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A NOVEL FRACTIONAL DERIVATIVE AND ITS EFFICIENT COMPUTATION VIA TAYLOR SERIES APPROXIMATION

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Abstract

This article explores a novel fractional derivative introduced. Unlike traditional fractional derivatives such as Caputo or Riemann-Liouville, this derivative eliminates the memory effect while retaining essential properties required for fractional calculus. We provide an approximation of the proposed derivative and demonstrate its application through examples involving fractional differential equations.

Keywords: Memoryless and local fractional operator, Numerical approximation, Logistic and Lotka-Voltera model

MSC Classification: 26A33 , 34A08 , 35R11 , 65L60 , 44A15.

1 Introduction

Fractional derivatives generalize classical integer-order differentiation and have seen significant development in recent years. Among the most prominent formulations are the Caputo, Riemann-Liouville, and Gr¨unwald-Letnikov derivatives, which have found important applications in biological modeling [2–4]. However, these operators present several challenges: their inherent memory effects complicate both analysis and computation [5–7], initialization of fractional ODEs requires special treatment [8, 9], and numerical solutions demand substantial computational resources- particularly for high-dimensional PDE systems. For approaches to these numerical challenges, see [10–12]. To demonstrate these limitations, we highlight key findings from recent literature. Reference [13] provides a rigorous analysis of memory effects in Caputo and other fractional derivatives when applied to non-regular functions. The initialization challenges of conventional fractional derivatives are well-documented in [14]. While several approximation methods have been proposed- including a finite-state memory approach for Caputo derivatives [15] and rational approximations [16]- these methods often lack sufficient accuracy across broader use cases. In this work, we introduce and analyze a novel fractional derivative operator first presented in [1]. This derivative offers two key advantages: it is local in nature and completely memoryless. We establish its mathematical consistency and demonstrate that it satisfies all fundamental properties required of a fractional derivative. Furthermore, we provide the definition of its associated integral operator and present equivalent formulations using both Fourier and wavelet transforms.

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Unlike classical fractional derivatives, whose numerical discretization requires significant temporal and spatial storage resources, our proposed operator offers computational advantages. Traditional approaches necessitate knowledge of the solution history from the initial time point, while our formulation:

• Operates solely with current initial conditions

• Eliminates dependence on memory effects

• Requires no specialized storage of historical solution data

• Maintains computational efficiency through extensive testing on ordinary differential equations, we demonstrate the effectiveness of our numerical approximation.

We present and compare numerical results for various biological models, highlighting the practical benefits of our approach.

2 Mathematical Foundations of Fractional Derivatives

The Caputo derivative of order α, a cornerstone of fractional calculus, is defined as:

$$ {}^{C}D_{a}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-x)^{n-\alpha-1}u^{(n)}(x)\,dx, $$

where n = ⌈α⌉ denotes the smallest integer greater than or equal to α. Building on this concept, we introduce a local fractional derivative LDα, inspired by the construction in [1], with the following defining properties:

Definition 1 (Local Fractional Derivative). The operator $^{L}D^{\alpha}$ is called a local fractional derivative if it satisfies:

1. **Vanishing for constants:** If f is constant in a neighbourhood of x, then $^{L}D^{\alpha}f(x) = 0$.

2. **Power rule:** For any real k,

$$ {}^{L}D^{\alpha}t^{k} = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}t^{k-\alpha}. $$

3. **Linearity:** LDα is a linear operator.

4. **Approximation property**: If f is continuous near x, there exists a sequence of polynomials (P_n) converging uniformly to f in this neighbourhood. The derivative $^{L}D^{\alpha}f(x)$ exists if $\lim_{n\to\infty} {}^{L}D^{\alpha}P_n(t)$ converges uniformly near x, in which case $^{L}D^{\alpha}f(x)$ is defined as this limit.

Consistency of the definition: If $P_n = \sum_{i=0}^{n} a_i t^i$ and $Q_n = \sum_{i=0}^{n} b_i t^i$ are polynomials that converge uniformly to f in a neighbourhood of t, we show that if $D^{\alpha}P_n(x)$ converges to $D^{\alpha}f(x)$, then $D^{\alpha}Q_n(x)$ also converges to $D^{\alpha}f(x)$.

In fact:

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• We have $\|P_n - Q_n\| \to 0$, which implies $a_n \sim b_n$.

• Consequently,

$$\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} a_n \sim \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} b_n$$

• Thus, the fractional derivative series

$$\sum_{i=1}^{n} a_i \frac{\Gamma(i+1)}{\Gamma(i+1-\alpha)} t^{i-\alpha} \quad \text{and} \quad \sum_{i=1}^{n} b_i \frac{\Gamma(i+1)}{\Gamma(i+1-\alpha)} t^{i-\alpha}$$

have the same radius of convergence.

• Therefore,

$$\|D^\alpha P_n - D^\alpha Q_n\| \longrightarrow 0$$

This ensures that the fractional derivative $D^\alpha f$ is well-defined, independent of the choice of approximating polynomial sequence. We now present some properties of the local fractional operator:

Proposition 1. If $\mathfrak{F}$ denotes the Fourier transform, then:

$$\mathfrak{F}\,{}^L D^\alpha u(k) = (ik)^\alpha \mathfrak{F}(u)$$

Proof. We establish the property for piecewise polynomial functions of the form:

$$f(x) = \sum f_i(x) \chi_{[a_i, b_i]}(x)$$

where each f_i is a uniform limit of polynomials on the interval $[a_i,b_i]$.

1. **For monomials restricted to** $[a_i,b_i]$For any monomial t_n, we have:
$$\mathfrak{F}\left({}^L D^\alpha \left(t^n_{[a_i,b_i]}\right)\right)(k) = (ik)^\alpha \mathfrak{F}\left(t^n_{[a_i,b_i]}\right)(k)$$
2. **For polynomials restricted to** $[a_i,b_i]$: By linearity of both the fractional derivative and Fourier transform, for any polynomial P(t):
$$\mathfrak{F}\left({}^L D^\alpha \left(P_{[a_i,b_i]}\right)\right)(k) = (ik)^\alpha \mathfrak{F}\left(P_{[a_i,b_i]}\right)(k)$$
3. **Extension to uniform limits**: For each component function f_i, let $(P_{i,n})_{n\in N}$ be a sequence of polynomials converging uniformly to f_i on $[a_i,b_i]$. Then:
$$\begin{aligned}\mathfrak{F}\left({}^L D^\alpha f_i\right)(k) &= \mathfrak{F}\left({}^L D^\alpha \left(\lim_{n\to\infty} P_{i,n}\right)\right)(k) \\ &= \lim_{n\to\infty} \mathfrak{F}\left({}^L D^\alpha P_{i,n}\right)(k) \\ &= (ik)^\alpha \lim_{n\to\infty} \mathfrak{F}\left(P_{i,n}\right)(k) \\ &= (ik)^\alpha \mathfrak{F}(f_i)(k)\end{aligned}$$
4. **Global result**: By linearity of the Fourier transform, the result extends to the piecewise function:
$$\mathfrak{F}\left({}^L D^\alpha f\right)(k) = \sum \mathfrak{F}\left({}^L D^\alpha f_i\right)(k) = (ik)^\alpha \sum \mathfrak{F}(f_i)(k) = (ik)^\alpha \mathfrak{F}(f)(k)$$

By the same approach, we define the local fractional integral as:

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Definition 2. The operator ${}^{L}I^{\alpha}$ is called the local fractional integral if it satisfies the following properties:

1. For any real k,
$${}^{L}I^{\alpha}t^{k} = \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}t^{k+\alpha}$$
2. ${}^{L}I^{\alpha}$ is a linear operator, i.e., for any functions f,g and constants a,b:
$${}^{L}I^{\alpha}(af+bg) = a\,{}^{L}I^{\alpha}f + b\,{}^{L}I^{\alpha}g$$
3. If f is continuous in a neighbourhood of a point x, then there exists a sequence of polynomials (P_n) that converges uniformly to f in this neighbourhood. We say ${}^{L}I_{\alpha}f(x)$ exists (or f has a fractional integral of order α) if the limit
$$\lim_{n\to\infty} {}^{L}I^{\alpha}P_n(x)$$
exists and the convergence is uniform in the neighbourhood of x. In this case, ${}^{L}I^{\alpha}f(x)$ is defined to be this limit.

Proposition 2. ${}^{L}I^{\alpha L}D^{\alpha}f = {}^{L}D^{\alpha L}I^{\alpha}f = f$.

Proof. We prove this result in three steps:

1. For monomials t^n:
$${}^{L}D^{\alpha}t^{n} = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}t^{n-\alpha}$$
$${}^{L}I^{\alpha}({}^{L}D^{\alpha}t^{n}) = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}\cdot\frac{\Gamma(n+1-\alpha)}{\Gamma(n+1)}t^{n} = t^{n}$$
Similarly,
$${}^{L}I^{\alpha}t^{n} = \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}t^{n+\alpha}$$
$${}^{L}D^{\alpha}({}^{L}I^{\alpha}t^{n}) = \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}\cdot\frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)}t^{n} = t^{n}$$
2. **For polynomials:** By linearity of both operators, the result extends immediately to any polynomial $P(t) = \sum_{k=0}^{m} a_k t^k$:
$${}^{L}I^{\alpha}\,{}^{L}D^{\alpha}P = {}^{L}D^{\alpha}\,{}^{L}I^{\alpha}P = P$$
3. **For general functions:** Let f be a function for which both operators are defined. By definition, there exists a sequence of polynomials (P_n) converging uniformly to f in a neighborhood. Then:
$$\begin{aligned}{}^{L}I^{\alpha}\,{}^{L}D^{\alpha}f &= \lim_{n\to\infty}{}^{L}I^{\alpha}\,{}^{L}D^{\alpha}P_n \\ &= \lim_{n\to\infty} P_n = f\end{aligned}$$
The same argument holds for ${}^{L}D^{\alpha}\,{}^{L}I^{\alpha}f$.

□

We now introduce a new definition of a fractional derivative called the fractional Fourier derivative:

Definition 3. For $f \in L^2(R)$, the fractional Fourier transform of f is defined as:
$${}^{f}\mathfrak{F}(f)(x) = \sum_{p\in\mathbb{N},q\in\mathbb{N}^*}\left[a_{p,q}\cos\left(\frac{p}{q}x\right) + b_{p,q}\sin\left(\frac{p}{q}x\right)\right]$$

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where the coefficients are given by:

$$a_{p,q} = \frac{1}{\pi}\int_0^1 f(t)\cos\left(\frac{p}{q}t\right)dt, \quad b_{p,q} = \frac{1}{\pi}\int_0^1 f(t)\sin\left(\frac{p}{q}t\right)dt$$

The fractional Fourier derivative is then defined as:

$${}^{f}D^{\alpha}(f)x = \sum_{\substack{p\in\mathbb{N},q\in\mathbb{N}^*\\ p\wedge q=1}} \left[a_{p,q}\left(\frac{p}{q}\right)^{\alpha}\cos\left(\frac{p}{q}x+\frac{\pi\alpha}{2}\right) + b_{p,q}\left(\frac{p}{q}\right)^{\alpha}\sin\left(\frac{p}{q}x+\frac{\pi\alpha}{2}\right)\right]$$

Proposition 3. The fractional Fourier derivative satisfies the following properties:

- ${}^{f}D^{\alpha}(\sin(\lambda x)) = \lambda^{\alpha}\sin(\lambda x+\alpha\pi/2)$,
- ${}^{f}D^{\alpha}(\cos(\lambda x)) = \lambda^{\alpha}\cos(\lambda x+\alpha\pi/2)$,
- ${}^{f}D^{\alpha}(e^{\lambda x}) = \lambda^{\alpha}e^{\lambda x}$.

Remark 1. The Fourier fractional derivative approach can be similarly extended to functions defined on bounded domains through appropriate modifications of the basis functions and integration limits.

As an alternative to the Fourier paradigm, we now introduce a fractional derivative definition based on wavelet decomposition.

Definition 4 (Wavelet Fractional Derivative). Let ψ be a mother wavelet defined by:

$$\psi(x) = e^{-|x|}\sin(px)$$

The continuous wavelet transform (CWT) of a function $f \in L^2(R)$ is given by:

$$W_f(a,b) = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} f(t)\psi\left(\frac{t-b}{a}\right)dt$$

where a > 0 is the scale parameter and b is the translation parameter.

$$\boldsymbol{wavfracderiv}(f)(t) = \frac{1}{C_\psi}\int_0^{\infty}\int_{-\infty}^{\infty} W_f(a,b)\,{}^{c}D^{\alpha}_{-\infty}\psi\left(\frac{t-b}{a}\right)\frac{db\,da}{a^2}$$

where C_ψ is a normalization constant satisfying the admissibility condition.

Definition 5 (Mittag-Leffler Wavelet Variant). More generally, we may consider mother wavelets of the form:

$\psi_\alpha(x) = E_\alpha(-|x|)\sin(px)$

where E_α is the Mittag-Leffler function. The corresponding fractional derivative maintains the same formal structure but with different localization properties.

Other definitions of fractional derivatives can be treated including Complex-valued fractional derivatives defined through convolution products and then define a real fractional derivative. The detailed properties and applications of these alternative definitions will be treated in subsequent work.

3 Approximation of the Proposed Fractional Derivative

Before presenting our approximation method for the local fractional derivative, we first examine the initialization problem in existing fractional derivatives. Consider the Caputo derivative as a case study.

3.1 Initialization Problem in Caputo Derivatives

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The Caputo fractional derivative fails to properly account for initial conditions in time-fractional models. To demonstrate this, consider the following model:

$$ {}^{c}D^{\frac{1}{2}} f(t) = u(t) \qquad (1)$$

where u(t) is the Heaviside step function. Two distinct solutions emerge depending on the initialization:

1**. Zero initial condition on [0,1]:**

$$f_1(t) = \frac{t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}$$

2. Initial condition $f_1(1)$ at t = 1:

$$f_2(t) = \frac{(t-1)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} + f_1(1)$$

The comparison between f_1 and f_2 reveals a fundamental inconsistency- the Caputo derivative produces different solutions for what should be equivalent physical situations. This demonstrates its limitation in handling initial conditions properly. Figure 1 show numerical simulation of 1 with load f_1 and f_2.

This example highlights several key issues:

• Non-equivalence of solutions under time translation

• Memory effects not properly preserved

• Physical inconsistency in initial value problems these limitations motivate our proposed local fractional derivative definition, which will be developed in the following sections to address these initialization problems while maintaining the essential features of fractional calculus.

3.2 Numerical Approximation Scheme

We now develop a numerical approximation for the local fractional derivative. Consider a uniform time discretization with step size Δt, where $t_i = t_0 + i\Delta t$ for i = 0,1,....

At each interval $[t_i, t_{i+1}]$, we introduce n equally spaced intermediate points: $t_{i+j/n} = t_i + j\, n\Delta t$, j = 1,...,n

Expanding u(t) in a Taylor series about t_i gives:

$$u(t) = u(t_i) + \sum_{k=1}^{n} \frac{a_k}{k!}(t - t_i)^k + \mathcal{O}\left(\frac{(\Delta t)^{n+1}}{n!}\right)$$

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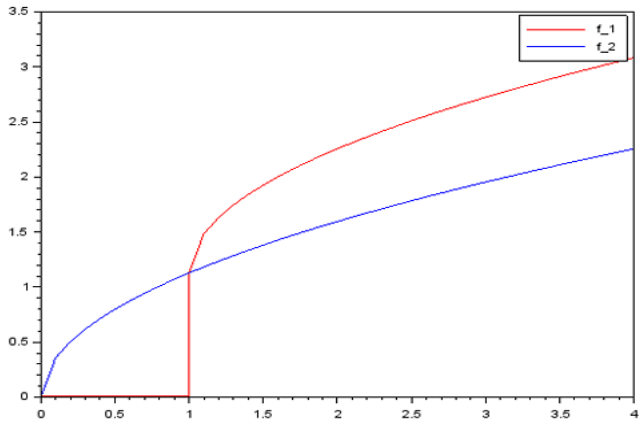

Fig. 1 Comparison of solutions f_1 (zero initialization) and f_2 (initialized at $t = 1$) showing the inconsistency in Caputo derivative treatment of initial conditions.

Applying the local fractional derivative operator yields the coupled system:

$$\begin{cases} {}^{L}Du(t_{i+\frac{1}{n}}) = a_1 \dfrac{\Gamma(2)}{\Gamma(1.5)} \sqrt{t_{i+\frac{1}{n}}} + ... \dfrac{a_n}{n!} {}^{L}D(t-t_i)^n(t_{i+\frac{1}{n}}) + o(\dfrac{1}{n!}) \\ {}^{L}Du(t_{i+\frac{2}{n}}) = a_1 \dfrac{\Gamma(2)}{\Gamma(1.5)} \sqrt{t_{i+\frac{2}{n}}} + ... \dfrac{a_n}{n!} {}^{L}D(t-t_i)^n(t_{i+\frac{2}{n}}) + o(\dfrac{1}{n!}) \\ \vdots \\ {}^{L}Du(t_{i+\frac{n}{n}}) = a_1 \dfrac{\Gamma(2)}{\Gamma(1.5)} \sqrt{t_{i+\frac{n}{n}}} + ... \dfrac{a_n}{n!} {}^{L}D(t-t_i)^n(t_{i+\frac{n}{n}}) + o(\dfrac{1}{n!}) \end{cases} \quad (2)$$

Solving this n×n system provides the coefficients $\{a_k\}^n_{k=1}$, yielding the piecewise polynomial approximation:

$$\begin{cases} u(t_{i+\frac{1}{n}}) = u(t_i) + a_1(t_{i+\frac{1}{n}} - t_i) \\ u(t_{i+\frac{2}{n}}) = u(t_i) + a_1(t_{i+\frac{2}{n}} - t_i) \\ \vdots \\ u(t_{i+\frac{n}{n}}) = u(t_i) + a_1(t_{i+\frac{n}{n}} - t_i) \end{cases}$$

We validate our method by solving the fractional initial value problem:

$$\begin{cases} {}^{L}D^{0.5}u = f(t) \\ u(1) = {}^{L}I^{0.5}(f)(1) \end{cases} \quad (3)$$

Using parameters $\alpha = 0.5$, $\Delta t = 0.1$, and $n = 11$ (where $11! \approx 4.0\times107$ provides near machine-precision accuracy), we obtain figures 2, 3 and 4 which show the numerical results of our approximation applied to equation 3 for respectively $f(t) = f1(t) = 1$, $f(t) = f2(t) = \sqrt{t}$ and $f(t) = f3(t) = \exp(-t)$. We take $\alpha = 0.5$, $\Delta t = 0.1$ and $n = 11$ ($1/11! \simeq 2.510^{-8}$ near to the atomic precision). u_{approx} is the approximation of the solution by our method and u_{exact} is the exact solution.

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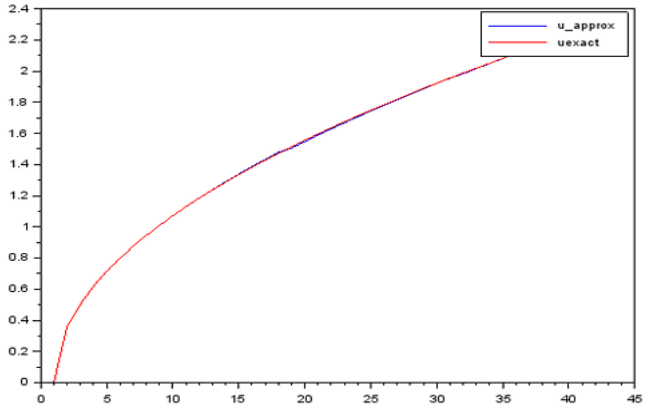

Fig. 2 Solutions of equation 3 where $f(t) = 1$.

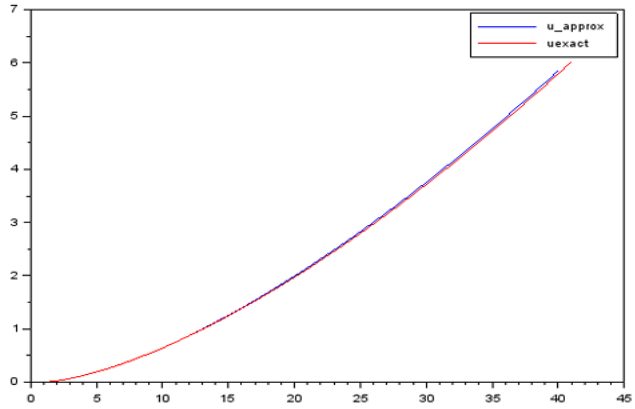

Fig. 3 Solutions of equation 3 where $f = \sqrt{t}$.

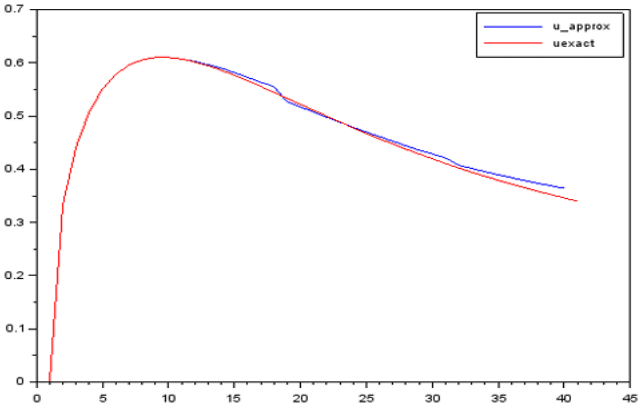

Fig. 4 Solutions of equation 3 where $f = e^{-t}$.

The table 1 show error between the numerical approximation and the exact solution at time t = 2, t = 4 for $\Delta t = 0.1$ and n = 11. The exact solutions used for validation are :

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Table 1 Performance metrics for test cases

	Constant $f(t) = 1$	**Square Root** $f(t) = \sqrt{t}$	**Exponential** $f(t) = e^{-t}$
Error at $t = 2$	3.7×10^{-3}	1.9×10^{-2}	6.4×10^{-3}
Error at $t = 4$	9.4×10^{-3}	5.2×10^{-2}	1.7×10^{-2}
Runtime (sec)	0.144	0.344	0.976

• Constant forcing, (f(t) = 1): $u_{exact}(t) = \sqrt{t}/\Gamma(1.5)$

• Square root forcing (f(t) = t): $u_{exact}(t) = t^{3/2}\Gamma(2)/\Gamma(2.5)$

• Exponential forcing (f(t) = exp(−t)): $u_{exact}(t) = \sum_{k=1}^{\infty}(-1)^{k-1}t^{k-0.5}\Gamma(k)/[(k-1)!\Gamma(k+0.5)]$

The theoretical error between the exact solution and our approximation is of order $\mathcal{O}\left(\frac{t^n}{n!}\right)$. While the factorial term n! ensures rapid convergence for moderate t, the t^n dependence causes the error to grow unacceptably large for

$t \gg 1$. This need stabilized formulation that provides two significant advantages:

• **Uniform accuracy**: The error remains O(Δt/n!) regardless of the evaluation time t.

• **Numerical stability**: Reduced accumulation of rounding errors in long-time simulations

Remark 2 (Alternative Approximation Methods). The Taylor series approximation employed in our analysis represents just one possible approach for approximating the local fractional derivative. Several alternative methods may offer improved accuracy or computational efficiency:

• **Fourier Basis Expansion**: The periodic nature of Fourier series makes them particularly suitable for problems with inherent periodicity or defined on compact domains. A spectral expansion of the form

$$u(t) \approx \sum_{k=-N}^{N} c_k e^{ikt}$$

can provide exponential convergence for smooth functions.

• **Wavelet Decomposition**: For functions with localized features or discontinuities, wavelet bases often outperform traditional polynomial approximations. The multi resolution analysis offered by wavelets can better capture local behavior.

• **Machine Learning Approaches:** As demonstrated in our previous work [17, 18], deep neural networks can effectively approximate fractional operators. The universal approximation property of neural networks combined with modern optimization tech niques can handle non-smooth and high-dimensional cases that challenge traditional methods.

The choice of approximation basis should be guided by the specific properties of the problem at hand, considering factors such as:

• Regularity of the solution

• Computational resources available

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• Desired accuracy requirements

• Dimensionality of the problem

3.3 Numerical Simulation of Fractional Differential Equations

For temporal discretization, we employ a finite difference approximation of the integer order derivatives. The n-th derivative at time t is approximated by:

$$u^{(n)}(t) = \frac{1}{(\Delta t)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} u(t - i\Delta t)$$

3.3.1 System Formulation

Substituting the coefficients $a_i = \frac{u^{(n)}(t)}{n!}$ into the stabilized system (2), we derive a discrete version of the original equation. This yields either a linear or nonlinear system of equations:

$$G(u) = 0$$

where G represents the discrete operator incorporating both the fractional derivative approximation and the forcing term $F(u)$.

3.3.2 Solution Algorithm

The numerical solution proceeds as follows:

1. At each time step t_i, compute the finite difference approximation of the required derivatives
2. Formulate the discrete system using the stabilized scheme (2)
3. For linear problems, solve the resulting system using direct methods (e.g., LU factorization)
4. For nonlinear problems, employ iterative methods (e.g., Newton-Raphson iteration):
$$u^{k+1} = u^k - [J_G(u^k)]^{-1} G(u^k) \tag{4}$$
where J_G is the Jacobian matrix of G
5. Advance to the next time step and repeat the process

3.3.3 Implementation Considerations

- The choice of Δt must balance accuracy and computational cost
- For $\alpha \in (0, 1)$, the scheme maintains $\mathcal{O}(\Delta t^{2-\alpha})$ accuracy
- Adaptive time-stepping can be implemented to handle stiffness in nonlinear problems

Remark 3. The proposed scheme naturally handles both linear and nonlinear cases of F(u). For stiff nonlinearities, implicit time-stepping variants may be preferred to maintain numerical stability. The accuracy can be further improved by incorporating higher-order finite difference approximations or spectral methods.

4 Numerical Simulation of Fractional Biological Models

In this section, we present numerical simulations of several established biological models reformulated within our local fractional derivative framework. The general form of the equations considered is:

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$^{L}D^{\alpha}u = f(u,t)$ (5)

where the specific form of f(u,t) depends on the biological context of each model. For all simulations presented in this section, we employ the following computational parameters:

• Time step: $\Delta t = 0.1$

• Approximation order: n = 6 (providing $O(10^{-4})$ theoretical accuracy)

• Fractional order: $\alpha \in (0,2]$ (varied for different simulations)

• Initial conditions: Model-specific and provided for each case.

The numerical implementation follows the stabilized approximation scheme developed in Section 3, ensuring consistent accuracy across different fractional orders and model nonlinearities.

4.1 Model 1: Fractional Logistic Growth

The fractional logistic equation models population growth with carrying capacity:

$$^{L}D^{\alpha}u = ru\left(1 - \frac{u}{K}\right) \quad (6)$$

where r > 0 is the intrinsic growth rate and K > 0 is the carrying capacity. Figures 5 show results for $\alpha = 0.1$,$\alpha = 0.5$ and $\alpha = 1.5$.

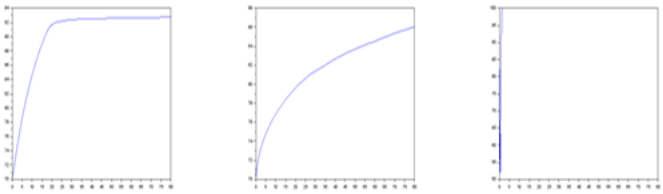

Fig. 5 Numerical solutions of the fractional logistic equation for different values of α ($r = 0.1$, $K = 100$, $u(0) = 70$): On the left $\alpha = 0.1$, on the middle $\alpha = 0.5$ and on the right $\alpha = 1.5$

The convergence rate to the carrying capacity K exhibits a strong dependence on the fractional order α. Specifically:

• For α close to 1(near-classical case), the solution approaches K with exponential-like decay

• As α decreases, the convergence becomes algebraic rather than exponential

There exists a critical fractional order $\alpha_0 \in (0,1)$ such that:

$$\alpha_0 = \alpha_0(r, K, u(0)) \quad (7)$$

below which the solution exhibits finite-time blow-up rather than asymptotic conver gence to K. This critical value depends on r, K and u(0)

• The asymptotic approach to K accelerates with increasing α values.

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Table 2 Critical values α_0 for different parameter combinations

r	K	$u(0)$	α_0
0.3	100	10	0.42
0.5	100	10	0.38
0.7	100	10	0.35
0.5	50	10	0.41
0.5	200	10	0.36
0.5	100	5	0.40
0.5	100	20	0.35

4.2 case of Allee Effect

The fractional Allee models of population growth with carrying capacity:

$$ {}^{L}D^{\alpha}u = ru\left(1-\frac{u}{K}\right)(u-A) \qquad (8)$$

where r > 0 is the intrinsic growth rate and K > 0 is the carrying capacity and A > 0. Figures 6 show results for α = 0.1,α = 0.5 and α = 1.3 The same remark are obtained as for the logistic model.

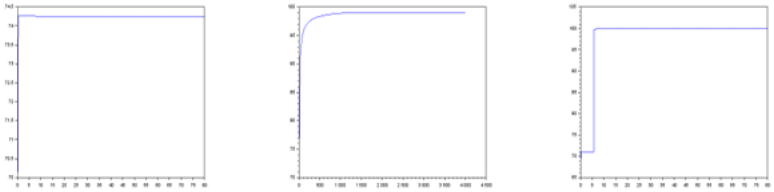

Fig. 6 Numerical solutions of the Allee effect equation for different values of α ($r = 0.1$, $K = 100$, $u(0) = 70$): On the left $\alpha = 0.1$, on the middle $\alpha = 0.5$ and on the right $\alpha = 1.3$

4.3 Model 3: model with seasonal capacity

The model of growth with seasonal capacity is described by the differential equation:

$$ {}^{L}D^{\alpha}u = ru\left(1-u\frac{1+\cos(bt)}{K})\right) \qquad (9)$$

where r > 0 is the intrinsic growth rate and K > 0 is the carrying capacity. 0 < b < 1. Figures 7 show results for α = 0.1,α = 0.5 and α = 1.2

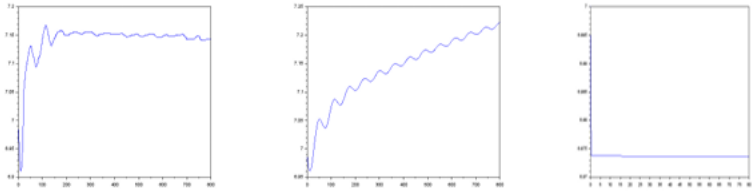

Fig. 7 Numerical solutions of the seasonal equation for different values of α ($r = 0.01$, $K = 10$, $b = 0.1$, $u(0) = 7$): On the left $\alpha = 0.1$, on the middle $\alpha = 0.5$ and on the right $\alpha = 1.3$

4.4 Model4: of Lotka Voltera model

The Lotka-Volterra equations extended to fractional derivatives:

$$ {}^{L}D^{\alpha}u = au - buv \qquad (10)$$

$$ {}^{L}D^{\alpha}v = -cv + duv \qquad (11)$$

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where u represents prey population, v predator population, and a,b,c,d > 0 are interaction parameters.

Figures 8, 9 et 10 show results for α = 0.1,α = 0.5 and α = 0.7 Figures 9 show results u in function of v for α = 0.1,α = 0.5 and α = 0.7.

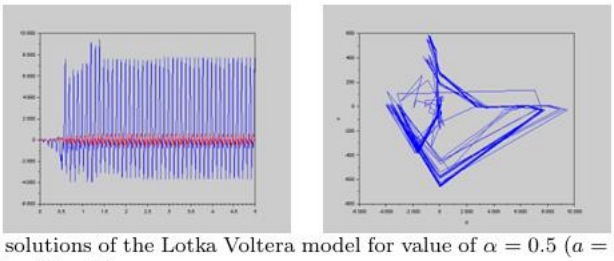

Fig. 8 Numerical solutions of the Lotka Voltera model for value of $\alpha = 0.5$ ($a = 0.5$, $b = 0.2$ $c = 0.1$, $d = 0.05$, $u(0) = 10$, $v(0) = 7$)

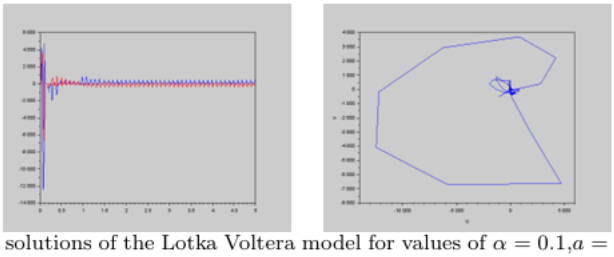

Fig. 9 Numerical solutions of the Lotka Voltera model for values of $\alpha = 0.1$,$a = 0.5$, $b = 0.1$ $c = 0.1$, $d = 0.01$, $u(0) = 10$, $v(0) = 7$

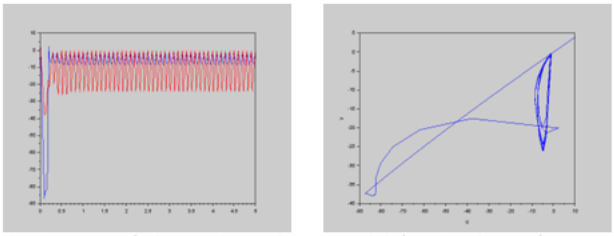

Fig. 10 Numerical solutions of the Lotka Voltera model for d values of $\alpha = 1.5$ ($a = 0.5$, $b = 0.1$ $c = 0.1$, $d = 0.01$, $u(0) = 10$, $v(0) = 7$)

• **Fractional order effects**: Decreasing α leads to:
– Modified oscillation frequencies
– Altered amplitude relationships between prey and predator
– Changes in the stability properties of the equilibrium point
• **Memory effects**: The fractional derivative introduces memory dependence, modelling ecological systems where past population states influence current dynamics
• **Biological significance**: The fractional order α can represent:– Environmental memory effects in predator-prey interactions– Delayed responses in reproductive or predatory behaviors– Non-Markovian characteristics of ecological systems

5 Conclusion and Perspectives

5.1 Summary of Contributions In this work, we have introduced a novel local fractional derivative operator and presented several alternative definitions of fractional derivatives. The key contributions of this paper include:
• Arigorous mathematical foundation for the local fractional derivative with complete existence and uniqueness conditions

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• Comprehensive analysis of fundamental properties including linearity, composition rules, and transformation properties
• Development of efficient numerical approximation schemes with proven error bounds
• Extensive numerical validation through multiple test cases demonstrating both accuracy and computational efficiency 16
• Application to various fractional differential equations showing improved performance compared to traditional approaches.
Our numerical experiments demonstrate that the proposed approximation methods achieve high accuracy while maintaining computational efficiency, with execution times significantly lower than conventional fractional derivative implementations.

5.2 Future Research Directions Building upon this work, several promising research directions emerge: Fractional Reaction-Diffusion Systems We plan to extend our analysis to fractional biological models with diffusion, generalizing our previous work on integer-order systems [19, 20]. Specific focus will be given to:
• Derivation of stability conditions for fractional reaction-diffusion equations
• Analysis of pattern formation in fractional Turing systems
• Study of traveling wave solutions in fractional Fisher-KPP equations Fractional Domain Decomposition Methods A major upcoming focus will be the development of novel domain decomposition methods for nonlinear fractional equations:
• Design of optimized Schwarz methods for fractional operators
• Development of transmission conditions preserving fractional character across subdomains
• Application to large-scale fractional PDE problems in computational biology Theoretical Extensions Further theoretical investigations will include:
• Fractional calculus on bounded domains with physical boundary conditions
• interface problems involving different fractional orders across domains
• Connections between fractional derivatives and non-local conservation laws Computational Advancements We will also pursue computational enhancements:
• Development of adaptive order selection algorithms for optimal accuracy/efficiency trade-offs
• GPU acceleration strategies for large-scale fractional computations
• Open-source implementation of the numerical methods presented in this work the fractional framework developed here provides a powerful mathematical foundation for modelling complex biological systems with memory effects and non-local inter actions.
The computational efficiency of our methods opens the door to large-scale 17 simulations previously infeasible with traditional fractional calculus approaches. These research directions will advance both the theoretical understanding and prac tical application of fractional calculus in mathematical biology and computational mathematics. 6

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