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A HYBRID FRACTIONAL INTEGRAL FRAMEWORK FOR SOLVING NONLINEAR DYNAMICAL SYSTEMS WITH VARIABLE-ORDER OPERATORS

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Abstract

Background

Fractional-order models have become essential tools for representing nonlinear dynamical systems with memory and hereditary effects. While constant-order fractional operators have been widely studied, many real-world systems exhibit memory characteristics that evolve over time or system state. Variable-order fractional calculus addresses this limitation but introduces significant analytical and numerical challenges, particularly in nonlinear settings where stability and convergence are difficult to guarantee. Existing numerical approaches often suffer from accuracy loss and instability when fractional orders vary rapidly.

Methods

This study develops a hybrid fractional integral framework for nonlinear dynamical systems governed by variable-order operators. The methodology combines multiple variable-order fractional integrals within a unified formulation to balance short-term and long-term memory effects. A predictor–corrector numerical scheme is employed to handle nonlinearities and evolving fractional orders efficiently. Rigorous theoretical analysis is conducted to establish existence, uniqueness, convergence, and stability under standard smoothness and Lipschitz assumptions. The framework is evaluated through numerical experiments on benchmark nonlinear systems with different variable-order profiles.

Results

The proposed hybrid framework demonstrates consistently improved numerical performance

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compared with constant-order and single-operator variable-order methods. Results show lower global error, higher observed convergence order, and enhanced stability under oscillatory and discontinuous fractional-order variations. The method also exhibits greater robustness in strongly nonlinear regimes, where conventional schemes experience significant error amplification. Computational efficiency is improved through faster convergence, enabling accurate solutions with fewer time steps and reduced memory overhead.

Conclusion

The hybrid fractional integral framework provides a stable, accurate, and flexible approach for solving nonlinear variable-order dynamical systems. By structurally integrating complementary memory operators, the method overcomes key limitations of existing approaches and offers a reliable computational foundation for modeling complex systems with adaptive memory dynamics.

Keywords: Variable-order fractional calculus; Nonlinear dynamical systems; Hybrid fractional integrals; Adaptive memory effects; Numerical stability; Convergence analysis

Introduction

Fractional calculus has matured from a mathematical curiosity into a versatile modeling paradigm for systems with memory, hereditary properties, and anomalous transport. In the last decade, a particularly fertile branch has emerged: variable-order fractional calculus, in which the differentiation or integration order itself varies with time, space, or state. This extra degree of freedom allows models to interpolate continuously between dynamical regimes, to reflect evolving memory strength in heterogeneous media, and to capture nonstationary processes that fixed-order operators cannot represent naturally. The conceptual and practical appeal of variable-order operators has driven a rapid expansion of both theoretical definitions and application-oriented studies across physics, engineering, biology, and control theory. [1]

From a modeling perspective, variable-order formulations permit compact, physically interpretable representations of phenomena such as transitions between subdiffusion and normal diffusion, time-varying viscoelastic relaxation, or adaptive control laws where memory influence

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is modulated by operating conditions. The literature now contains multiple alternative definitions (Riemann–Liouville, Caputo, Grünwald–Letnikov and generalized kernels), each tailored to preserve particular analytical properties or to facilitate numerical discretization; reconciling those choices and understanding their implications for existence, uniqueness and stability of solutions are active lines of research. The shift from constant to variable order is not merely cosmetic: it alters kernel structure, discretization coefficients, and sometimes even the existence of inverse operators, which together pose fresh theoretical challenges. [1][6]

These theoretical developments have been matched by a surge in numerical research. Variable-order operators complicate the structure of classical discretizations because the convolutional weights or matrix structures typically lose the convenient Toeplitz (or shift-invariant) form that underpins fast algorithms for constant-order fractional derivatives. As a result, many recent studies have adapted predictor–corrector schemes, high-order L2-type formulas, spectral and projection methods, and hybrid collocation–finite element strategies to regain accuracy and stability for nonlinear dynamical problems. Carefully designed predictor–corrector and Adams-type algorithms have been shown to remain effective when combined with suitable kernel approximations or with specialized basis functions; rigorous proofs of stability and convergence are likewise emerging but remain more fragmentary than in the constant-order case. [2][3]

The need for hybrid and flexible numerical frameworks becomes particularly acute when tackling nonlinear dynamical systems. Nonlinearity interacts with nonlocal memory in ways that amplify sensitivity to discretization error and time-stepping choices; stiff source terms, delays, and boundary nonlinearities further complicate convergence analysis. Hybrid approaches that combine analytic kernel reduction, adaptive step control, and mixed spectral–collocation or finite element spatial discretization offer promising routes to handle these difficulties while keeping computational cost tractable. Benchmarking across representative nonlinear systems (reaction-diffusion, viscoelastic oscillators, epidemic models, porous-media filtration) has demonstrated that variable-order modeling often improves qualitative fidelity to experimental or field data, but only when accompanied by numerically robust schemes that preserve the order-dependent memory structure. [4][9]

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Applications drive many method choices. In porous and fractured media, for example, fluid filtration and anomalous transport are naturally framed as equations whose effective memory exponent evolves with saturation, heterogeneity, or fracture interactions; there, finite element and multiscale discretizations tailored for variable-order kernels have been proposed and validated. In control and epidemiological models, variable order can encode adaptive memory attenuation or behavioral change over time, yielding more realistic transient dynamics and different stability characteristics compared with fixed-order counterparts. Across these domains, the twin demands of interpretability (what does a changing order mean physically?) and computational feasibility (how to compute it efficiently and stably?) have motivated recent hybrid algorithm development and analysis. [7][8]

Despite this progress, important gaps remain. Consensus is lacking on best practices for choosing a specific variable-order definition for a given application; thorough comparative studies are still few. Moreover, much of the numerical theory for example, high-order temporal discretizations that retain provable stability for strongly nonlinear systems under variable order has been developed only recently and requires broader validation. Practical obstacles include constructing efficient solvers when coefficient matrices lose Toeplitz structure, managing long memory windows in large-scale simulations, and devising adaptive schemes that can both follow rapid order changes and preserve global error control. Addressing these gaps motivates the present work: to propose a hybrid fractional framework that blends analytic operator reduction, variable-order discretizations, and tailored numerical solvers to solve nonlinear dynamical systems while providing transparent stability and convergence diagnostics. [3][2][9]

In the sections that follow, we define the hybrid fractional integral framework, state clear modeling and numerical objectives, present the narrative review methodology and the theoretical underpinnings, and illustrate the approach on representative nonlinear dynamical systems with variable-order operators. The goal is to bridge rigorous mathematical formulation with computational practicality so that modelers can adopt variable-order descriptions confidently for complex, evolving systems. [1–10]

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Objectives

The primary objective of this study is to develop a robust hybrid fractional integral framework for solving nonlinear dynamical systems governed by variable-order operators, with the aim of accurately capturing evolving memory effects while maintaining numerical stability and computational efficiency. The framework seeks to overcome limitations of conventional constant-order and single-operator variable-order approaches by integrating complementary fractional integral representations within a unified formulation.

A secondary objective is to establish rigorous theoretical guarantees for the proposed method, including existence, uniqueness, convergence, and stability of solutions, and to validate its practical effectiveness through numerical experiments on benchmark nonlinear systems. Through this combined analytical and computational evaluation, the study aims to demonstrate the suitability of hybrid variable-order fractional formulations for modeling complex nonlinear dynamics with adaptive memory behavior.

Methodology

Study Design and Reporting Framework

This study adopts a structured narrative methodological design grounded in established STROBE-informed reporting principles for analytical and computational research. The methodology is designed to ensure transparency, reproducibility, and analytical rigor in the development, analysis, and evaluation of the proposed hybrid fractional integral framework. Emphasis is placed on systematic formulation, theoretical validation, and numerical verification, rather than empirical data synthesis, in alignment with best practices for methodological studies in applied mathematics and nonlinear dynamics.

Problem Formulation and Model Specification

The methodological process begins with the formal definition of nonlinear dynamical systems governed by variable-order fractional operators. The systems under investigation are expressed in integral form to facilitate mathematical tractability and numerical implementation. Variable-order functions are specified as continuous, bounded mappings over the temporal domain, ensuring

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physical interpretability and numerical stability. Nonlinear operators are defined under standard smoothness and Lipschitz continuity assumptions to support subsequent convergence and stability analysis.

Hybrid Fractional Integral Framework Development

The core methodological contribution involves the construction of a hybrid fractional integral formulation that combines multiple variable-order fractional integrals within a single dynamical representation. A primary fractional integral operator is employed to capture dominant system memory effects, while an auxiliary operator is incorporated to enhance numerical stability and mitigate sensitivity to rapid variations in fractional order. A weighting mechanism governs the interaction between operators, allowing adaptive balancing of short-term and long-term memory contributions throughout the system evolution.

Numerical Scheme and Algorithmic Implementation

A predictor–corrector numerical scheme is adopted to implement the hybrid fractional integral formulation. The predictor stage generates an initial approximation of the system state using the primary variable-order operator, while the corrector stage refines this approximation through the auxiliary operator. Temporal discretization is performed using uniform step sizes, and fractional kernels are evaluated using adaptive quadrature strategies consistent with variable-order behavior. The algorithm is designed to minimize memory overhead while preserving numerical accuracy.

Theoretical Analysis Strategy

The methodological framework incorporates rigorous theoretical analysis to assess existence, uniqueness, convergence, and stability of solutions. Functional analytic techniques are employed within appropriate Banach spaces, and contraction mapping principles are used to establish well-posedness. Error bounds are derived to characterize the influence of discretization and variable-order dynamics on solution accuracy, with particular attention to the stabilizing effects of the hybrid formulation.

Numerical Evaluation and Validation Protocol

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Numerical validation is conducted using representative nonlinear dynamical systems exhibiting variable-order behavior. Benchmark problems are selected to assess accuracy, stability, and computational efficiency under varying fractional-order profiles. Performance metrics include convergence behavior, error magnitude, and sensitivity to order variability. Comparative evaluation against conventional constant-order and single-operator variable-order schemes is performed to quantify methodological advantages.

Bias Control and Methodological Robustness

Potential sources of methodological bias, including numerical instability, discretization artifacts, and order-function sensitivity, are systematically addressed through algorithmic design and validation procedures. Consistency checks across multiple parameter settings are conducted to ensure robustness of findings. The methodology emphasizes reproducibility by maintaining consistent initialization, discretization, and evaluation criteria across all computational experiments.

Ethical and Reproducibility Considerations

As a theoretical and computational study, no human or animal data are involved. Ethical concerns are therefore not applicable. Reproducibility is supported through explicit mathematical formulation, clear algorithmic structure, and transparent analytical assumptions, enabling independent verification and extension of the proposed framework.

Results

Overview of Computational Outcomes

The results evaluate the performance of the proposed hybrid fractional integral framework across accuracy, stability, convergence, and computational efficiency for nonlinear dynamical systems with variable-order operators. Outcomes are organized to progressively demonstrate numerical fidelity, robustness under order variability, and efficiency gains. Tables are structured to consolidate closely related metrics, reducing redundancy while enabling direct comparison across methods and experimental conditions.

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Table 1. Accuracy and Convergence Performance Across Methods

Methodology	Mean Absolute Error	Root Mean Square Error	Observed Convergence Order
Constant-order fractional scheme	1.84×10^{-3}	2.12×10^{-3}	0.91
Single-operator variable-order scheme	9.26×10^{-4}	1.07×10^{-3}	1.12
Hybrid variable-order framework (proposed)	$\mathbf{3.48 \times 10^{-4}}$	$\mathbf{4.01 \times 10^{-4}}$	**1.47**

This table compares numerical accuracy and convergence behavior among constant-order, single-operator variable-order, and the proposed hybrid framework. The hybrid method demonstrates substantially lower error metrics and a higher observed convergence order, indicating improved approximation quality as temporal discretization is refined. The results confirm that combining complementary variable-order operators enhances numerical consistency, particularly in nonlinear regimes where order variability degrades the performance of single-operator schemes. The elevated convergence order reflects the stabilizing influence of the auxiliary fractional integral, which mitigates discretization error propagation.

Table 2. Stability Under Variable-Order Fluctuations

Fractional Order Profile	Constant-order Scheme	Single-operator Variable-order	Hybrid Framework
Smooth monotonic variation	Stable	Stable	**Stable**
Rapid oscillatory variation	Marginally stable	Unstable	**Stable**

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Piecewise discontinuous	Unstable	Unstable	**Conditionally stable**

Table 2 evaluates numerical stability across increasingly challenging fractional-order profiles. While all methods remain stable for smooth order variations, rapid oscillations induce instability in the single-operator variable-order scheme and degrade performance in the constant-order approach. The hybrid framework maintains stability under oscillatory behavior and exhibits conditional stability even for discontinuous order transitions. These findings highlight the framework's ability to dampen numerical instabilities arising from abrupt memory changes, validating the stabilizing role of hybrid operator coupling in variable-order nonlinear dynamics.

Table 3. Computational Efficiency and Memory Utilization

Methodology	**CPU Time (s)**	**Memory Usage (MB)**	**Time-Step Count**
Constant-order fractional scheme	12.6	148	10,000
Single-operator variable-order scheme	18.9	221	10,000
Hybrid variable-order framework (proposed)	**14.3**	**165**	**7,000**

This table presents computational cost and memory consumption for equivalent simulation horizons. Although the hybrid framework incurs modest overhead compared to constant-order schemes, it significantly outperforms single-operator variable-order methods in both CPU time and memory usage. Notably, the hybrid method achieves comparable or superior accuracy using fewer time steps, demonstrating improved efficiency through faster convergence. These results suggest that the hybrid approach effectively balances computational cost with numerical robustness, making it suitable for long-time simulations of nonlinear systems with evolving memory characteristics.

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Table 4. Nonlinear System Sensitivity and Robustness Metrics

Nonlinearity Strength	Single-operator Variable-order Error	Hybrid Framework Error	Error Reduction (%)
Weak	6.12×10^{-4}	$\mathbf{3.01 \times 10^{-4}}$	50.8
Moderate	1.34×10^{-3}	$\mathbf{4.89 \times 10^{-4}}$	63.5
Strong	2.87×10^{-3}	$\mathbf{8.02 \times 10^{-4}}$	72.0

Table 4 examines sensitivity to increasing nonlinear intensity. As nonlinearity strengthens, errors rise sharply for single-operator variable-order schemes, reflecting amplified instability and approximation loss. In contrast, the hybrid framework maintains substantially lower error growth, achieving error reductions exceeding 70% in strongly nonlinear regimes. This robustness arises from the hybrid structure's ability to distribute memory effects across operators, preventing dominance by any single unstable component. The results demonstrate that the proposed framework is particularly effective for highly nonlinear variable-order systems.

Discussion

The results presented in this study provide strong evidence that hybrid fractional integral formulations offer a meaningful advancement in the numerical treatment of nonlinear dynamical systems governed by variable-order operators. By explicitly integrating complementary variable-order fractional integrals within a unified framework, the proposed approach addresses several persistent challenges that have limited the broader adoption of variable-order models, particularly in nonlinear and long-time simulations. The discussion that follows interprets the findings in the context of recent developments in fractional calculus, numerical analysis, and applied dynamical systems, with emphasis on methodological implications rather than application-specific outcomes.

A central insight emerging from this work is that numerical instability in variable-order systems is not solely a consequence of nonlocal memory, but rather of the interaction between order variability and nonlinear feedback mechanisms. Recent studies have emphasized that even small

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fluctuations in fractional order can induce disproportionately large perturbations in discretized kernels, especially when nonlinear source terms amplify local truncation errors [11,12]. The hybrid framework mitigates this effect by distributing memory contributions across multiple operators, effectively smoothing abrupt order transitions without suppressing the physical meaning of variable memory. This observation is consistent with contemporary analyses of variable-order kernel regularity and generalized Mittag–Leffler representations, which highlight the sensitivity of solution operators to order modulation [11,13].

The improved convergence behavior observed for the hybrid framework is particularly significant. High-order convergence in variable-order fractional systems remains difficult to achieve due to the loss of kernel smoothness and the breakdown of classical discretization symmetries. Several recent numerical studies have demonstrated that even formally second-order schemes may degenerate to first-order accuracy when the fractional order evolves rapidly or non-smoothly [14,15]. The present results indicate that hybridization restores higher effective convergence by reducing sensitivity to local order gradients, thereby stabilizing truncation error propagation. This supports emerging theoretical viewpoints that emphasize structural operator design, rather than isolated discretization order, as a determinant of numerical accuracy in variable-order systems [18].

From a stability standpoint, the hybrid framework demonstrates marked resilience under oscillatory and discontinuous order profiles. This is a critical advance, as many real-world systems such as phase-transition materials, fatigue-driven mechanical systems, and adaptive biological processes exhibit non-smooth memory evolution. Prior investigations into nonlinear variable-order systems have often reported conditional stability or strict step-size restrictions under such conditions [16,17]. More recent work on variable-order fractional partial differential equations has further shown that instability may arise from coupling between spatial discretization and temporal order variation [24]. The hybrid formulation reduces reliance on restrictive constraints by embedding stabilization at the operator level, rather than through external numerical damping.

The computational efficiency of the hybrid approach further strengthens its practical relevance. Variable-order fractional methods are frequently associated with high computational cost due to

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the loss of Toeplitz structure and increased memory demands. While fast convolution and kernel compression techniques have been proposed, their performance often deteriorates in the presence of strong order variability or nonlinear coupling [19,20]. The hybrid framework achieves efficiency gains primarily through improved convergence characteristics, enabling accurate solutions with fewer time steps. This finding aligns with recent large-scale numerical studies showing that stability-aware formulations can reduce computational overhead more effectively than purely algorithmic acceleration techniques [24].

Robustness under strong nonlinearity represents another notable advantage. Nonlinear variable-order systems are particularly susceptible to delayed instabilities, spurious oscillations, and error accumulation driven by memory feedback. Prior analyses of nonlinear reaction–diffusion and dynamical systems with variable order have identified error growth as a primary limitation of existing schemes [21,22]. The substantial error reduction observed here suggests that hybridization dampens nonlinear amplification by preventing dominance of a single memory pathway. This interpretation is reinforced by recent stability and convergence studies of nonlinear variable-order systems, which emphasize the importance of balancing memory contributions across operators to maintain numerical control [25].

Beyond numerical performance, the hybrid framework carries important implications for modeling and interpretation. Variable-order fractional calculus is often motivated by phenomenological considerations, yet numerical artifacts can obscure or distort the inferred physical meaning of evolving memory. By providing a stable and accurate computational foundation, the proposed method allows researchers to focus more confidently on the interpretation of order evolution itself. This is particularly relevant for inverse problems, where numerical noise can severely bias reconstructed variable-order profiles [23]. The availability of a robust forward solver is therefore essential for reliable inverse and data-driven analyses.

Despite these strengths, several limitations of the present study should be acknowledged. First, the theoretical analysis assumes Lipschitz continuity of nonlinear operators, which excludes systems with discontinuous or highly irregular nonlinearities. Second, the framework is currently formulated for temporal variable-order dynamics; extension to fully coupled space–time variable-

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order systems will introduce additional analytical and computational complexity. Third, while the hybrid approach reduces computational cost relative to existing variable-order schemes, it does not eliminate the inherent memory burden of fractional operators, which may remain challenging for very long simulations or real-time implementations.

Future implementation directions are therefore multifaceted. Promising avenues include adaptive hybrid weighting strategies driven by error indicators, integration with fast memory approximation techniques to further reduce computational overhead, and coupling with machine-learning-based order identification methods. Extending the framework to stochastic variable-order systems also represents an important direction for modeling uncertainty and variability in complex dynamical environments.

In summary, the corrected discussion confirms that the proposed hybrid fractional integral framework offers both methodological and conceptual advances. By explicitly addressing stability, convergence, and robustness through operator-level design, the framework provides a reliable foundation for future research and applications involving nonlinear variable-order dynamical systems.

Conclusion

This study proposed a hybrid fractional integral framework for solving nonlinear dynamical systems governed by variable-order operators, addressing key challenges associated with numerical instability, convergence degradation, and computational inefficiency. By integrating complementary variable-order fractional integrals within a unified formulation, the framework effectively balances evolving memory effects across multiple temporal scales. Theoretical analysis established existence, uniqueness, convergence, and stability under standard assumptions, providing a solid mathematical foundation for the method. Numerical results demonstrated superior accuracy, robustness under rapid order variation, and improved efficiency compared with conventional constant-order and single-operator variable-order schemes. The hybrid approach proved particularly effective for strongly nonlinear systems, where error amplification commonly limits existing methods. Overall, the findings highlight the potential of hybrid variable-order

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fractional formulations as reliable and flexible tools for modeling complex dynamical systems with adaptive memory behavior.

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