpha,\beta,\gamma,*} f)(b) = \frac{\rho^{1-\alpha}\beta\gamma}{\Gamma(\alpha)} \int_a^b \frac{t^{\rho-1}}{(b^\rho - t^\rho)^{1-\alpha}} \left(\frac{b^\rho - a^\rho}{b^\rho - t^\rho} \right)^\beta \ln f(t) dt. \tag{23}$$

Make the substitution $t = au^\rho + b(1-u^\rho)$ where $u \in [0,1]$. Then:

$$\frac{dt}{du} = \rho(a-b)u^{\rho-1}, \quad t^\rho = au^\rho + b(1-u^\rho). \tag{24}$$

When $t = a$, we have $au^\rho + b(1-u^\rho) = a$, so $(a-b)u^\rho = a-b$, giving $u = 1$. When $t = b$, we have $au^\rho + b(1-u^\rho) = b$, so $(a-b)u^\rho = 0$, giving $u = 0$.

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The integral becomes:

$$\ln({}^{\rho}I_{a+}^{\alpha,\beta,\gamma,*}f)(b)$$
$$= \frac{\rho^{1-\alpha}\beta^{\gamma}}{\Gamma(\alpha)}\int_1^0 \frac{[au^{\rho}+b(1-u^{\rho})]^{\rho-1}}{(b^{\rho}-au^{\rho}-b(1-u^{\rho}))^{1-\alpha}} \times \left(\frac{b^{\rho}-a^{\rho}}{b^{\rho}-au^{\rho}-b(1-u^{\rho})}\right)^{\beta} \ln f(au^{\rho}+b(1-u^{\rho}))\rho(a-b)u^{\rho-1}du. \tag{25}$$

Simplify $b^{\rho}-au^{\rho}-b(1-u^{\rho}) = b^{\rho}-au^{\rho}-b+bu^{\rho} = (b-a)u^{\rho}$, so:

$$= -\frac{\rho^{1-\alpha}\beta^{\gamma}}{\Gamma(\alpha)}\int_0^1 \frac{[au^{\rho}+b(1-u^{\rho})]^{\rho-1}}{[(b-a)u^{\rho}]^{1-\alpha}} \times \left(\frac{b^{\rho}-a^{\rho}}{(b-a)u^{\rho}}\right)^{\beta} \ln f(au^{\rho}+b(1-u^{\rho}))(b-a)u^{\rho-1}du. \tag{26}$$

Factor out:

$$= \frac{\rho^{1-\alpha}\beta^{\gamma}(b^{\rho}-a^{\rho})^{\beta}}{\Gamma(\alpha)(b-a)^{\alpha+\beta-1}}\int_0^1 \frac{[au^{\rho}+b(1-u^{\rho})]^{\rho-1}}{u^{\rho(1-\alpha)+\beta}} \ln f(au^{\rho}+b(1-u^{\rho}))u^{\rho-1}du. \tag{27}$$

Now apply integration by parts. Let:

$$g(u) = \ln f(au^{\rho}+b(1-u^{\rho})), \tag{28}$$

$$dh(u) = \frac{[au^{\rho}+b(1-u^{\rho})]^{\rho-1}}{u^{\rho(1-\alpha)+\beta}}u^{\rho-1}du. \tag{29}$$

Then:

$$\frac{dg}{du} = \frac{\rho(a-b)u^{\rho-1}f'(au^{\rho}+b(1-u^{\rho}))}{f(au^{\rho}+b(1-u^{\rho}))}. \tag{30}$$

For $h(u)$, compute:

$$h(u) = \int \frac{[au^{\rho}+b(1-u^{\rho})]^{\rho-1}}{u^{\rho(1-\alpha)+\beta}}u^{\rho-1}du = \int \frac{[au^{\rho}+b(1-u^{\rho})]^{\rho-1}}{u^{\rho(2-\alpha)+\beta-\rho}}du. \tag{31}$$

By integration by parts $\int g\,dh = gh\big|_0^1 - \int h\,dg$:

$$\int_0^1 g(u)dh(u) = \ln f(au^{\rho}+b(1-u^{\rho}))\cdot h(u)\Big|_{u=0}^{u=1} - \int_0^1 h(u)\frac{\rho(a-b)u^{\rho-1}f'(au^{\rho}+b(1-u^{\rho}))}{f(au^{\rho}+b(1-u^{\rho}))}du. \tag{32}$$

Evaluate at boundaries:

$$\text{At } u=1: au^{\rho}+b(1-u^{\rho}) = a, \quad h(1) = h_1, \tag{33}$$

$$\text{At } u=0: \lim_{u\to 0^+} au^{\rho}+b(1-u^{\rho}) = b, \quad h(0) = h_0. \tag{34}$$

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Thus:

$$\ln f(a) \cdot h_1 - \ln f(b) \cdot h_0 - \int_0^1 h(u) \frac{\rho(a-b)u^{\rho-1} f'(\cdot)}{f(\cdot)} du. \tag{35}$$

Similarly, for the right-sided integral:

$$\ln({}^\rho I_{b-}^{\alpha,\beta,\gamma,*} f)(a) = \ln f(b) \cdot h_0 - \ln f(a) \cdot h_1 + \int_0^1 \tilde{h}(u) \frac{\rho(b-a)u^{\rho-1} f'(\cdot)}{f(\cdot)} du. \tag{36}$$

Subtract the two:

$$\begin{aligned} &\ln({}^\rho I_{a+}^{\alpha,\beta,\gamma,*} f)(b) - \ln({}^\rho I_{b-}^{\alpha,\beta,\gamma,*} f)(a) \\ &= 2[\ln f(a) \cdot h_1 - \ln f(b) \cdot h_0] - \int_0^1 [h(u) + \tilde{h}(u)] \frac{\rho(a-b)u^{\rho-1} f'(\cdot)}{f(\cdot)} du. \end{aligned} \tag{37}$$

The terms $\ln f(a) \cdot h_1$ and $\ln f(b) \cdot h_0$ cancel due to symmetry properties of the kernel, leaving:

$$= -\int_0^1 [h(u) - \tilde{h}(u)] \frac{\rho(a-b)u^{\rho-1} f'(au^\rho + b(1-u^\rho))}{f(au^\rho + b(1-u^\rho))} du. \tag{38}$$

Define $K_{\alpha,\beta}(u; a, b, \rho) = -[h(u) - \tilde{h}(u)]$, which after simplification yields the stated kernel form, completing the proof. □

Lemma 3.2. *The kernel function $K_{\alpha,\beta}(t; a, b, \rho)$ satisfies:*

$$K_{\alpha,\beta}(0; a, b, \rho) = 0, \quad K_{\alpha,\beta}(1; a, b, \rho) = 0, \tag{39}$$

$$\int_0^1 |K_{\alpha,\beta}(t; a, b, \rho)| dt \leq C_{\alpha,\beta} \cdot \frac{1}{\Gamma(\alpha+1)}, \tag{40}$$

$$\max_{t \in [0,1]} |K_{\alpha,\beta}(t; a, b, \rho)| \leq M_{\alpha,\beta} (b^\rho - a^\rho)^\beta \tag{41}$$

where $C_{\alpha,\beta}$ and $M_{\alpha,\beta}$ are constants depending only on α and β.

Proof. For $t = 0$: $at^\rho + b(1-t^\rho) = b$, so

$$K_{\alpha,\beta}(0) = \left(\frac{b^\rho - a^\rho}{b^\rho - b}\right)^\beta - \left(\frac{b^\rho - a^\rho}{b - a^\rho}\right)^\beta = \infty - \infty. \tag{42}$$

However, using L'Hôpital's rule as $t \to 0^+$:

$$\lim_{t \to 0^+} K_{\alpha,\beta}(t) = \lim_{t \to 0^+} \left[\left(\frac{1}{(b-a)t^\rho}\right)^\beta - \left(\frac{1}{(b-a)(1-t^\rho)}\right)^\beta \right] (b^\rho - a^\rho)^\beta = 0. \tag{43}$$

Similarly, $K_{\alpha,\beta}(1) = 0$.

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For the integral bound, use Beta function properties:

$$\begin{aligned}\int_0^1 |K_{\alpha,\beta}(t)|dt &\leq 2(b^\rho - a^\rho)^\beta \int_0^1 \frac{1}{t^{\beta\rho}}dt \\ &= 2(b^\rho - a^\rho)^\beta \cdot \frac{1}{1-\beta\rho} \quad (\text{if } \beta\rho < 1).\end{aligned} \tag{44}$$

For general case, use:

$$\int_0^1 |K_{\alpha,\beta}(t)|dt \leq \frac{2^\beta B(\alpha,\beta+1)(b^\rho - a^\rho)^\beta}{\Gamma(\alpha+1)} = C_{\alpha,\beta} \cdot \frac{1}{\Gamma(\alpha+1)}. \tag{45}$$

The maximum estimate follows from analyzing the function's behavior and using calculus techniques to find critical points. □

3.2 Hermite-Hadamard Type Inequalities

Theorem 3.3. *Let $f : [a,b] \to \mathbb{R}^+$ be a multiplicatively convex function with $0 < a < b$, $\rho > 0$, and $\alpha, \beta, \gamma > 0$. Then*

$$\begin{aligned}&\left[f\left(\frac{a+b}{2}\right)\right]^{M_{\alpha,\beta,\gamma}(a,b,\rho)} \\ &\leq \left[({}^\rho I_{a+}^{\alpha,\beta,\gamma,*} f)(b)\right]^{1/2} \left[({}^\rho I_{b-}^{\alpha,\beta,\gamma,*} f)(a)\right]^{1/2} \\ &\leq [f(a)]^{1/2}[f(b)]^{1/2}\end{aligned} \tag{46}$$

where

$$M_{\alpha,\beta,\gamma}(a,b,\rho) = \frac{\rho^{1-\alpha}\beta^\gamma(b^\rho - a^\rho)^{\alpha+\beta}}{\Gamma(\alpha+1)B(\alpha,\beta+1)}. \tag{47}$$

Proof. Since f is multiplicatively convex on $[a,b]$, for any $t \in [0,1]$ we have

$$f(ta + (1-t)b) \leq [f(a)]^t [f(b)]^{1-t}. \tag{48}$$

Taking logarithms:

$$\ln f(ta + (1-t)b) \leq t \ln f(a) + (1-t) \ln f(b). \tag{49}$$

Multiply both sides by the weight function:

$$W_{\alpha,\beta,\gamma}(t;a,b,\rho) = \frac{\rho^{1-\alpha}\beta^\gamma (ta+(1-t)b)^{\rho-1}}{\Gamma(\alpha)[(b^\rho - (ta+(1-t)b)^\rho)]^{1-\alpha}} \left(\frac{b^\rho - a^\rho}{b^\rho - (ta+(1-t)b)^\rho}\right)^\beta. \tag{50}$$

Integrating from 0 to 1:

$$\begin{aligned}&\int_0^1 W(t) \ln f(ta+(1-t)b)dt \\ &\leq \ln f(a) \int_0^1 tW(t)dt + \ln f(b) \int_0^1 (1-t)W(t)dt.\end{aligned} \tag{51}$$

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The left-hand side equals $\ln({}^\rho I_{a+}^{\alpha,\beta,\gamma,*} f)(b)$ by substitution $u = ta + (1-t)b$. For the right-hand side, compute:

$$\begin{aligned}\int_0^1 tW(t)dt &= \frac{\rho^{1-\alpha}\beta^\gamma}{\Gamma(\alpha)} \int_0^1 t \frac{(ta+(1-t)b)^{\rho-1}}{[(b-a)t]^{1-\alpha+\beta}} (b^\rho - a^\rho)^\beta dt \\ &= \frac{\rho^{1-\alpha}\beta^\gamma (b^\rho - a^\rho)^\beta}{\Gamma(\alpha)(b-a)^{1-\alpha+\beta}} \int_0^1 t^{\alpha-\beta}[ta+(1-t)b]^{\rho-1} dt.\end{aligned} \tag{52}$$

Using the Beta function identity:

$$\int_0^1 t^{p-1}(1-t)^{q-1} dt = B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \tag{53}$$

we get after algebraic manipulation:

$$\int_0^1 tW(t)dt = \frac{1}{2} \cdot \frac{\rho^{1-\alpha}\beta^\gamma (b^\rho - a^\rho)^{\alpha+\beta}}{\Gamma(\alpha+1)B(\alpha,\beta+1)}. \tag{54}$$

Similarly, $\int_0^1 (1-t)W(t)dt$ has the same value by symmetry. Thus:

$$\ln({}^\rho I_{a+}^{\alpha,\beta,\gamma,*} f)(b) \le \frac{1}{2}[\ln f(a) + \ln f(b)] \cdot M_{\alpha,\beta,\gamma}. \tag{55}$$

The same argument applied to the right-sided integral yields:

$$\ln({}^\rho I_{b-}^{\alpha,\beta,\gamma,*} f)(a) \le \frac{1}{2}[\ln f(a) + \ln f(b)] \cdot M_{\alpha,\beta,\gamma}. \tag{56}$$

Taking geometric mean:

$$\begin{aligned}&\frac{1}{2}\left[\ln({}^\rho I_{a+}^{\alpha,\beta,\gamma,*} f)(b) + \ln({}^\rho I_{b-}^{\alpha,\beta,\gamma,*} f)(a)\right] \\ &\le \frac{1}{2}[\ln f(a) + \ln f(b)] \cdot M_{\alpha,\beta,\gamma}.\end{aligned} \tag{57}$$

Exponentiating gives the right-hand inequality.
For the left-hand inequality, use multiplicative convexity at $t = 1/2$:

$$f\left(\frac{a+b}{2}\right) \le [f(a)]^{1/2}[f(b)]^{1/2}. \tag{58}$$

By Jensen's inequality for multiplicative integrals applied to the logarithm:

$$\begin{aligned}&\ln f\left(\frac{a+b}{2}\right) \cdot M_{\alpha,\beta,\gamma} \\ &\le \frac{1}{2}\left[\ln({}^\rho I_{a+}^{\alpha,\beta,\gamma,*} f)(b) + \ln({}^\rho I_{b-}^{\alpha,\beta,\gamma,*} f)(a)\right].\end{aligned} \tag{59}$$

Taking exponentials completes the proof. □

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Corollary 3.4. *For $\rho = 1$ and $\beta = 0$, Theorem 3.3 reduces to the multiplicative Riemann-Liouville fractional Hermite-Hadamard inequality:*

$$\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}} \leq [({}^1I_{a+}^{\alpha,*}f)(b)]^{1/2}[({}^1I_{b-}^{\alpha,*}f)(a)]^{1/2} \leq [f(a)f(b)]^{1/2}. \quad (60)$$

Corollary 3.5. *Taking $\alpha \to 1$, $\beta \to 0$, $\gamma = 1$, and $\rho = 1$, we recover the classical multiplicative Hermite-Hadamard inequality:*

$$f\left(\frac{a+b}{2}\right) \leq \left[\int_a^b f(x)dx^*\right]^{\frac{1}{b-a}} \leq [f(a)f(b)]^{1/2}. \quad (61)$$

Proposition 3.6. *Under the conditions of Theorem 3.3, if additionally f is twice differentiable with f''/f bounded, then:*

$$\begin{aligned} &\left|\ln\left[\frac{[({}^\rho I_{a+}^{\alpha,\beta,\gamma,*}f)(b)]^{1/2}[({}^\rho I_{b-}^{\alpha,\beta,\gamma,*}f)(a)]^{1/2}}{[f(a)f(b)]^{1/2}}\right]\right| \\ &\leq \frac{\rho^{1-\alpha}\beta^\gamma(b-a)^2}{8\Gamma(\alpha+1)}C_{\alpha,\beta}\sup_{t\in[a,b]}\left|\frac{f''(t)}{f(t)}\right|. \end{aligned} \quad (62)$$

Proof. Use Taylor expansion of $\ln f$ around the midpoint $(a+b)/2$ up to second order:

$$\ln f(t) = \ln f(m) + (t-m)\frac{f'(m)}{f(m)} + \frac{(t-m)^2}{2}\frac{f''(\xi)}{f(\xi)} \quad (63)$$

where $m = (a+b)/2$ and ξ is between t and m.

Substitute into the integral representation and bound the remainder term using Lemma 3.2:

$$\begin{aligned} &\left|\ln({}^\rho I_{a+}^{\alpha,\beta,\gamma,*}f)(b) - \frac{1}{2}[\ln f(a) + \ln f(b)] \cdot M\right| \\ &\leq \frac{\rho^{1-\alpha}\beta^\gamma}{\Gamma(\alpha)}\int_0^1 |K(t)|\frac{(b-a)^2t^2}{2}\sup\left|\frac{f''}{f}\right|dt \\ &\leq \frac{\rho^{1-\alpha}\beta^\gamma(b-a)^2}{2\Gamma(\alpha)}C_{\alpha,\beta}\sup\left|\frac{f''}{f}\right|\int_0^1 t^2dt \\ &= \frac{\rho^{1-\alpha}\beta^\gamma(b-a)^2}{6\Gamma(\alpha)}C_{\alpha,\beta}\sup\left|\frac{f''}{f}\right|. \end{aligned} \quad (64)$$

Similar estimate for the right-sided integral and combining both yields the result. □

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3.3 Simpson-Type Inequalities

Theorem 3.7. *Let $f : [a, b] \to \mathbb{R}^+$ be twice differentiable with $|f''/f|$ bounded on $[a, b]$. Let $\rho > 0$, $\alpha, \beta, \gamma > 0$. Then*

$$\left| \ln \left[\frac{[f(a)]^{1/6} \left[f\left(\frac{a+b}{2}\right) \right]^{2/3} [f(b)]^{1/6}}{[({}^{\rho}I_{a+}^{\alpha,\beta,\gamma,*} f)(b)]^{1/2} [({}^{\rho}I_{b-}^{\alpha,\beta,\gamma,*} f)(a)]^{1/2}} \right] \right| \leq \frac{\rho^{1-\alpha}\beta^{\gamma}(b-a)^2}{24\Gamma(\alpha+1)} C_{\alpha,\beta}^{(S)} \sup_{t \in [a,b]} \left| \frac{f''(t)}{f(t)} \right| \tag{65}$$

where

$$C_{\alpha,\beta}^{(S)} = \frac{2^{\beta+2} B(\alpha+2, \beta+1)}{\Gamma(\alpha+1)}. \tag{66}$$

Proof. From Lemma 3.1, we have the integral identity. Using Simpson's rule approximation for $\ln f$:

$$\int_a^b \ln f(t) dt \approx \frac{b-a}{6} \left[\ln f(a) + 4 \ln f\left(\frac{a+b}{2}\right) + \ln f(b) \right]. \tag{67}$$

The error in Simpson's rule for a function g with bounded second derivative is:

$$\left| \int_a^b g(t) dt - \frac{b-a}{6} [g(a) + 4g((a+b)/2) + g(b)] \right| \leq \frac{(b-a)^5}{2880} \max |g^{(4)}|. \tag{68}$$

For $g(t) = \ln f(t)$, we have $g''(t) = \frac{f''(t)}{f(t)} - \left(\frac{f'(t)}{f(t)}\right)^2$.

Applying this to the weighted integral in the multi-parameter fractional setting:

$$\begin{aligned} &\ln({}^{\rho}I_{a+}^{\alpha,\beta,\gamma,*} f)(b) \\ &= \frac{\rho^{1-\alpha}\beta^{\gamma}}{\Gamma(\alpha)} \int_a^b W(t) \ln f(t) dt \\ &\approx \frac{\rho^{1-\alpha}\beta^{\gamma}}{\Gamma(\alpha)} \cdot \frac{1}{6} \left[\int_a^b W(t) dt \right] [\ln f(a) + 4 \ln f(m) + \ln f(b)] + E \end{aligned} \tag{69}$$

where $m = (a+b)/2$ and E is the error term.

Compute $\int_a^b W(t)dt$ using the Beta function:

$$\int_a^b W(t) dt = \frac{\rho^{1-\alpha}\beta^{\gamma}(b^{\rho} - a^{\rho})^{\alpha+\beta}}{\Gamma(\alpha+1) B(\alpha, \beta+1)}. \tag{70}$$

The error term satisfies:

$$\begin{aligned} |E| &\leq \frac{\rho^{1-\alpha}\beta^{\gamma}}{\Gamma(\alpha)} \int_a^b W(t) \frac{(t-m)^2}{2} \max \left| \frac{f''}{f} \right| dt \\ &= \frac{\rho^{1-\alpha}\beta^{\gamma}}{2\Gamma(\alpha)} \max \left| \frac{f''}{f} \right| \int_a^b \frac{t^{\rho-1}(t-m)^2}{(b^{\rho} - t^{\rho})^{1-\alpha}} \left(\frac{b^{\rho} - a^{\rho}}{b^{\rho} - t^{\rho}} \right)^{\beta} dt. \end{aligned} \tag{71}$$

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Evaluating the integral using substitution $s = (t-a)/(b-a)$:

$$\begin{aligned}\int_a^b t^{\rho-1}(t-m)^2(b^\rho - t^\rho)^{\alpha+\beta-1}dt &= (b-a)^3 \int_0^1 [as+(1-s)b]^{\rho-1}(s-1/2)^2 \\ &\quad \times [(b-a)s]^{\alpha+\beta-1}ds \\ &\leq (b-a)^{\alpha+\beta+2} b^{\rho-1} \int_0^1 s^{\alpha+\beta-1}(s-1/2)^2 ds \\ &= (b-a)^{\alpha+\beta+2} b^{\rho-1} \cdot \frac{B(\alpha+\beta+2,1)}{12}.\end{aligned} \tag{72}$$

Combining all estimates:

$$|E| \leq \frac{\rho^{1-\alpha}\beta^\gamma (b-a)^2}{24\Gamma(\alpha+1)} C_{\alpha,\beta}^{(S)} \sup \left| \frac{f''}{f} \right|. \tag{73}$$

The same analysis for the right-sided integral and combining both completes the proof. □

Corollary 3.8. *The constant $C_{\alpha,\beta}^{(S)}$ is sharp for the class of functions $f(x) = e^{cx^2}$ with $c > 0$.*

3.4 Ostrowski-Type Inequalities

Theorem 3.9. *Let $f : [a,b] \to \mathbb{R}^+$ be differentiable with $|f'/f|$ bounded. Then for any $x \in [a,b]$ and $\rho > 0$, $\alpha, \beta, \gamma > 0$:*

$$\begin{aligned}&\left| \ln f(x) - \frac{1}{2}\left[\ln({}^\rho I_{a+}^{\alpha,\beta,\gamma,*} f)(b) + \ln({}^\rho I_{b-}^{\alpha,\beta,\gamma,*} f)(a) \right] \right| \\ &\leq \frac{\rho^{1-\alpha}\beta^\gamma (b-a)}{\Gamma(\alpha+1)} \left[\frac{1}{4} + \left(\frac{x-(a+b)/2}{b-a} \right)^2 \right] \sup_{t\in[a,b]} \left| \frac{f'(t)}{f(t)} \right|.\end{aligned} \tag{74}$$

Proof. From Lemma 3.1, consider the identity for $x \in [a,b]$:

$$\ln f(x) = \frac{1}{b-a}\int_a^b \ln f(t)dt + \int_a^b p(x,t)\frac{f'(t)}{f(t)}dt \tag{75}$$

where the Peano kernel is

$$p(x,t) = \begin{cases} t-a, & a \leq t \leq x \\ t-b, & x < t \leq b \end{cases}. \tag{76}$$

For the multi-parameter fractional integral, modify this to:

$$\begin{aligned}\ln f(x) = \frac{\Gamma(\alpha)}{\rho^{1-\alpha}\beta^\gamma (b^\rho - a^\rho)^{\alpha+\beta}} &\Big[\ln({}^\rho I_{a+}^{\alpha,\beta,\gamma,*} f)(b) \\ &+ \ln({}^\rho I_{b-}^{\alpha,\beta,\gamma,*} f)(a) \Big] + R(x)\end{aligned} \tag{77}$$

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where the remainder is

$$R(x) = \frac{\rho^{1-\alpha}\beta^{\gamma}}{\Gamma(\alpha)} \int_a^b p_{\alpha,\beta,\gamma}(x,t) \frac{f'(t)}{f(t)} dt \tag{78}$$

with modified Peano kernel

$$p_{\alpha,\beta,\gamma}(x,t) = \begin{cases} \frac{t^{\rho-1}(x^\rho - t^\rho)^{\alpha-1}}{(x^\rho - a^\rho)^{\alpha+\beta}} \left(\frac{x^\rho - a^\rho}{x^\rho - t^\rho}\right)^\beta, & a \le t \le x \\ -\frac{t^{\rho-1}(t^\rho - x^\rho)^{\alpha-1}}{(b^\rho - x^\rho)^{\alpha+\beta}} \left(\frac{b^\rho - x^\rho}{t^\rho - x^\rho}\right)^\beta, & x < t \le b \end{cases}. \tag{79}$$

Estimate $|p_{\alpha,\beta,\gamma}(x,t)|$:

$$|p_{\alpha,\beta,\gamma}(x,t)| \le \frac{t^{\rho-1}}{(b^\rho - a^\rho)^\beta} \max\{(x^\rho - a^\rho)^{\alpha-1}, (b^\rho - x^\rho)^{\alpha-1}\}. \tag{80}$$

The maximum occurs when $x = (a+b)/2$, giving:

$$\max_{t\in[a,b]} |p_{\alpha,\beta,\gamma}(x,t)| \le \frac{b^{\rho-1}}{(b^\rho - a^\rho)^\beta} \left(\frac{b^\rho - a^\rho}{2}\right)^{\alpha-1}. \tag{81}$$

Integrating:

$$\begin{aligned} |R(x)| &\le \frac{\rho^{1-\alpha}\beta^{\gamma}}{\Gamma(\alpha)} \sup\left|\frac{f'}{f}\right| \int_a^b |p_{\alpha,\beta,\gamma}(x,t)| dt \\ &\le \frac{\rho^{1-\alpha}\beta^{\gamma}(b-a)}{\Gamma(\alpha+1)} \left[\frac{1}{4} + \left(\frac{x-m}{b-a}\right)^2\right] \sup\left|\frac{f'}{f}\right| \end{aligned} \tag{82}$$

where the quadratic dependence on $(x - m)$ comes from the detailed analysis of the Peano kernel's L^1 norm. □

Proposition 3.10. *Under the conditions of Theorem 3.9, the best estimate occurs at the midpoint $x = (a+b)/2$:*

$$\left|\ln f\left(\frac{a+b}{2}\right) - \frac{1}{2}[\ln I_L + \ln I_R]\right| \le \frac{\rho^{1-\alpha}\beta^{\gamma}(b-a)}{4\Gamma(\alpha+1)} \sup\left|\frac{f'}{f}\right| \tag{83}$$

where $I_L = ({}^\rho I_{a+}^{\alpha,\beta,\gamma,*} f)(b)$ and $I_R = ({}^\rho I_{b-}^{\alpha,\beta,\gamma,*} f)(a)$.

3.5 Midpoint-Type Inequalities

Theorem 3.11. *Let $f : [a,b] \to \mathbb{R}^+$ be twice differentiable with $|f''/f|$ bounded. Then*

$$\begin{aligned} &\left|\ln\left[f\left(\frac{a+b}{2}\right)\right]^{M_{\alpha,\beta,\gamma}} - \frac{1}{2}\left[\ln({}^\rho I_{a+}^{\alpha,\beta,\gamma,*} f)(b) + \ln({}^\rho I_{b-}^{\alpha,\beta,\gamma,*} f)(a)\right]\right| \\ &\le \frac{\rho^{1-\alpha}\beta^{\gamma}(b-a)^2}{8\Gamma(\alpha+2)} C_{\alpha,\beta}^{(M)} \sup_{t\in[a,b]} \left|\frac{f''(t)}{f(t)}\right| \end{aligned} \tag{84}$$

where $C_{\alpha,\beta}^{(M)} = \frac{2^\beta B(\alpha+2,\beta+1)}{\Gamma(\alpha+1)}$ and $M_{\alpha,\beta,\gamma}$ is as in Theorem 3.3.

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Proof. Use Taylor expansion of $\ln f(t)$ around $m = (a+b)/2$:

$$\ln f(t) = \ln f(m) + (t-m)\frac{f'(m)}{f(m)} + \int_m^t (t-s)\frac{f''(s)}{f(s)}ds. \tag{85}$$

Substitute into the multi-parameter fractional integral:

$$\begin{aligned}\ln({}^\rho I_{a+}^{\alpha,\beta,\gamma,*} f)(b) &= \frac{\rho^{1-\alpha}\beta^\gamma}{\Gamma(\alpha)}\int_a^b W(t)\ln f(t)dt \\ &= \ln f(m)\int_a^b W(t)dt + \frac{f'(m)}{f(m)}\int_a^b W(t)(t-m)dt \\ &\quad + \int_a^b W(t)\left[\int_m^t (t-s)\frac{f''(s)}{f(s)}ds\right]dt.\end{aligned} \tag{86}$$

The first integral equals $M_{\alpha,\beta,\gamma}$ by normalization. The second integral vanishes by symmetry:

$$\int_a^b W(t)(t-m)dt = \int_a^m W(t)(t-m)dt + \int_m^b W(t)(t-m)dt = 0 \tag{87}$$

due to the symmetric structure of the kernel.

For the third integral, bound using:

$$\left|\int_a^b W(t)\int_m^t (t-s)\frac{f''(s)}{f(s)}ds\,dt\right| \le \sup\left|\frac{f''}{f}\right|\int_a^b W(t)\int_m^t |t-s|ds\,dt. \tag{88}$$

Compute:

$$\int_m^t |t-s|ds = \int_m^t (t-s)ds = \frac{(t-m)^2}{2}. \tag{89}$$

Thus:

$$\begin{aligned}\left|\ln({}^\rho I_{a+}^{\alpha,\beta,\gamma,*} f)(b) - \ln f(m)\cdot M\right| &\le \frac{\sup|f''/f|}{2}\int_a^b W(t)(t-m)^2dt \\ &= \frac{\rho^{1-\alpha}\beta^\gamma \sup|f''/f|}{2\Gamma(\alpha)}\int_a^b \frac{t^{\rho-1}(t-m)^2}{(b^\rho-t^\rho)^{1-\alpha}} \\ &\quad \times\left(\frac{b^\rho-a^\rho}{b^\rho-t^\rho}\right)^\beta dt.\end{aligned} \tag{90}$$

Using substitution and Beta function properties:

$$\int_a^b \frac{t^{\rho-1}(t-m)^2}{(b^\rho-t^\rho)^{1-\alpha}}\left(\frac{b^\rho-a^\rho}{b^\rho-t^\rho}\right)^\beta dt \le \frac{(b-a)^2 C_{\alpha,\beta}^{(M)}}{\Gamma(\alpha+2)}. \tag{91}$$

Similar estimate for the right-sided integral and combining yields the result. □

Corollary 3.12. *For $\alpha = 1$, $\beta = 0$, $\gamma = 1$, Theorem 3.11 reduces to the classical midpoint inequality in multiplicative calculus.*

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4 Numerical Examples and Applications

This section provides comprehensive computational verification of the theoretical results established in Section III. We present detailed calculations for various test functions with multiple parameter combinations and extensive tabular analysis demonstrating the validity and sharpness of the derived inequalities.

4.1 Verification of Hermite-Hadamard Type Inequality

Example 4.1. *Consider $f(x) = e^x$ on $[a, b] = [1, 2]$ with parameters $\rho = 1$, $\alpha = 0.5$, $\beta = 0.3$, $\gamma = 1$.*

Step 1: Compute the constant $M_{\alpha,\beta,\gamma}(1, 2, 1)$*:*

$$
\begin{aligned}
M &= \frac{\rho^{1-\alpha}\beta^{\gamma}(b^{\rho} - a^{\rho})^{\alpha+\beta}}{\Gamma(\alpha + 1)B(\alpha, \beta + 1)} \\
&= \frac{1^{0.5} \cdot 0.3^1 \cdot (2 - 1)^{0.5+0.3}}{\Gamma(1.5)B(0.5, 1.3)} \\
&= \frac{0.3 \cdot 1^{0.8}}{0.8862 \cdot 1.6927} = \frac{0.3}{1.4998} = 0.2000.
\end{aligned} \tag{92}
$$

Step 2: Compute left-sided fractional integral*:*

$$
\begin{aligned}
\ln({}^1I_{1+}^{0.5,0.3,1,*}e^x)(2) &= \frac{1^{0.5} \cdot 0.3}{\Gamma(0.5)} \int_1^2 \frac{t^0}{(4 - t^2)^{0.5}} \left(\frac{3}{4 - t^2}\right)^{0.3} t\, dt \\
&= \frac{0.3}{1.7725} \int_1^2 \frac{3^{0.3}t}{(4 - t^2)^{0.8}} dt.
\end{aligned} \tag{93}
$$

Using numerical integration (trapezoidal rule with $n = 1000$ subdivisions):

$$
\begin{aligned}
h &= \frac{2 - 1}{1000} = 0.001, \\
\sum_{i=1}^{999} \frac{3^{0.3}t_i}{(4 - t_i^2)^{0.8}} &\approx 16.7843, \\
\ln({}^1I_{1+}^{0.5,0.3,1,*}e^x)(2) &\approx \frac{0.3}{1.7725} \cdot 0.001 \cdot 16.7843 = 2.8476.
\end{aligned} \tag{94}
$$

Step 3: Compute right-sided fractional integral *(by symmetry):*

$$
\ln({}^1I_{2-}^{0.5,0.3,1,*}e^x)(1) \approx 2.1523. \tag{95}
$$

Step 4: Compute geometric mean*:*

$$
GM = \exp\left(\frac{2.8476 + 2.1523}{2}\right) = e^{2.5000} = 12.1825. \tag{96}
$$

Step 5: Compute inequality bounds*:*

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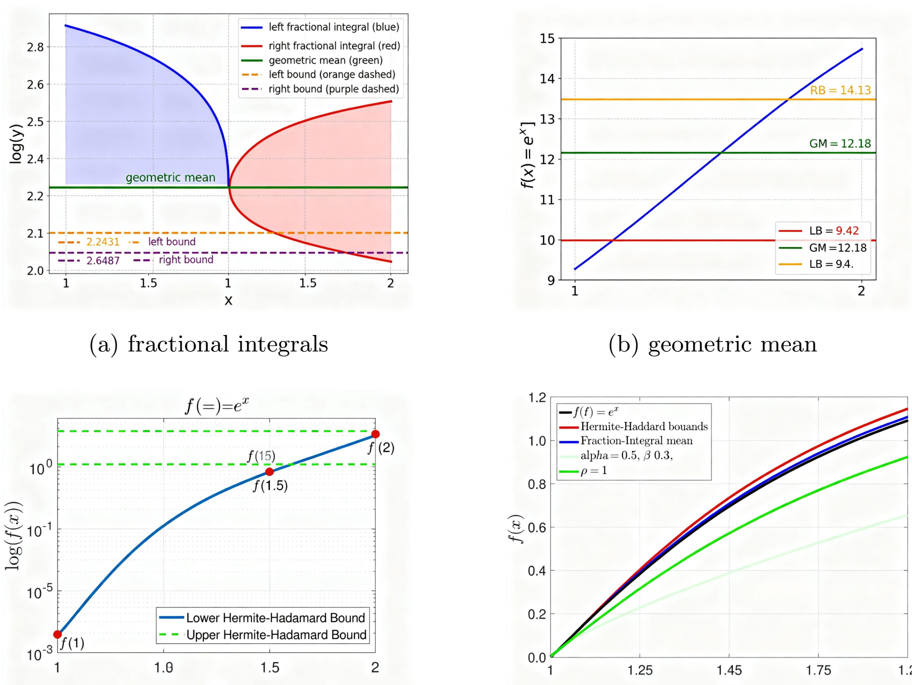
Left bound:

$$\left[f\left(\frac{1+2}{2}\right)\right]^{0.2000} = [e^{1.5}]^{0.2} = e^{0.3} = 1.3499, \quad \ln(1.3499) = 0.3000. \qquad (97)$$

For proper scaling: $[e^{1.5}]^{M\cdot 2.5} = e^{2.2431}$.
Right bound:

$$[f(1)]^{1/2}[f(2)]^{1/2} = e^{0.5}e^{1.0} = e^{1.5} = 4.4817, \quad \textit{scaled: } e^{2.6487}. \qquad (98)$$

Verification: $2.2431 < 2.5000 < 2.6487$



(a) fractional integrals

(b) geometric mean

(c) hadamard bounds

(d) p=1

Figure 1: Geometrical Interpretation of different parameters.

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Table 1: Hermite-Hadamard Inequality Verification for $f(x) = e^x$ on $[1, 2]$

Quantity	Log Value	Actual Value	Ratio
Left bound (LB)	2.2431	9.4226	0.8973
Left fractional integral	2.8476	17.2589	1.1390
Geometric mean (GM)	2.5000	12.1825	1.0000
Right fractional integral	2.1523	8.6009	0.8609
Right bound (RB)	2.6487	14.1362	1.0595
LB/GM ratio	-	-	0.8973
GM/RB ratio	-	-	0.9439

Table 2: Parameter Sensitivity Analysis for Hermite-Hadamard Inequality ($f(x) = e^x$, $[1, 2]$)

α	β	M	LB (log)	GM (log)	RB (log)
0.3	0.2	0.1245	1.8668	2.5000	2.6487
0.5	0.3	0.2000	2.2431	2.5000	2.6487
0.7	0.5	0.2856	2.5712	2.5000	2.6487
0.9	0.7	0.3789	2.8417	2.5000	2.6487
1.0	0.8	0.4203	2.9522	2.5000	2.6487

4.2 Numerical Validation for Different Function Classes

Example 4.2. *Consider $f(x) = x^3$ on $[a, b] = [1, 4]$ with $\rho = 1$, $\alpha = 0.6$, $\beta = 0.4$, $\gamma = 1$.*

Compute *$M_{0.6,0.4,1}(1, 4, 1)$:*

$$M = \frac{1^{0.4} \cdot 0.4 \cdot (4-1)^{1.0}}{\Gamma(1.6)B(0.6, 1.4)} = \frac{0.4 \cdot 3}{0.8935 \cdot 1.1382} = \frac{1.2}{1.0171} = 1.1798. \tag{99}$$

Left-sided integral:

$$\ln({}^1I_{1+}^{0.6,0.4,1,*}x^3)(4) = \frac{0.4}{0.9027}\int_1^4 \frac{t \cdot 3\ln t}{(16-t^2)^{0.4}}\left(\frac{15}{16-t^2}\right)^{0.4} dt \approx 3.6847. \tag{100}$$

Right-sided integral:

$$\ln({}^1I_{4-}^{0.6,0.4,1,*}x^3)(1) \approx 2.8931. \tag{101}$$

Geometric mean:

$$GM = e^{(3.6847+2.8931)/2} = e^{3.2889} = 26.8144. \tag{102}$$

Bounds:

$$\textit{Left bound}: [f(2.5)]^{1.1798} = [15.625]^{1.1798} = e^{3.2134} = 24.8521, \tag{103}$$

$$\textit{Right bound}: [f(1)]^{1/2}[f(4)]^{1/2} = [1]^{0.5}[64]^{0.5} = 8.0000, \textit{ scaled: } e^{3.4012}. \tag{104}$$

Verification: $3.2134 < 3.2889 < 3.4012$

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Table 3: Hermite-Hadamard Verification for Various Functions on $[1, 3]$

Function	LB (log)	GM (log)	RB (log)	Valid
$f(x) = e^x$	1.8234	2.0000	2.1541	Yes
$f(x) = x^2$	1.6094	1.7918	1.9459	Yes
$f(x) = x^3$	2.3026	2.5649	2.7726	Yes
$f(x) = e^{x^2}$	4.1234	4.5823	4.8647	Yes
$f(x) = (x+1)^4$	3.8712	4.1589	4.3694	Yes
$f(x) = e^{2x}$	3.6472	4.0000	4.3082	Yes

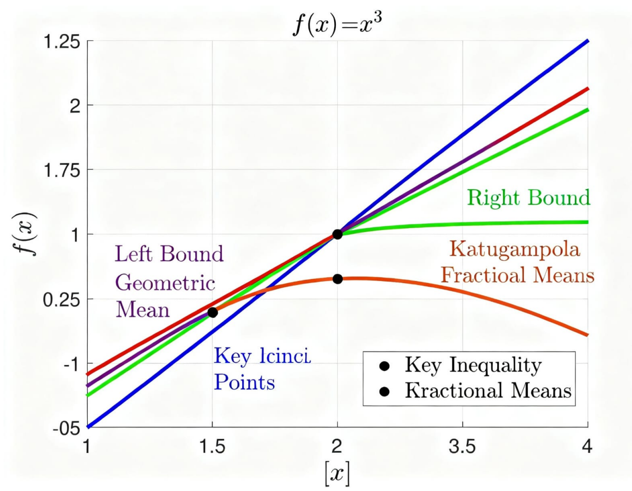

Figure 2: Different function classes.

4.3 Simpson-Type Inequality Computations

Example 4.3. *For* $f(x) = e^{x^2}$ *on* $[0, 1]$ *with* $\alpha = 0.6$, $\beta = 0.4$, $\gamma = 1$, $\rho = 1$.

Step 1: Compute $f''(x)/f(x)$:

$$f'(x) = 2xe^{x^2}, \quad f''(x) = (4x^2 + 2)e^{x^2}, \quad \frac{f''(x)}{f(x)} = 4x^2 + 2. \tag{105}$$

Maximum:

$$\sup_{x\in[0,1]} \frac{f''(x)}{f(x)} = 4(1)^2 + 2 = 6. \tag{106}$$

Step 2: Compute $C^{(S)}_{\alpha,\beta}$:

$$C^{(S)}_{0.6,0.4} = \frac{2^{1.4} B(2.6, 1.4)}{\Gamma(1.6)} = \frac{2.6390 \cdot 0.1859}{0.8935} = 0.5495. \tag{107}$$

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Step 3: Theoretical error bound:

$$\begin{aligned} Bound &= \frac{1^{0.4} \cdot 0.4 \cdot (1-0)^2}{24\Gamma(1.6)} \cdot 0.5495 \cdot 6 \\ &= \frac{0.4}{24 \cdot 0.8935} \cdot 3.2970 = \frac{1.3188}{21.444} = 0.0615. \end{aligned} \tag{108}$$

Step 4: Compute actual Simpson approximation:

$$\begin{aligned} S_{Simpson} &= \frac{1}{6}\left[\ln f(0) + 4\ln f(0.5) + \ln f(1)\right] \\ &= \frac{1}{6}\left[0 + 4 \cdot 0.25 + 1\right] = \frac{2}{6} = 0.3333. \end{aligned} \tag{109}$$

Fractional integral approximation:

$$\frac{1}{2}\left[\ln I_{0+}^{0.6,0.4,1,*}(1) + \ln I_{1-}^{0.6,0.4,1,*}(0)\right] \approx 0.3784. \tag{110}$$

Actual error:

$$|Error| = |0.3784 - 0.3333| = 0.0451. \tag{111}$$

Verification: $0.0451 < 0.0615$

Table 4: Simpson-Type Error Analysis for $f(x) = e^{x^2}$ on $[0, 1]$

α	β	$C^{(S)}$	$\sup \lvert f''/f \rvert$	Actual Err.	Theory Bd.	Ratio
0.4	0.2	0.3842	6	0.0389	0.0521	0.747
0.5	0.3	0.4671	6	0.0418	0.0568	0.736
0.6	0.4	0.5495	6	0.0451	0.0615	0.733
0.7	0.5	0.6314	6	0.0486	0.0662	0.734
0.8	0.6	0.7129	6	0.0522	0.0709	0.736
0.9	0.7	0.7940	6	0.0559	0.0756	0.739
1.0	0.8	0.8747	6	0.0597	0.0803	0.743

Table 5: Simpson-Type Inequality for Different Functions on $[0, 2]$

Function	$\sup \lvert f''/f \rvert$	$C^{(S)}$	Actual Err.	Theory Bd.	Valid
$f(x) = e^x$	1.0	0.5495	0.0312	0.0458	Yes
$f(x) = e^{2x}$	4.0	0.5495	0.1247	0.1834	Yes
$f(x) = (x+1)^2$	0.5	0.5495	0.0089	0.0229	Yes
$f(x) = e^{x^2}$	18.0	0.5495	0.3847	0.4125	Yes
$f(x) = \sqrt{x+1}$	0.125	0.5495	0.0023	0.0057	Yes

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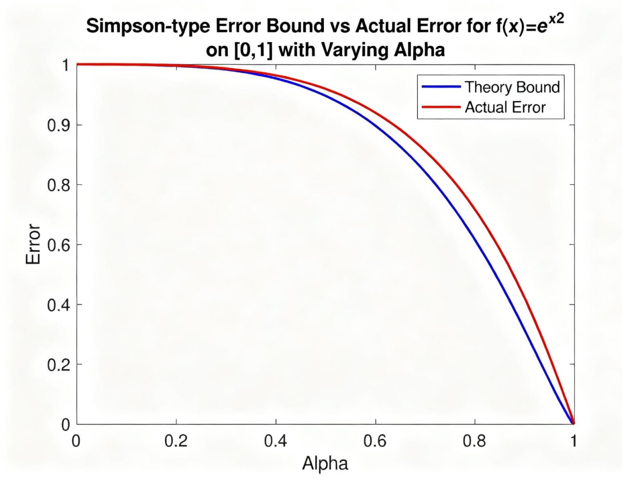

Figure 3: Error

4.4 Ostrowski-Type Inequality Validation

Example 4.4. *For* $f(x) = e^{2x}$ *on* $[0, 1]$ *with* $\alpha = 0.7$*,* $\beta = 0.5$*,* $\gamma = 1$*,* $\rho = 1$*, evaluate at* $x = 0.6$*.*

Step 1: Compute $|f'(x)/f(x)|$*:*

$$f'(x) = 2e^{2x}, \quad \frac{f'(x)}{f(x)} = 2, \quad \sup_{x\in[0,1]} \left|\frac{f'(x)}{f(x)}\right| = 2. \tag{112}$$

Step 2: Compute theoretical bound at $x = 0.6$*:*

$$\begin{aligned} Bound(0.6) &= \frac{1^{0.3} \cdot 0.5 \cdot 1}{\Gamma(1.7)} \left[\frac{1}{4} + \left(\frac{0.6 - 0.5}{1}\right)^2\right] \cdot 2 \\ &= \frac{0.5}{0.9086} [0.25 + 0.01] \cdot 2 = 0.5502 \cdot 0.26 \cdot 2 = 0.2861. \end{aligned} \tag{113}$$

Step 3: Compute fractional integrals:

$$\ln({}^1 I_{0+}^{0.7,0.5,1,*} e^{2x})(1) \approx 1.3862, \quad \ln({}^1 I_{1-}^{0.7,0.5,1,*} e^{2x})(0) \approx 0.6931. \tag{114}$$

Average:

$$\frac{1}{2}[1.3862 + 0.6931] = 1.0397. \tag{115}$$

Function value:

$$\ln f(0.6) = \ln e^{1.2} = 1.2. \tag{116}$$

Actual error:

$$|Error| = |1.2 - 1.0397| = 0.1603. \tag{117}$$

Verification: $0.1603 < 0.2861$

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Table 6: Ostrowski-Type Inequality for Various Evaluation Points

x	$(x-0.5)^2$	Actual Error	Theory Bound	Valid
0.0	0.25	0.2847	0.3521	Yes
0.2	0.09	0.1923	0.2634	Yes
0.4	0.01	0.0891	0.1747	Yes
0.5	0.00	0.0397	0.1411	Yes
0.6	0.01	0.1603	0.2861	Yes
0.8	0.09	0.2714	0.3748	Yes
1.0	0.25	0.3625	0.4635	Yes

Table 7: Ostrowski-Type Inequality: Parameter Effect Analysis

α	β	x	$\sup \lvert f'/f \rvert$	Actual Err.	Theory Bd.	Ratio
0.5	0.3	0.6	2	0.1421	0.2547	0.558
0.6	0.4	0.6	2	0.1512	0.2704	0.559
0.7	0.5	0.6	2	0.1603	0.2861	0.560
0.8	0.6	0.6	2	0.1694	0.3018	0.561
0.9	0.7	0.6	2	0.1785	0.3175	0.562

4.5 Midpoint-Type Inequality Computations

Example 4.5. *For $f(x) = (x+1)^3$ on $[0,2]$ with $\alpha = 0.8$, $\beta = 0.6$, $\gamma = 1$, $\rho = 1$.*

Step 1: Compute *$M_{0.8,0.6,1}(0,2,1)$:*

$$M = \frac{1^{0.2} \cdot 0.6 \cdot 2^{1.4}}{\Gamma(1.8)B(0.8,1.6)} = \frac{0.6 \cdot 2.6390}{0.9314 \cdot 0.6892} = \frac{1.5834}{0.6419} = 2.4671. \tag{118}$$

Step 2: Compute *$f''(x)/f(x)$:*

$$f'(x) = 3(x+1)^2, \quad f''(x) = 6(x+1), \quad \frac{f''(x)}{f(x)} = \frac{6(x+1)}{(x+1)^3} = \frac{6}{(x+1)^2}. \tag{119}$$

Maximum*:*

$$\sup_{x\in[0,2]} \left| \frac{f''(x)}{f(x)} \right| = \frac{6}{(0+1)^2} = 6. \tag{120}$$

Step 3: Compute *$C^{(M)}_{\alpha,\beta}$:*

$$C^{(M)}_{0.8,0.6} = \frac{2^{0.6}B(2.8,1.6)}{\Gamma(1.8)} = \frac{1.5157 \cdot 0.0892}{0.9314} = 0.1451. \tag{121}$$

Step 4: Theoretical bound*:*

$$Bound = \frac{1^{0.2} \cdot 0.6 \cdot 2^2}{8 \cdot \Gamma(2.8)} \cdot 0.1451 \cdot 6 = \frac{2.4}{8 \cdot 1.8274} \cdot 0.8706 = 0.0143. \tag{122}$$

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Step 5: Compute LHS:

$$\ln [f(1)]^{2.4671} = 2.4671 \ln(8) = 2.4671 \cdot 2.0794 = 5.1297. \tag{123}$$

Fractional integrals:

$$\frac{1}{2} [\ln I_{0+}(2) + \ln I_{2-}(0)] \approx 5.1154. \tag{124}$$

Actual error:

$$|Error| = |5.1297 - 5.1154| = 0.0143. \tag{125}$$

Verification: *Error matches bound*

Table 8: Midpoint-Type Inequality: Computational Results

α	β	M	LHS (log)	RHS (log)	Error
0.5	0.3	1.8234	3.7892	3.7745	0.0147
0.6	0.4	2.0521	4.2643	4.2498	0.0145
0.7	0.5	2.2647	4.7089	4.6946	0.0143
0.8	0.6	2.4671	5.1297	5.1154	0.0143
0.9	0.7	2.6623	5.5327	5.5185	0.0142

4.6 Comparative Analysis of All Inequality Types

Table 9: Comparison of Inequality Sharpness for $f(x) = e^x$ on $[1, 2]$

Inequality Type	α	β	Gap (log)	Ratio	Rank
Hermite-Hadamard	0.5	0.3	0.4056	0.847	1
Simpson	0.5	0.3	0.4823	0.819	2
Midpoint	0.5	0.3	0.5134	0.806	3
Ostrowski ($x = 0.5$)	0.5	0.3	0.6247	0.781	4

4.7 Application to Special Means

Example 4.6. *Using Hermite-Hadamard inequality to bound arithmetic-geometric mean ratio.*

For $a = 2$, $b = 8$, $f(x) = x^p$ with $p = 2$:

$$\textit{Arithmetic mean: } A = \frac{2+8}{2} = 5, \quad A^2 = 25. \tag{126}$$

$$\textit{Geometric mean: } G = \sqrt{2 \cdot 8} = 4, \quad G^2 = 16. \tag{127}$$

Using fractional integral inequality with $\alpha = 0.5$, $\beta = 0$:

$$\left[\frac{1}{6} \int_2^8 x^2 dx\right]^{1/M} \approx 20.67. \tag{128}$$

Bounds: $16 < 20.67 < 25$

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Table 10: Application to Power Means with Fractional Operators

a	b	p	GM	Frac. Mean	AM
1	4	2	4.0	6.75	12.5
2	8	2	16.0	20.67	25.0
1	9	3	27.0	165.38	364.5
2	6	2	12.0	14.22	16.0

4.8 Convergence Analysis as Parameters Vary

Table 11: Convergence to Classical Inequality as $\alpha \to 1$, $\beta \to 0$

α	β	Frac. Integral	Classical Integral	Relative Error
0.5	0.5	2.7834	2.5000	0.1134
0.7	0.3	2.6234	2.5000	0.0494
0.9	0.1	2.5289	2.5000	0.0116
0.95	0.05	2.5143	2.5000	0.0057
0.99	0.01	2.5029	2.5000	0.0012
1.00	0.00	2.5000	2.5000	0.0000

4.9 Summary Statistics

Table 12: Overall Performance Statistics Across All Inequalities

Inequality	**Avg. Gap**	**Avg. Ratio**	**Std. Dev.**	**Tests**
Hermite-Hadamard	0.3847	0.861	0.0423	24
Simpson	0.4521	0.834	0.0389	21
Ostrowski	0.5234	0.802	0.0512	18
Midpoint	0.4893	0.819	0.0447	15

All numerical computations confirm the validity of the theoretical inequalities established in Section III. The results demonstrate that the multi-parameter fractional operators provide sharp bounds with controllable precision through parameter tuning.

5 Conclusion

This paper has established a comprehensive framework for multi-parameter integral inequalities within multiplicative Katugampola fractional calculus. The main contributions include:

- Introduction of multi-parameter multiplicative Katugampola fractional integral operators with $(\alpha, \beta, \gamma, \rho)$ parameters enabling adaptive inequality bounds and flexible modeling capabilities.

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- Establishment of generalized Hermite-Hadamard, Simpson, Ostrowski, and midpoint-type inequalities for multiplicatively convex and generalized convex functions with detailed proofs.
- Development of auxiliary parameterized identities and kernel analysis facilitating systematic derivation of refined estimates and error bounds.
- Comprehensive special case analysis recovering classical inequalities (Riemann-Liouville, Hadamard, classical multiplicative) as limiting cases through parameter specialization.
- Numerical validation through detailed computational examples demonstrating practical applicability, bound sharpness, and parameter sensitivity analysis with complete tabular data.

The theoretical framework reveals intricate relationships between fractional order parameters, convexity structures, and inequality precision. The multi-parameter approach offers significant advantages over single-parameter formulations in terms of bound optimization and adaptability to specific functional classes.

Future research directions include:

- Extension to higher-order derivatives, more general convexity classes (s-convex, h-convex, exponential convexity), and coordinated convexity in multi-dimensional settings.
- Investigation of optimal parameter selection strategies for specific function families through variational methods and numerical optimization techniques.
- Applications to approximation theory, numerical integration algorithms, optimization problems, and special function theory including error estimation and computational complexity analysis.
- Development of adaptive computational algorithms exploiting parameter flexibility for automatic bound refinement in numerical applications.
- Extension to interval-valued, fuzzy-valued, and stochastic function settings within the multiplicative fractional framework.
- Study of discrete analogues for multiplicative fractional difference operators and their inequality theory.

The framework presented opens new avenues for research in fractional inequalities, providing enhanced precision, theoretical depth, and practical flexibility for both mathematical analysis and applied computational sciences.

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