International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

DEFORMABLE EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

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Abstract

This paper introduces a few new outcomes of deformable fractional calculus of multivariable functions for a recently proposed deformable derivative. It proposes a deformable form of Euler's Theorem on homogeneous functions. As an application, the proposed results are applied to homogeneous functions.

Math. Subj. Classification 2020:

26A33: Fractional derivatives and integrals

Keywords: Deformable derivative, Deformable Euler's Theorem, Multivariate deformable Calculus, Generalized derivative, Optimization in Applied Mathematics.

1. Introduction

The theory of fractional calculus is a well-known topic among researchers. This theory is an extension of ordinary calculus, which deals with the differentiation and integration of fractional order. Also, the calculus of multivariable is essential to solving many problems in mathematics, science, and engineering. The concept of partial differentiation is used to find the derivative of the functions with more than one variable [5, 6].

Fractional calculus includes famous definitions like the Riemann-Liouville operator, Grunwald-Letnikov operator, Caputo operator, Riesz fractional operator, and many more [7]. In the majority, the integral forms of the fractional derivatives define the fractional derivative. R. Khalil, in 2014, presented a conformable fractional derivative based on the limit approach.

Received: August 02, 2025

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
However, this definition does not deal with non-positive numbers [4]. Thus, to overcome these shortcomings, a deformable derivative is introduced in [2, 9].

The remaining article will proceed in the following manner. Section 2 will discuss basic concepts and preliminaries of deformable fractional calculus. Section 3 will present the deformable partial derivative (DPD) with some important outcomes on homogeneous functions. Section 4 will introduce the deformable form of Euler's Theorem and its extension to higher-order derivatives.

2. Preliminaries

Definition 2.1. For a given real-valued function t in (a,b), the *deformable derivative* of fractional order α is

(1) $$\lim_{\delta \to 0} \frac{(1+\delta(1-\alpha)t(\theta+\delta\alpha)-t(\theta)}{\delta},$$

for all $0 \leqslant \alpha \leqslant 1$. If the above limit exists, the symbol $D^{\alpha}[t(\theta)]$ denotes the deformable derivative of t and is said to be α-differentiable at the point $\theta \in (a,b)$.

Remark 2.2. If t is differentiable at $\theta \in (a,b)$, then

(2) $$D^{\alpha}t(\theta) = (1-\alpha)\, t(\theta) + \alpha D t(\theta).$$

The above definition is evident for $\alpha = 0, 1$, i.e., $D^{0}[t(\theta)] = t(\theta)$ (function itself) and $D[t(\theta)] = t'(\theta)$ (first ordinary derivative).

For simplicity, $0 \leqslant \alpha \leqslant 1$ is taken in the remaining part of the article.

Theorem 2.3. *For some* $\alpha \in [0,1]$, *if* t *is* α*-differentiable at* $\theta_0 \in (a,b)$, *then* t *is continuous at* $\theta_0 \in (a,b)$.

Theorem 2.4. *Let* t_1 *and* t_2 *are* α*-differentiable in* (a,b). *Then, for* $\theta \in (a,b)$

(a) $D^{\alpha}(k_1t_1 + k_2t_2)(\theta) = aD^{\alpha}t_1(\theta) + bD^{\alpha}t_2(\theta)$, *for all* $k_1,k_2 \in \mathrm{R}$

(b) $D\alpha 1 \cdot D\alpha 2 t 1(\theta) = D\alpha 2 \cdot D\alpha 1 t 1(\theta)$.

(c) $D^{\alpha}(k) = (1-\alpha)k$, *for all constant functions* $t_1(\theta) = k$.

(d) $D^{\alpha}(t_1 \cdot t_2)(\theta) = (D^{\alpha}t_1)(\theta) \cdot t_2(\theta) + \alpha t_1(\theta) \cdot Dt_2(\theta)$.

Particularly, we have

(1) $D^{\alpha}(\theta^{x}) = (1-\alpha)\theta^{x} + \theta\alpha t^{x}-1,\ x \in R$.

(2) $D^{\alpha}(e^{\theta}) = (1-\alpha)e^{\theta} + \alpha e^{\theta}$.

(3) $D^{\alpha}(\sin\theta) = (1-\alpha)\sin^{\theta} + \alpha\cos\theta$.

(4) $D^{\alpha}(\ln\theta) = (1-\alpha)\ln\theta + \dfrac{\alpha}{\theta}$, $\theta > 0$.

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Theorem 2.5 ([1]). *(Chain Rule) Let t_1 and t_2 are two α-differentiable functions at $\theta \in (a,b)$ and $t(\theta) = t_1(t_2(\theta))$. Then $t(\theta)$ is α-differentiable at $\theta \in (a,b)$ such that*

(3) $$D^{\alpha}t(\theta) = (1-\alpha)(t_1(t_2(\theta))) + \alpha D(t_1(t_2(\theta))).$$

Remark 2.6. The sequential deformable derivative of T of order n, for $\alpha \in [0,1]$ and $n \in Z^+$ is defined as

(4) $$^{(n)}D^{\alpha}t(\theta) = D^{\alpha}D^{\alpha}.....D^{\alpha}t(\theta)(n - times).$$

Also, if for some α, t is continuously differentiable n-times at $\theta \in (a,b)$, the sequential deformable derivative of order n is continuous at $\theta \in (a,b)$.

3. Deformable partial derivative

This section introduces a real-valued multi-variable function's DPD. Also, some basic outcomes of homogeneous functions have been included.

Definition 3.1. For a given n-variate real-valued function $t(\theta)$, where $\theta = (\theta_1, \theta_2,..,\theta_n) \in R^n$, *ith* DPD of t of fractional order α is

(3) $$\lim_{\delta \to 0} \frac{(1+\delta(1-\alpha))t(\theta_1,..,\theta_i+\delta\alpha,..,\theta_n) - t(\theta_1,\theta_2,...,\theta_n)}{\delta}$$

for all $0 \leqslant \alpha \leqslant 1$. If the above limit exists, the symbol $D^{\alpha}_{t_i}t(\theta)$ denotes the DPD such that $D^{\alpha}_{t_i}t(\theta) = (1-\alpha)t(\theta) + \alpha D_{\theta i}t(\theta)$.

Remark 3.2. Let $t(\theta)$ be a n-variate real-valued function, on an open set S with $(\theta_1, \theta_2,..,\theta_n) \in S$. The function t is said to be $D^{\alpha(m)}(S,R)$ for $m \in Z^+$, if all DPDs of order $\leqslant m$ exist and continuous on S.

Definition 3.3 ([3]). For a given n-variate real-valued function $t(\theta)$ defined on a set S such that $(t\theta_1,t\theta_2,..,t\theta_n) \in S$ whenever $t > 0$ and $(\theta_1,\theta_2,..,\theta_n) \in S$. If

(4) $t(\lambda\theta_1,\lambda\theta_2,..,\lambda\theta_n) = \lambda^s f(\theta_1,\theta_2,..,\theta_n)$, $\forall$ $(\theta_1,\theta_2,..,\theta_n) \in S$ and $\lambda > 0$, then t is a homogeneous function (HF) of degree s.

Lemma 3.4. *An n-variate real-valued function $t(\theta)$ on a set S is HF of degree s iff*

(5) $$t(\theta_1,\theta_2,..,\theta_n) = \theta_1^s\phi\left(\frac{\theta_2}{\theta_1},\frac{\theta_3}{\theta_1},...,\frac{\theta_n}{\theta_1}\right) \; if \; \theta_1 > 0 .$$

Theorem 3.5. *Let t_1 and t_2 be two n-variate real-valued functions on a set S with $(\lambda\theta_1,\lambda\theta_2,..,\lambda\theta_n) \in S$, $\lambda > 0$ and $(\theta_1,\theta_2,..,\theta_n) \in S$. If t_1 and t_2 are HFs of degree s, then $t_1 + t_2$ and λt are also HFs of degree s on set S.*

Proof. We have

$$\begin{aligned}(t_1+t_2)(\lambda\theta_1,\lambda\theta_2,..,\lambda\theta_n) &= t_1(\lambda\theta_1,\lambda\theta_2,..,\lambda\theta_n) + t_2(\lambda\theta_1,\lambda\theta_2,..,\lambda\theta_n)\\ &= \lambda^s t_1(\theta_1,\theta_2,..,\theta_n) + \lambda^s t_2(\theta_1,\theta_2,..,\theta_n)\\ &= \lambda^s(t_1(\theta_1,\theta_2,..,\theta_n) + t_2(\theta_1,\theta_2,..,\theta_n))\end{aligned}$$

Received: August 02, 2025

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$= \lambda^s(h_1 + h_2)(\theta_1,\theta_2,..,\theta_n)$$

and

$$\begin{aligned}(kt)(\lambda\theta_1,\lambda\theta_2,..,\lambda\theta_n) &= kt(\lambda\theta_1,\lambda\theta_2,..,\lambda\theta_n)\\ &= k\lambda^s t(\theta_1,\theta_2,..,\theta_n)\\ &= \lambda^s\,(kt(\theta_1,\theta_2,..,\theta_n))\\ &= \lambda^s(kt)(\theta_1,\theta_2,..,\theta_n).\end{aligned}$$

This completes the proof.

Theorem 3.6. *Let t_1 and t_2 be two n-variate real-valued functions on a set S with $(\lambda\theta_1, \lambda\theta_2,..,\lambda\theta_n) \in S$, $\lambda > 0$ and $(\theta_1,\theta_2,..,\theta_n) \in S$. If t_1 and t_2 are HFs of degree s_1 and s_2, respectively, then $t_1 \cdot t_2$ and t_1/t_2 are HFs of degree $s_1 + s_2$ and $s_1 - s_2$, respectively, such that $t_2(\theta_1, \theta_2,..,\theta_n) = 0/$ for all $(\theta_1,\theta_2,..,\theta_n) \in S$.*

Proof. We have

$$\begin{aligned}(t_1 \cdot t_2)(\lambda\theta_1,\lambda\theta_2,..,\lambda\theta_n) &= t_1(\lambda\theta_1,\lambda\theta_2,..,\lambda\theta_n) \cdot t_2(\lambda\theta_1,\lambda\theta_2,..,\lambda\theta_n)\\ &= t^{s1}\lambda_1(\theta_1,\theta_2,..,\theta_n) \cdot \lambda^{s2}t_2(\theta_1,\theta_2,..,\theta_n)\end{aligned}$$

$$= \lambda^{s1+s2}\,(t_1(\theta_1,\theta_2,..,\theta_n) \cdot t_2(\theta_1,\theta_2,..,\theta_n)) = \lambda s1{+}s2(t1 \cdot t2)(\theta1,\theta2,..,\theta n)$$

and

$$\begin{aligned}\left(\frac{t_1}{t_2}\right)(\lambda\theta_1,\lambda\theta_2,..,\lambda\theta_n) &= \frac{t_1(\lambda\theta_1,\lambda\theta_2,..,\lambda\theta_n)}{t_2(\lambda\theta_1,\lambda\theta_2,..,\lambda\theta_n)}\\ &= \frac{\lambda^{s_1}t_1(\theta_1,\theta_2,..,\theta_n)}{\lambda^{s_2}t_2(\theta_1,\theta_2,..,\theta_n)}\\ &= \lambda^{s_1-s_2}\frac{t_1(\theta_1,\theta_2,..,\theta_n)}{t_1(\theta_1,\theta_2,..,\theta_n)}\\ &= \lambda^{s_1-s_2}\left(\frac{t_1}{t_2}\right)(\theta_1,\theta_2,..,\theta_n)\end{aligned}$$

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This completes the proof.

Theorem 3.7.

Let t_1 be a n-variate real-valued functions on a set S with $(\theta_1,\theta_2,..,\theta_n) \in S$ and $(\lambda\theta_1,\lambda\theta_2,..,\lambda\theta_n) \in S$ such that $\lambda > 0$. Also, t_2 is an n-variate real-valued function on a set S_1 such that $\lambda\theta \in S_2$ whenever $\lambda > 0$ and $\theta \in S_2$. If $t_1(S_1) \subset S_2$ and that t_1, t_2 are HFs of degree s_1 and s_2, respectively, then $t_2 \circ t_1$ is a HF of degree $s_1\, s_2$.

Proof. We have

$$(t_2 \circ t_1)(\lambda\theta_1,\lambda\theta_2,..,\lambda\theta_n) = t_2\,(t_1(\lambda\theta_1,\lambda\theta_2,..,\lambda\theta_n))$$

$$= t_2\,(\lambda^{s1}h_1(\theta_1,\theta_2,..,\theta_n)) = (\lambda^{s1})^{s2}t_2\,(t_1(\theta_1,\theta_2,..,\theta_n))$$

$$= \lambda^{s1s2}(t_2 \circ t_1)(\theta_1,\theta_2,..,\theta_n).$$

4. Deformable Euler's theorem

This section proposed Euler's theorem in this section by using DPDs on HFs. Also above theorem is extended for higher order derivatives on HFs [8]. Further, as an application above results are applied to an HF.

Theorem 4.1. (Deformable Euler's theorem) *Let t be an n-variate real-valued HF of degree s on an open set S and* $t \in D^{\alpha}(S,R)$*, then*

(6)
$$\sum_{i=1}^{n} \theta_i D^{\alpha}_{\theta_i} t = \left((1-\alpha) \sum_{i=1}^{n} \theta_i + \alpha s \right) t$$

Proof. Since t is a HF of degree s, then using (5), we get

(7)
$$t(\theta_1, \theta_2, .., \theta_n) = \theta_1^s \phi \left(\frac{\theta_2}{\theta_1}, \frac{\theta_3}{\theta_1}, ..., \frac{\theta_n}{\theta_1} \right).$$

On taking $\frac{\theta_i}{\theta_1} = y_i$, for all $i = 2,3,....,n$, then (7) becomes

(8)
$$t(\theta_1, \theta_2, .., \theta_n) = \theta_1^s \phi(y_2, y_3, ..., y_n) .$$

Since h is α-differentiable, on computing the DPD of (8) using chain rule with respect to θ_1 and then on multiplying by θ_1, we get

(9)
$$\theta_1 D^{\alpha}_{\theta_1} h = (1-\alpha)\theta_1^{s+1} \phi(y_2, y_3, ..., y_n) + \alpha s \theta_1^s \phi(y_2, y_3, ..., y_n) - \alpha \theta_1^{s-1} \sum_{i=2}^{n} \theta_i D^{\alpha}_{y_i} \phi$$

Now, on computing the DPD of (8) using chain rule with respect to θ_i and then on multiplying by θ_i for $i = 2,3,..,n$, we get

(10)
$$\theta_i D^{\alpha}_{\theta_i} h = (1-\alpha)\theta_i \theta_1^s \phi(y_2, y_3, ..., y_n) + \alpha \theta_1^{s-1} \theta_i D^{\alpha}_{y_i} \phi$$

On adding (9) and $(n-1)$ equations in (10), we get

$$\begin{aligned} \sum_{i=1}^{n} \theta_i D^{\alpha}_{\theta_i} t &= (1-\alpha)\theta_1^s \phi(y_2, y_3, ..., y_n) \sum_{i=1}^{n} \theta_i + \alpha s \theta_1^s \phi(y_2, y_3, ..., y_n) \\ &= \left((1-\alpha) \sum_{i=1}^{n} \theta_i + \alpha s \right) \theta_1^s \phi(y_2, y_3, ..., y_n) \\ &= \left((1-\alpha) \sum_{i=1}^{n} \theta_i + \alpha s \right) t. \end{aligned}$$

This completes the proof.

$$\sum_{i=1}^{3} \theta_i D^{\alpha}_{\theta_i} t = \left((1-\alpha) \sum_{i=1}^{3} \theta_i + 4\alpha \right) t$$

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \theta_i \theta_j {}^{(2)}D^{\alpha}{}_{\theta_i \theta_j} t = \left[(1-\alpha)^2 \left(\sum_{i=1}^{3} \theta_i \right)^2 + 8\alpha(1-\alpha) \left(\sum_{i=1}^{3} \theta_i \right) + 12\alpha^2 \right] t$$

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \theta_i \theta_j \theta_k {}^{(3)}D^{\alpha}{}_{\theta_i \theta_j \theta_k} h = \left[(1-\alpha)^3 \left(\sum_{i=1}^{3} \theta_i \right)^3 + 12\alpha(1-\alpha)^2 \left(\sum_{i=1}^{3} \theta_i \right)^2 + 36\alpha^2(1-\alpha) \left(\sum_{i=1}^{3} \theta_i \right) + 24\alpha^3 \right] t$$

Example 4.2. For a given HF $t(\theta_1, \theta_2, \theta_3) = \theta_1^4 + \theta_2^2 \theta_3^2$ of degree 4 and $t \in D^{\alpha(m)}$ on R^3, for all $m \in$

Z^+, we have and so on.

5. Conclusion

Some properties of homogeneous functions that involve their deformable partial derivatives are proposed and proven in this article, specifically, the homogeneity of the deformable partial derivatives of a homogeneous function. Moreover, a deformable version of Euler's theorem is also proposed. Furthermore, it is extended to higher-order partial derivatives. The findings of this study indicate that the results obtained in the deformable derivative case are consistent with those obtained in the ordinary case. Finally, this work applies to various areas of mathematical finance, energy-related problems, work systems, and more.

Conflict of Interest

The author declares no conflicts of interest.

Acknowledgement and Funding

The author declares that no financial support was received.

Data Availability

This manuscript has no associated data.

Ethical Conduct

Not applicable

References

[1] Thabet Abdeljawad, On conformable fractional calculus, *Journal of Computational and Applied Mathematics* **279** (2015), 57–66.

[2] Priyanka Ahuja, Fahed Zulfeqarr, and Amit Ujlayan, Deformable fractional derivative and its applications, *American institute of Physics(A.I.P.)*.

[3] A.P. Hiwarekar, Extension of eulers theorem on homogeneous function to higher derivatives, *Bulletin of the Marathwada Mathematical society* **10** (2009), 16–19.

[4] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, A new definition of fractional derivative, *Journal of Computational and Applied Mathematics* **264** (2014), 65 – 70.

[5] G. W. Leibniz, "letter from hanover, Germany to Johann G. F. A. L'Hospital, September 30, 1695", Leibniz Mathematische Schriften, Olms-Verlag, Hildeshein, Germany, 1962.

[6] Keith Oldham and Jerome Spanier, The fractional calculus theory and applications of differentiation and integration to arbitrary order, vol. 111, *Elsevier*, 1974.

[7] I. Podlubny, Fractional differential equations, *Academic Press, San Diego, California*, U.S.A., 1999.

[8] V. Shah and J. Sharma, Extension of eulers theorem on homogeneous function for finite variables and higher derivatives, *International journal of Engineering and Innovative Technology* **4** (2014), 84–87.

International Journal of Applied Mathematics

Volume 39 No. 1s, 2026

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

[9] Fahed Zulfeqarr, Amit Ujlayan, and Priyanka Ahuja, A generalization to ordinary derivative and its associated integral with some applications, *Punjab University Journal of Mathematics* **55,** No 4 (2023), 135 – 148.

Received: August 02, 2025