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COMMON FIXED-POINT THEOREMS FOR ψ −COMPATIBLE MAPPING OF TYPE(P)

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ABSTRACT

The purpose of this paper is to introduce the concept of a ψ-compatible mapping of type (P) defined via a suitable control function ψ. We establish a common fixed-point theorem in a complete metric space by employing this new notion. The obtained result generalizes the theorem of Pathak et al. [2], Sastry et al. [4], and further extends several related results available in the literature.

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1. INTRODUCTION

The notion of compatibility in metric spaces was developed as a generalization of commutativity in fixed-point theory. Classical fixed-point results often rely on strong assumptions, such as continuity, completeness, compactness, or monotonicity, which may not always hold in applications. Jungck [1] introduced the concept of *compatible mappings*, requiring continuity, non-decreasing, and the existence of a common fixed point of the involved mappings. However, in many situations, a pair of mappings f and g may fail to satisfy Jungck's compatibility condition, even when a sequence converging to a common fixed point exists. Such pairs are known as *non-compatible mappings*, a notion introduced by Pant [3]. For more works on the generalization of metric spaces, see [8-12, 14, 18-19, 22].

To overcome this limitation, Sastry et al. [4] studied the existence and uniqueness of fixed points for self-mappings on metric spaces by altering the distance between points using a suitable control function**,** thereby converting certain non-compatible mappings into compatible ones. In a related development, Khan, Swaleh, and Sessa [13] established fixed-point results for single self-mappings using altering distance functions. More recently, Sastry [4] extended this approach to pairs of self-mappings.

Motivated by these developments, in this paper, we introduce a new variant of compatibility-namely, the ψ-compatible mapping of type (P)**-** defined via an altering distance function ψ. Using this framework, we establish a common fixed-point theorem for four self-mappings in a complete metric space. Our result generalizes **t**he common fixed-point theorem of Pathak et al. [2], Sastry et al. [4], and extends several related results in the literature.

2.PRELIMINARIES

Definition 2.1 [1] Two self-mappings $\mathcal{G}, \mathcal{H}: (X,d) \to (X,d)$ of a metric space (X,d) are said to be compatible if and only if $\lim_{n\to\infty} d(\mathcal{G}\mathcal{H}x_n, \mathcal{H}\mathcal{G}x_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} \mathcal{G}x_n = \lim_{n\to\infty} \mathcal{H}x_n = t$ for some t in X.

Definition 2.2 [5] Two self-mappings $\mathcal{G}, \mathcal{H}: (X,d) \to (X,d)$ of a metric space (X,d) are said to be compatible of type (P) if and only if $\lim_{n\to\infty} d(\mathcal{G}\mathcal{G}x_n, \mathcal{H}\mathcal{H}x_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} \mathcal{G}x_n = \lim_{n\to\infty} \mathcal{H}x_n = t$ for some t in X.

Definition 2.3 [3] Two self-mappings $\mathcal{G}, \mathcal{H}: (X,d) \to (X,d)$ of a metric space (X,d) are said to be non-compatible mappings if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} \mathcal{G}x_n = \lim_{n\to\infty} \mathcal{H}x_n = t$ for some t in X but $\lim_{n\to\infty} d(\mathcal{G}\mathcal{H}x_n, \mathcal{H}\mathcal{G}x_n) \neq 0$ or not exist.

Definition 2.4 [5] Let a function $\psi: R^+ \to R^+$ is called an altering distance function if the following properties are satisfied.

(i) $\psi(t) = 0$ if and only if $t = 0$

(ii) ψ is monotonically non-decreasing.

(iii) ψ is continuous.

By Ψ we denote the set of all altering distance functions.

Definition 2.5 [4] Two self-mappings $\mathcal{G}, \mathcal{H}: (X,d) \to (X,d)$ of a metric space (X,d) are said to be ψ −compatible if and only if $\lim_{n\to\infty} \psi(d(\mathcal{G}\mathcal{H}x_n, \mathcal{H}\mathcal{G}x_n)) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} \mathcal{G}x_n = \lim_{n\to\infty} \mathcal{H}x_n = t$ for some t in X.

Definition 2.6 Two self-mappings $\mathcal{G}$ and $\mathcal{H}$ of a metric space (X,d) are said to be ψ −compatible mapping of type (P) if and only if

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$\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{G}x_n,\mathcal{H}\mathcal{H}x_n))=0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}\mathcal{G}x_n=\lim_{n\to\infty}\mathcal{H}x_n=t$ for some t in X.

Example 2.1 Let $X=R^+$ as usual, metric and the two self-mappings $\mathcal{G}$ and the $\mathcal{H}$ of a metric space (X,d) are defined by $\mathcal{G}(x)=(1-x)$ and $\mathcal{H}(x)=2x$ and the altering distance function defined by $\psi(t)=t^2-t$ satisfying the [2.5] conditions.

Suppose that $x_n=\left\{\frac{1}{3}+\frac{1}{n}\right\}$ be the sequence in $X=R^+$ then

$$\lim_{n\to\infty}\mathcal{G}x_n=\lim_{n\to\infty}(1-x_n)=\lim_{n\to\infty}\left(1-\frac{1}{3}-\frac{1}{n}\right)=\frac{2}{3}$$

$\lim_{n\to\infty}\mathcal{H}x_n=\lim_{n\to\infty}(2x_n)=\lim_{n\to\infty}2\left(\frac{1}{3}+\frac{1}{n}\right)=\frac{2}{3}$, thus

$\lim_{n\to\infty}\mathcal{G}x_n=\lim_{n\to\infty}\mathcal{H}x_n=\frac{2}{3}\in R^+$,

Now, $\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{G}x_n,\mathcal{H}\mathcal{H}x_n))=\lim_{n\to\infty}\psi\left(d\left(\mathcal{G}\left(\frac{2}{3}\right),\mathcal{H}\left(\frac{2}{3}\right)\right)\right)=\psi\left(d\left(\frac{1}{3},\frac{4}{3}\right)\right)=\psi(1)=0$

Therefore, $\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{G}x_n,\mathcal{H}\mathcal{H}x_n))=0$ Hence, the two self-mappings $\mathcal{G}$ and $\mathcal{H}$ are ψ −compatible mapping of type (P).

Proposition 2.1 Suppose that $\mathcal{G}$ and $\mathcal{H}$ are non-continuous self-mappings of a metric space (X,d), then $\mathcal{G}$ and $\mathcal{H}$ are ψ −compatible mappings if and only if they are ψ −compatible mappings of type (P).

Proof: - Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty}\mathcal{G}x_n=\lim_{n\to\infty}\mathcal{H}x_n=z$ for some z in X. Since $\mathcal{G}$, and $\mathcal{H}$ are non-continuous, then there may be $\lim_{n\to\infty}\mathcal{G}\mathcal{G}x_n=\lim_{n\to\infty}\mathcal{G}\mathcal{H}x_n\neq\mathcal{G}z$ and $\lim_{n\to\infty}\mathcal{H}\mathcal{G}x_n=\lim_{n\to\infty}\mathcal{H}\mathcal{H}x_n\neq\mathcal{H}z$ so, by the definition of ψ we can write $\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{G}x_n,\mathcal{G}z)=0$ and $\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{H}x_n,\mathcal{G}z))=0$ this implies that $\lim_{n\to\infty}\mathcal{G}\mathcal{G}x_n=\lim_{n\to\infty}\mathcal{G}\mathcal{H}x_n=\mathcal{G}z$ and in the similar argument $\lim_{n\to\infty}\mathcal{H}\mathcal{G}x_n=\lim_{n\to\infty}\mathcal{H}\mathcal{H}x_n=\mathcal{H}z$

Since the mapping $\mathcal{G}$ and $\mathcal{H}$, are ψ −compatible, then we have $\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{H}x_n,\mathcal{H}\mathcal{G}x_n))=0$

Now, $\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{G}x_n,\mathcal{H}\mathcal{H}x_n))\leq\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{G}x_n,\mathcal{G}\mathcal{H}x_n))+\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{H}x_n,\mathcal{H}\mathcal{H}x_n))$

$$\leq\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{G}x_n,\mathcal{G}\mathcal{H}x_n))+\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{H}x_n,\mathcal{H}\mathcal{G}x_n))+\lim_{n\to\infty}\psi(d(\mathcal{H}\mathcal{G}x_n,\mathcal{H}\mathcal{H}x_n))$$

$$=\lim_{n\to\infty}\psi(d(\mathcal{G}z,\mathcal{G}z))+\lim_{n\to\infty}\psi(0)+\lim_{n\to\infty}\psi(d(\mathcal{H}z,\mathcal{H}z))\to 0$$

So, that is the ψ −compatible mapping of type (P).

Conversely: - suppose that $\mathcal{G}$ and $\mathcal{H}$ are ψ −compatible mapping of type (P). That is $\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{G}x_n,\mathcal{H}\mathcal{H}x_n))=0$

Now, $\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{H}x_n,\mathcal{H}\mathcal{G}x_n))\leq\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{H}x_n,\mathcal{G}\mathcal{G}x_n))+\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{G}x_n,\mathcal{H}\mathcal{G}x_n))$

$$\leq\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{H}x_n,\mathcal{G}\mathcal{G}x_n))+\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{G}x_n,\mathcal{H}\mathcal{H}x_n))+\lim_{n\to\infty}\psi(d(\mathcal{H}\mathcal{H}x_n,\mathcal{H}\mathcal{G}x_n))$$

$$=\lim_{n\to\infty}\psi(d(\mathcal{G}z,\mathcal{G}z))+\lim_{n\to\infty}\psi(0)+\lim_{n\to\infty}\psi(d(\mathcal{H}z,\mathcal{H}z))\to 0$$

So, $\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{H}x_n,\mathcal{H}\mathcal{G}x_n))=0$

Therefore, $\mathcal{G}$ and $\mathcal{H}$ are ψ −compatible mapping.

Proposition 2.2 Suppose $\mathcal{G}$ and $\mathcal{K}$ are ψ −compatible mapping of type (P) and $\mathcal{G}z=\mathcal{K}z$ for some $z\in X$, then $\mathcal{G}\mathcal{G}z=\mathcal{G}\mathcal{K}z=\mathcal{K}\mathcal{G}z=\mathcal{K}\mathcal{K}z$.

Proof: -Since, $\mathcal{G}$ and $\mathcal{K}$ are ψ −compatible mapping of type (P), then

$\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{G}x_n,\mathcal{K}\mathcal{K}x_n))=0$…(1)

Suppose that $\{x_n\}$ is a sequence in X defined by $x_n=z$ for $n=1,2$ ... and $\mathcal{G}z=\mathcal{K}z=z$ for some $z\in X$. Then $\lim_{n\to\infty}\mathcal{G}x_n=\lim_{n\to\infty}\mathcal{K}x_n=\mathcal{G}z$

Since $\mathcal{G}$ and $\mathcal{K}$ are ψ −compatible mapping of type (P), then we have

$$\lim_{n\to\infty}\psi(d(\mathcal{G}\mathcal{G}x_n,\mathcal{K}\mathcal{K}x_n))=\psi\big(d(\mathcal{G}\mathcal{G}z,\mathcal{K}\mathcal{K}z)\big)=0$$

Therefore, $\big(d(\mathcal{G}\mathcal{G}z,\mathcal{K}\mathcal{K}z)\big)=0$

$\mathcal{G}\mathcal{G}z=\mathcal{K}\mathcal{K}z$ but we have $\mathcal{G}z=\mathcal{K}z$

This implies that $\mathcal{G}\mathcal{G}z=\mathcal{G}\mathcal{K}z=\mathcal{K}\mathcal{G}z=\mathcal{K}\mathcal{K}z$.

Proposition 2.3 Suppose that $\mathcal{G}$ and $\mathcal{K}$ are the self-mappings of a metric space (X,d) and let $\mathcal{G}$ and $\mathcal{K}$ be ψ −compatible mappings of type (P) such that $\lim_{n\to\infty}\mathcal{G}x_n=\lim_{n\to\infty}\mathcal{K}x_n=z$ for some $z\in X$. Then the following holds

(i) $\lim_{n\to\infty}\mathcal{K}\mathcal{K}x_n=\mathcal{G}z$ if $\mathcal{G}$ is non-continuous at z.

(ii) $\lim_{n\to\infty}\mathcal{G}\mathcal{G}x_n=\mathcal{K}z$ if $\mathcal{K}$ is non-continuous at z.

(iii) $\mathcal{G}\mathcal{K}z=\mathcal{K}\mathcal{G}z$ and $\mathcal{G}z=\mathcal{K}z$ if $\mathcal{G}$ and $\mathcal{K}$ are non-continuous at z.

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Proof: - (i) Suppose that $\mathcal{G}$ is non-continuous at z. Since we have $\lim_{n\to\infty} \mathcal{G}x_n = \lim_{n\to\infty} \mathcal{K}x_n = z$ for some $z \in X$, then $\lim_{n\to\infty} \mathcal{G}\mathcal{G}x_n = \mathcal{G}z$ by applying the property of ψ. Also, since $\mathcal{G}$ and $\mathcal{K}$ are ψ −compatible mappings of type (P), so $\lim_{n\to\infty} \psi(\, d(\mathcal{G}\mathcal{G}x_n, \mathcal{K}\mathcal{K}x_n)) = 0$.

Now, $\lim_{n\to\infty} \psi(\, d(\mathcal{K}\mathcal{K}x_n, \mathcal{G}z)) \leq \lim_{n\to\infty} \psi(\, d(\mathcal{K}\mathcal{K}x_n, \mathcal{G}\mathcal{G}x_n)) + \lim_{n\to\infty} \psi(\, d(\mathcal{G}\mathcal{G}x_n, \mathcal{G}z)) \to 0$

Therefore $\lim_{n\to\infty} \psi(\, d(\mathcal{K}\mathcal{K}x_n, \mathcal{G}z)) = 0$ implies $d(\mathcal{K}\mathcal{K}x_n, \mathcal{G}z)) = 0$ that is $\mathcal{K}\mathcal{K}x_n = \mathcal{G}z$. Proved.

(ii) Similarly, $\mathcal{K}$ is non-continuous at z, then $\lim_{n\to\infty} \mathcal{G}\mathcal{G}x_n = \mathcal{K}z$.

(iii) Since, $\mathcal{G}$ and $\mathcal{K}$ are non-continuous at z and $\mathcal{K}x_n \to z$ as $n \to \infty$ and $\mathcal{G}$ is non-continuous at z by (i) $\mathcal{K}\mathcal{K}x_n \to \mathcal{G}z$ as $n \to \infty$, on the other hand, since $\mathcal{K}x_n \to z$ as $n \to \infty$ and $\mathcal{K}$ is non-continuous at z, then $\mathcal{K}\mathcal{K}x_n \to \mathcal{K}z$ (by the property of ψ), thus we have $\mathcal{G}z = \mathcal{K}z$ by the uniqueness of the limit and by Prop.[2.2] , $\mathcal{K}\mathcal{G}z = \mathcal{G}\mathcal{K}z$.

3. Main Results

Theorem 3.1 Let (X, d) be a complete metric space and $\mathcal{G}, \mathcal{H}, \mathcal{K}$ and $\mathcal{L}$ be the self-mappings of a metric space (X, d) satisfying the following conditions.

(i) $\mathcal{G}X \subseteq \mathcal{L}X$ and $\mathcal{H}X \subseteq \mathcal{K}X$

(ii) $(\mathcal{G}, \mathcal{K})$ and $(\mathcal{H}, \mathcal{L})$ be ψ −compatible mapping of type (P),

(iii) There exists h in [0,1) such that $\psi(d(\mathcal{G}x, \mathcal{H}y) \leq hM_\psi(x, y)$ where, $M_\psi(x, y) =$

Max$\left\{\psi\big(d(\mathcal{K}x, \mathcal{L}y)\big), \psi\big(d(\mathcal{G}x, \mathcal{K}x)\big), \psi\big(d(\mathcal{H}y, \mathcal{L}y)\big), \frac{1}{2}\left[\psi\big(d(\mathcal{G}x, \mathcal{L}y)\big) + \psi\big(d(\mathcal{H}y, \mathcal{K}x)\big)\right]\right\}$ then $\mathcal{G}, \mathcal{H}, \mathcal{K}$ and $\mathcal{L}$ have a unique common fixed-point.

Proof: - Suppose that $\{u_n\}$ and $\{v_n\}$ be the sequences in X.Then by (i) we can define, for $n = 0,1,2,3$ ...

$$v_{2n} = \mathcal{G}u_{2n} = \mathcal{L}u_{2n+1}$$

$$v_{2n+1} = \mathcal{H}u_{2n+1} = \mathcal{K}u_{2n+2} \qquad [3.1.1]$$

Now, we shall show that $\{v_n\}$ is a Cauchy sequence. From (ii) we have

$$\begin{aligned}
\psi\big(d(v_{2n}, v_{2n+1})\big) &= \psi\big(d(\mathcal{G}u_{2n}, \mathcal{H}u_{2n+1})\big) \\
&\leq h\, M_\psi(u_{2n}, u_{2n+1}) \\
&= \text{h Max}\left\{\begin{matrix}\psi\big(d(\mathcal{K}u_{2n}, \mathcal{L}u_{2n+1})\big), \psi\big(d(\mathcal{G}u_{2n}, \mathcal{K}u_{2n})\big), \psi\big(d(\mathcal{H}u_{2n+1}, \mathcal{L}u_{2n+1})\big), \\ \frac{1}{2}\left[\psi\big(d(\mathcal{G}u_{2n}, \mathcal{L}u_{2n+1})\big) + \psi\big(d(\mathcal{H}u_{2n+1}, \mathcal{K}u_{2n})\big)\right]\end{matrix}\right\} \\
&= h \text{ Max}\left\{\begin{matrix}\psi\big(d(v_{2n-1}, v_{2n})\big), \psi\big(d(v_{2n}, v_{2n-1})\big), \psi\big(d(v_{2n+1}, v_{2n})\big), \\ \frac{1}{2}\left[\psi\big(d(v_{2n+1}, v_{2n-1})\big)\right]\end{matrix}\right\} \\
&= \text{h Max}\left\{\begin{matrix}\psi\big(d(v_{2n-1}, v_{2n})\big), \psi\big(d(v_{2n+1}, v_{2n})\big), \psi\big(d(v_{2n+1}, v_{2n})\big), \\ \frac{1}{2}\left[\psi\big(d(v_{2n+1}, v_{2n-1})\big)\right]\end{matrix}\right\} \\
&\leq h \text{ Max}\left\{\begin{matrix}\psi\big(d(v_{2n-1}, v_{2n})\big), \psi\big(d(v_{2n+1}, v_{2n})\big), \psi\big(d(v_{2n+1}, v_{2n})\big), \\ \frac{1}{2}\left[\psi\big(d(v_{2n+1}, v_{2n})\big) + \psi\big(d(v_{2n}, v_{2n-1})\big)\right]\end{matrix}\right\} \\
&\leq h \text{ Max}\left\{\begin{matrix}\psi\big(d(v_{2n-1}, v_{2n})\big), \psi\big(d(v_{2n+1}, v_{2n})\big), \psi\big(d(v_{2n+1}, v_{2n})\big), \\ \frac{1}{2}\left[\psi(2\, Max\{d(v_{2n+1}, v_{2n}), d(v_{2n}, v_{2n-1})\}\right]\end{matrix}\right\} \\
&\leq h \text{ Max}\left\{\begin{matrix}\psi\big(d(v_{2n-1}, v_{2n})\big), \psi\big(d(v_{2n+1}, v_{2n})\big), \\ \psi\, Max\{d(v_{2n+1}, v_{2n}), d(v_{2n}, v_{2n-1})\}\end{matrix}\right\} \\
&= h\, \psi\big(d(v_{2n-1}, v_{2n})\big)
\end{aligned}$$

That is $\psi\big(d(v_{2n}, v_{2n+1})\big) \leq h\, \psi\big(d(v_{2n-1}, v_{2n})\big) \qquad [3.1.2]$

In a similar way, we can show that

$$\psi\big(d(v_{2n-1}, v_{2n})\big) \leq h\, \psi\big(d(v_{2n-2}, v_{2n-1})\big) \qquad [3.1.3]$$

From the above two inequalities [3.1.2] and [3.1.3] we get.

$$\psi\big(d(v_n, v_{n+1})\big) \leq h^n\, \psi\big(d(v_0, v_1)\big)$$

Also, we have for every positive integer p

$$\psi\left(d(v_n, v_{n+p})\right) \leq \psi\left[d(v_n, v_{n+1}) + d(v_{n+1}, v_{n+2}) + \cdots + d(v_{n+p-1,} v_{n+p})\right]$$

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$$\leq \psi[(1 + h + h^2 \ldots + h^{p-1})\, h^n\, d(v_0, v_1)]$$
$$\leq \psi\left[\left(\frac{1}{1-h}\right) h^n\, d(v_0, v_1)\right]$$

Now, for given $\epsilon > 0$ there exist, positive number N such that

$$\psi\left[\left(\frac{1}{1-h}\right) h^n\, d(v_0, v_1)\right] < \psi(\epsilon) \text{ for all } n \geq N$$

This implies that,

$d(v_n, v_{n+p}) < \epsilon$ for all $n \geq N$ therefore $\{v_n\}$ is a Cauchy sequence in X.

Since, X is complete metric space there exist a point z in X such that $v_n \to z$ as $n \to \infty$,

Now from [3.1.1]

$$v_{2n} = \mathcal{G}u_{2n} = \mathcal{L}u_{2n+1} \to z \qquad [3.1.4]$$
$$v_{2n+1} = \mathcal{H}u_{2n+1} = \mathcal{K}u_{2n+2} \to z$$

Since, the mapping $\mathcal{G}$ and $\mathcal{K}$ be the ψ −compatible mapping of type (P) so,

$$\lim_{n\to\infty} \psi(\, d(\mathcal{G}\mathcal{G}x_n, \mathcal{K}\mathcal{K}x_n)) = 0$$

Then by the prop. [2.6]

$$\lim_{n\to\infty} \psi(\, d(\mathcal{K}z, \mathcal{G}z)) = 0$$
$$\Rightarrow d(\mathcal{K}z, \mathcal{G}z)) = 0 \Rightarrow \mathcal{K}z = \mathcal{G}z = t \quad \text{(say)} \qquad [3.1.5]$$

Since $\mathcal{G}X \subseteq \mathcal{L}X$ there is a point w in X such that $\mathcal{L}w = \mathcal{G}z$ then by [3.1.5]

$$\mathcal{L}w = \mathcal{G}z = \mathcal{K}z = t \qquad [3.1.6]$$

Now, we shall show that $t = \mathcal{H}w$, so let on the contrary that $t \neq \mathcal{H}w$, then the inequality (iii) we write,

$$\psi\big(d(t, \mathcal{H}w)\big) = \psi\big(d(\mathcal{G}z, \mathcal{H}w)\big) \leq h\, M_\psi(z, w) = h\, \psi(d(\mathcal{H}w, \mathcal{L}w)) = h\, \psi(d(\mathcal{H}w, \mathcal{G}z)) \qquad = h\, \psi(d(\mathcal{H}w, t)$$

Therefore, $\psi\big(d(t, \mathcal{H}w)\big) \leq h\, \psi(d(\mathcal{H}w, t))$ which is contradiction so, $t = \mathcal{H}w$ and hence by using [3.1.6]

$$\mathcal{H}w = \mathcal{G}z = \mathcal{K}z = \mathcal{L}w = t \qquad [3.1.7]$$

Also, by proposition [2.2] we have the relation

$$\mathcal{G}\mathcal{G}z = \mathcal{G}\mathcal{K}z = \mathcal{K}\mathcal{G}z = \mathcal{K}\mathcal{K}z$$

By using the relation [3.1.6] we get the relation

$$\mathcal{G}t = \mathcal{K}t \qquad [3.1.8]$$

Similarly, by taking $\mathcal{H}$ and $\mathcal{L}$ are the ψ −compatible mapping of type (P) so, by Prop. [2.2]

$$\mathcal{H}\mathcal{H}w = \mathcal{H}\mathcal{L}w = \mathcal{L}\mathcal{H}w = \mathcal{L}\mathcal{L}w$$

Again, by using [3.1.7] we get the relation

$$\mathcal{H}t = \mathcal{L}t \qquad [3.1.9]$$

Now, we shall show that $\mathcal{G}t = t$. For this, suppose on the contrary that $\mathcal{G}t \neq t$ then by using the inequality (iii) we get,

$$\psi\big(d(t, \mathcal{G}t)\big) = \psi\big(d(\mathcal{H}w, \mathcal{G}\mathcal{G}z)\big) \leq h\, M_\psi(\mathcal{G}z, w) = h\, \psi(d(\mathcal{G}z, \mathcal{G}\mathcal{G}z)) = \mathrm{h}\, \psi(\mathrm{d}(t, \mathcal{G}t))$$

Therefore $\psi\big(d(t, \mathcal{G}t)\big) \leq h\, \psi(d(t, \mathcal{G}t))$ which is contradiction, hence $\mathcal{G}t = t$

Then by [3.1.8] we get $\mathcal{G}t = \mathcal{K}t = t$ [3.2.0]

Therefore, t is the common fixed-point of $\mathcal{G}$ and $\mathcal{K}$

Now, we have to show that $\mathcal{H}t = t$. Suppose, on the contrary, that $\mathcal{H}t \neq t$ then by (iii) and using the relation [3.1.7] and [3.1.9]

$$\psi\big(d(t, \mathcal{H}t)\big) = \psi\big(d(\mathcal{H}w, \mathcal{H}\mathcal{H}w)\big) = \psi(\mathrm{d}(\mathcal{G}z, \mathcal{H}\mathcal{H}w)) \leq \mathrm{h}\, M_\psi(z, \mathcal{H}w) = h\psi(d(\mathcal{H}w, \mathcal{H}\mathcal{H}w)) = \psi\big(d(t, \mathcal{H}t)\big)$$

So, $\psi\big(d(t, \mathcal{H}t)\big) \leq \psi(d(t, \mathcal{H}t))$ Which is contradiction therefore, $\mathcal{H}t = t$

By [3.2.0] we get

$$\mathcal{G}t = \mathcal{K}t = \mathcal{H}t = t \qquad [3.2.1]$$

Also, we have to show that $\mathcal{L}t = \mathcal{H}t$. From the given relation Prop. [2.2]

We have the relation $\mathcal{L}\mathcal{H}w = \mathcal{H}\mathcal{H}w$ implies that $\mathcal{L}t = \mathcal{H}t$ [3.2.2]

So, from [3.2.1] and [3.2.2] we get,

$\mathcal{G}t = \mathcal{H}t = \mathcal{K}t = \mathcal{L}t = t$ So, t is a common fixed point of $\mathcal{G}, \mathcal{H}, \mathcal{K}$ and $\mathcal{L}$

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Uniqueness of common fixed-point:

Let t' be the other common fixed-point of $\mathcal{G}, \mathcal{H}, \mathcal{K}$ and $\mathcal{L}$, then we shall show that

$\psi(d(t,t')) = 0$.

Since t and t' be the two common fixed-point of $\mathcal{G}, \mathcal{H}, \mathcal{K}$ and $\mathcal{L}$ then clearly,

$\mathcal{G}t = \mathcal{H}t = \mathcal{K}t = \mathcal{L}t = t$ and $\mathcal{G}t' = \mathcal{H}t' = \mathcal{K}t' = \mathcal{L}t' = t'$

By inequality (iii) we get,

$$\begin{aligned}\psi(d(t,t')) &= \psi(d(\mathcal{G}t, \mathcal{G}t')) \\ &\leq h\, M_{\psi}(t,t') \\ &= h \operatorname{Max}\left\{\psi(d(\mathcal{K}t, \mathcal{L}t')), \psi(d(\mathcal{G}t, \mathcal{K}t)), \psi(d(\mathcal{H}t', \mathcal{L}t')), \tfrac{1}{2}\left[\psi(d(\mathcal{G}t, \mathcal{L}t')) + \psi(d(\mathcal{H}t', \mathcal{K}t))\right]\right\} \\ &= h \operatorname{Max}\left\{\psi(d(t,t')), \psi(d(t,t)), \psi(d(t',t')), \tfrac{1}{2}\left[\psi(d(t,t')) + \psi(d(t',t))\right]\right\} \\ &= \mathrm{h}\, \psi(d(t,t'))\end{aligned}$$

That is $\psi(d(t,t')) \leq h\, \psi(d(t,t'))$ which is contradiction, so

$\psi(d(t,t')) = 0$ implies $d(t,t') = 0$

So, $t = t'$, hence the common fixed-point is unique.

Example 3.1 Let (X, d)be the metric space where $X = [0,2]$ with the usual metric and the altering distance function defined by $\psi: R^{+} \to R^{+}$such that $\psi(t) = t^2 - \frac{t}{2}$ satisfying the criterion of this function.

Let $\mathcal{G}, \mathcal{H}, \mathcal{K}$, and $\mathcal{L}$ be the self-mappings of the metric space (X, d) defined by

$$\mathcal{G}(x) = \begin{cases} 1 - x & for\ 0 \leq \mathrm{x} < \frac{1}{2} \\ 1 & for\ \frac{1}{2} \leq x \leq 2 \end{cases}$$

$$\mathcal{H}(x) = \begin{cases} \frac{1}{2} - x & for\ 0 \leq x < \frac{1}{2} \\ 1 & for\ \frac{1}{2} \leq x \leq 2 \end{cases}$$

$$\mathcal{K}(x) = \begin{cases} x & for\ 0 \leq x < \frac{1}{2} \\ 1 & for\ \frac{1}{2} \leq x \leq 2 \end{cases}$$

$$\mathcal{L}(x) = \begin{cases} 0 & for\ 0 \leq x < \frac{1}{2} \\ 1 & for\ \frac{1}{2} \leq x \leq 2 \end{cases}$$

Now, we take a sequence $\{x_n\}$ in X such that $x_n = \frac{1}{2} - \frac{1}{n}$, for $n \in \mathbb{N}$

The condition (i) of the theorem clearly satisfied, For the (ii) condition to show the mapping $\mathcal{G}$ and $\mathcal{K}$ be the $\psi -$ compatible mapping of type (P)

$\lim_{n\to\infty} \mathcal{G}x_n =. \lim_{n\to\infty}(1 - x_n) = \lim_{n\to\infty}(1 - \left\{\frac{1}{2} - \frac{1}{n}\right\}) = \frac{1}{2}$

$\lim_{n\to\infty} \mathcal{K}x_n =. \lim_{n\to\infty}(x_n) = \lim_{n\to\infty}(x_n) = \lim_{n\to\infty}\left\{\frac{1}{2} - \frac{1}{n}\right\} = \frac{1}{2}$ that is

$\lim_{n\to\infty} \mathcal{G}x_n = \lim_{n\to\infty} \mathcal{K}x_n = \frac{1}{2}$.

Now, $\psi(d(\mathcal{G}\mathcal{G}x_n, \mathcal{K}\mathcal{K}x_n)) = \psi\left(d\left(\mathcal{G}\left(\frac{1}{2}\right), \mathcal{K}\left(\frac{1}{2}\right)\right)\right) = \psi(d(1,1)) = \psi(0) = 0$, Hence, $(\mathcal{G}, \mathcal{K})$ are the $\psi -$ compatible mapping of type (P)

Again, for the pair $(\mathcal{H}, \mathcal{L})$

$\lim_{n\to\infty} \mathcal{H}x_n =. \lim_{n\to\infty}\left(\frac{1}{2} - x_n\right) = \lim_{n\to\infty}\left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{n}\right\}\right) = 0$

$\lim_{n\to\infty} \mathcal{L}x_n =. \lim_{n\to\infty} S\left(\frac{1}{2} - \frac{1}{n}\right) = 0$ that is

$\lim_{n\to\infty} \mathcal{H}x_n = \lim_{n\to\infty} \mathcal{L}x_n = 0$.

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Now, $\psi(d(\mathcal{H}\mathcal{H}x_n, \mathcal{L}\mathcal{L}x_n)) = \psi\left(d(\mathcal{H}(0), \mathcal{L}(0))\right) = \psi\left(d\left(\frac{1}{2}, 0\right)\right) = \psi\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{4} = 0$, Hence, $(\mathcal{H}, \mathcal{L})$ are $\psi -$ compatible mapping of type (P).

Therefore, $(\mathcal{G}, \mathcal{K})$ and $(\mathcal{H}, \mathcal{L})$ are $\psi -$ compatible mapping of type (P).

Now. for the (iii) condition

$\psi(d(\mathcal{G}x, \mathcal{H}y) \leq \mathrm{h}M_{\psi}(x,y)$ for all $x, y \in X$ and $h \in [0,1)$. Where, $M_{\psi}(x,y) = \text{Max}\left\{\psi(d(\mathcal{K}x, \mathcal{L}y)), \psi(d(\mathcal{G}x, \mathcal{K}x)), \psi(d(\mathcal{H}y, \mathcal{L}y)), \frac{1}{2}\left[\psi(d(\mathcal{G}x, \mathcal{L}y)) + \psi(d(\mathcal{H}y, \mathcal{K}x))\right]\right\}$

That is $\psi(d(\mathcal{G}x, \mathcal{H}y) \leq \text{Max}\left\{\begin{array}{c}\psi(d(\mathcal{K}x, \mathcal{L}y)), \psi(d(\mathcal{G}x, \mathcal{K}x)), \psi(d(\mathcal{H}y, \mathcal{L}y)), \\ \frac{1}{2}\left[\psi(d(\mathcal{G}x, \mathcal{L}y)) + \psi(d(\mathcal{H}y, \mathcal{K}x))\right]\end{array}\right\}$

$$\psi(d(P(0), Q(2)) \leq \text{Max}\left\{\begin{array}{c}\psi\left(d(\mathcal{K}(0), \mathcal{L}(2))\right), \psi\left(d(\mathcal{G}(0), \mathcal{K}(0))\right), \psi\left(d(\mathcal{H}(2), \mathcal{L}(2))\right), \\ \frac{1}{2}\left[\psi(d(\mathcal{G}(0), \mathcal{L}(2))) + \psi(d(\mathcal{H}(2), \mathcal{K}(0)))\right]\end{array}\right\}$$

$0 \leq \frac{h}{2}$ for $0, 2 \in \mathrm{X} = [0,2]$ and $h \in [0,1)$,hence the condition (iii) is true.

Therefore, $\mathcal{G}, \mathcal{H}, \mathcal{K}$ and $\mathcal{L}$ satisfy all the conditions of the above theorem 3.1 and there exist a point $1 \in X = [0,2]$ such that $\mathcal{G}(1) = \mathcal{H}(1) = \mathcal{K}(1) = \mathcal{L}(1) = 1$. Therefore 1 is unique common fixed point of four self- mapping $\mathcal{G}, \mathcal{H}, \mathcal{K}$ and $\mathcal{L}$.

5.Conclusion

We used the altering distance function to introduce the idea of ψ-compatible mapping of type (P) in complete metric space in this paper. We also proved the common fixed-point theorem, which shows that there is a unique common fixed point for four self-mappings under the right contractive condition, and we did this with an example. This theory makes fixed-point theory more useful in the context of metric spaces and provides a unified way to connect classical deterministic results with the structure of metric spaces through the changing distance function ψ.

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