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TRIPLED FIXED POINT THEOREMS IN C^*-ALGEBRA VALUED S_b-METRIC SPACES

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Abstract:
In this paper, we establish unique common tripled fixed point theorems for pairs of mappings that satisfy a specific rational contraction condition in C^*-AV-S_b-MS .Several illustrative examples are provided to demonstrate the validity of the hypotheses underlying our results .Furthermore ,we explore an application in homotopy theory ,where we establish both the existence and uniqueness of solutions to a class of nonlinear integral equations .

Key words: Tripled Fixed-point Theorems in $C^* - algebra$ Valued S_b -metric Spaces, Examples to Theorems, Applications to Integral Equations and Homotopy.

2000 Mathematics Subject Classification: 2010 MSC :54H25,47H10,54E50.

1.INTRODUCTION

Polish mathematician Stephen Banach [1] developed the most notable fixed- point theorem in 1992. It is often referred to as the Banach contraction principle and is used extensively in a variety of mathematical contexts.
The study of fixed- point theory in metric spaces is thought to have led to the discovery of the Banach contraction principle. Different generalizations of metric spaces were introduced by a number of mathematicians worldwide ([2], [3], [4], [5], [6]). The concept of C^*-algebra-valued metric spaces was first introduced by Ma et al. in 2014 [7]. They presented the notion of C^*-algebra-valued b-metric spaces in 2015 and looked at a few results [8]. In order to find similar principles, Razavi and Masiha also explored C^*-algebra-valued b-metric spaces [9].
Combining the ideas of S and b-metric spaces, Sedghi et al. [10] established S_b-metric spaces and proved common fixed -point results in these spaces. Several authors have produced a wealth of results on S_b-metric spaces (e.g., [11] [19]). principles) in an e ort to improve.
Razavi et al. introduced the concept of C^*-algebra-valued S_b-metric space [20] in 2023, drawing inspiration from the work of Souayah and Mlaiki in [11]. They also established some common fixed-point results in this space [21].
This work presents a unique common tripled fixed-point theorem for pairs of mappings that satisfy a specific rational contraction condition in C^*-algebra-valued S_b-metric spaces. We can also provide appropriate and relevant examples for both integral equations and homotopy.
First, we recall some basic results.

2.PRELIMINARIES

This section provides a short introduction to some realities about the theory of C^*-algebras[24].First, suppose that $\mathfrak{U}$ is a unital C^*-algebra with the unit $1_{\mathfrak{U}}$ .Set $\mathfrak{U}_h$={s∈ $\mathfrak{U}$; s=s^* } .The element s∈ $\mathfrak{U}$ is said to be positive ,and we write s≥ $0_{\mathfrak{U}}$ if and only if s=s^* and $\sigma(s) \subseteq [0,\infty), in\ which$ $0_{\mathfrak{U}}$ in $\mathfrak{U}$ is the zero element and spectrum of s is $\sigma(s)$ .On $\mathfrak{U}_h$,we can find a

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natural partial ordering given by $\ell \preccurlyeq \varphi$ if and only if $\varphi - \ell \succcurlyeq 0_{\mathfrak{U}}$ .we denote with $\mathfrak{U}_+$= {s∈ $\mathfrak{U}$; s≥ $0_{\mathfrak{U}}$} and $\mathfrak{U}'$={s∈ $\mathfrak{U}$; st=ts $\forall t \in \mathfrak{U}$ }.

Definition 2.1. ([20]): Let G be a non-empty set and κ ∈ $\mathfrak{A}'$ with ||κ|| ≥ 1. Suppose that a mapping S_b: G 3 → $\mathfrak{A}$ be a function satisfying the following properties

(S_{b_1}) :S_b (α, β, γ) $\succcurlyeq 0_{\mathfrak{A}}$for all α, β, γ ∈ G,

(S_{b_2}): S_b (α, β, γ) = 0 ⇔ α = β = γ,

(S_{b_3}): S_b (α, β, γ) ≤ κ(S_b (α, α, θ) + S_b (β, β, θ) + S_b (γ, γ, θ)) for all α, β, γ, θ ∈ G.

Then the function $S_b$is called a C ∗ -algebra valued S_b-metric on G and the pair (G, $\mathfrak{A}$, S_b) is called a C ∗ -algebra valued S_b-metric space (C ∗ -AV-S_bMS) with a co-efficient κ.

Example 2.1. ([7]). Let G = R and $\mathfrak{A}$ = M2(R) be all 2 × 2 matrices with the usual operations of addition, scalar multiplication, and matrix multiplication. It is clear that ||P|| = $\sqrt{\sum_{i,j=1}^{2}\left|P_{i,j}\right|^2}$defines a norm on $\mathfrak{A}$, where P = (p_{ij}) ∈ $\mathfrak{A}$.

∗: $\mathfrak{A} \to \mathfrak{A}$ defines an involution on $\mathfrak{A}$ and where $\mathfrak{A}$ ∗ = $\mathfrak{A}$. Then, $\mathfrak{A}$ is a C ∗ -algebra. For P = (p_{ij}) and Q = (q_{ij}) in $\mathfrak{A}$, a partial order on $\mathfrak{A}$ can be given as follows: P ≼ Q ⇐⇒ $p_{ij}-q_{ij}$ ≤ 0 ∀ i , j = 1, 2. Let (G, d) be a b-metric space where, ||κ|| ≥ 1 and Sb : G 3 → M2(R), fulfilling Sb(p, q, r) = $\begin{bmatrix} d(p, q) + d(q, r) + d(r, p) & 0 \\ 0 & d(p, q) + d(q, r) + d(r, p) \end{bmatrix}$.

Then, clearly (G, $\mathfrak{A}$, Sb) is a C ∗ -AV-S_b −MS.

Definition 2.2. ([20]): A C ∗ -AV-S_b −MS is symmetric if S_b (α,β,γ) = S_b (α,γ,β)= S_b $(\beta,\gamma,\alpha) = S_b\ (\beta,\alpha,\gamma) = S_b\ (\gamma,\alpha,\beta)$ $= S_b\ (\gamma,\beta,\alpha)$∀ α, β, γ ∈ G

Definition 2.3. ([20]): Let (G, $\mathfrak{A}$, S_b) be a C ∗ -AV-Sb-MS and $\{X_n\}$ be a sequence in G:

(1) If for all p ∈ N, ||S_b (X_{n+p}, X_{n+p}, X_n)|| → 0, where n → ∞, then $\{X_n\}$ is a Cauchy sequence in G.

(2) If ||S_b $(X_n, X_n$, χ)|| → 0, where n → ∞, then $\{X_n\}$ converges to χ, and we present it with $\lim_{n\to\infty} X_n$ = χ.

(3) If every Cauchy sequence is convergent in G , then (G, $\mathfrak{A}$, S_b) is a complete C ∗ -AV-Sb-MS.

Definition 2.4. ([20]): Suppose that $(\mathcal{G}_1, \mathfrak{A}_1, S_{b_1})$ and $(\mathcal{G}_2, \mathfrak{A}_2, S_{b_2})$ are C ∗ -AV-S_b MS, and let Γ: $(\mathcal{G}_1, \mathfrak{A}_1, S_{b_1}) \to (\mathcal{G}_2, \mathfrak{A}_2, S_{b_2})$ be a function. Then, Γ is continuous at a point χ ∈ $\mathcal{G}_1$if, for every sequence, $\{X_n\}$ in $\mathcal{G}_1$, $S_b(X_n, X_n$, χ) →$0_{\mathfrak{A}}$, (n → ∞) implies S_b (Γ(X_n), Γ(X_n), γ(χ)) → 0 $_{\mathfrak{A}}$, where n → ∞. A function Γ is continuous at $\mathcal{G}_1$if and only if it is continuous at all χ ∈ $\mathcal{G}_1$.

Lemma 2.5([23]) : Suppose that $\mathfrak{A}$ is a unital C ∗ -algebra with a unit $1_{\mathfrak{A}}$:

(1) If $\{x_n\}_{n=1}^{\infty}$ ∞⊆ $\mathfrak{A}$ and $\lim_{n\to\infty} x_n$ =$0_{\mathfrak{A}}$, then for any χ ∈ $\mathfrak{A}$, $\lim_{n\to\infty}$ χ ∗ χ_n χ=$0_{\mathfrak{A}}$.

(2) If χ, ξ ∈ $\mathfrak{A}_h$and s ∈$\mathfrak{A}'_+$ then χ ≼ ξ yields sχ ≼ sξ in which $\mathfrak{A}'_+ = \mathfrak{A}_+ \cap \mathfrak{A}'$.

(3) If χ ∈ $\mathfrak{A}_+$ with ||χ|| < $\frac{1}{2}$then $1_{\mathfrak{A}}$− χ is invertible, and ||χ ($1_{\mathfrak{A}}$− χ) −1 || < 1.

(4) If χ, ξ ∈ $\mathfrak{A}_+$ such that χξ = ξχ, then χξ ≽$0_{\mathfrak{A}}$.

Lemma 2.6: If $(\mathcal{G}, \mathfrak{A}, S_b)$ be a C^*AVS_bMS with $\|k\| \geq 1$ and suppose that $\{æ_p\}$ is a $C^* - AV{-}S_b$-convergent to ℓ , then we have

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$\frac{1}{2k}\, S_b(\wp,\wp,\ell) \preccurlyeq \lim_{p\to\infty}\inf S_b(\wp,\wp,æ_p) \preccurlyeq \lim_{p\to\infty}\sup S_b(\wp,\wp,æ_p) \preccurlyeq 2\mathrm{k} S_b(\wp,\wp,\ell)$ for all $\wp \in \mathcal{G}$. In particular, if $\ell = \wp$,then we have $\lim_{p\to\infty} S_b(\ell_p,\ell_p,\wp)=0_{\mathfrak{U}}$.

Proof : Using condition (S_{b_3}) of Definition 2.1, we have

$$S_b(\wp,\wp,æ_p) \preccurlyeq 2\mathrm{k} S_b(\wp,\wp,\ell) + \mathrm{k} S_b(æ_p,æ_p,\ell)$$

and

$$S_b(\wp,\wp,\ell) \preccurlyeq 2\mathrm{k} S_b(\wp,\wp,æ_p) + \mathrm{k} S_b(\ell,\ell,æ_p)$$

Taking the upper limit as $p \to \infty$ in the first inequality and the lower limit as $p \to \infty$ in the second inequality we obtain the desired result.

3. MAIN RESULTS

3.1 Some Applications of Tripled Fixed-point Theorem in C^*-Algebra Valued S_b-Metric Spaces

The objective of this section is to prove the existence of common tripled fixed points under the specific rational contractive conditions and to illustrate the applicability of our findings through relevant examples drawn from existing results.

Definition 3.1.1: Let $(\mathcal{L},\mathcal{A},\varrho_\varsigma)$ be a complete C^*-AV-S_b-MS with $\|\varsigma\| \geq 1$ .Let $\Omega: \mathcal{L}^3 \to \mathcal{L}$ be a mapping, an element $(æ,œ,\mathfrak{s}) \in \mathcal{L}^3$ is called tripled fixed point of Ω if $\Omega(æ,œ,\mathfrak{s}) = æ$, $\Omega(œ,\mathfrak{s},æ) = œ$, $\Omega(\mathfrak{s},æ,œ) = \mathfrak{s}$.

Theorem 3.1.1. Let $(\mathcal{L},\mathcal{A},\varrho_\varsigma)$ be a complete C^*-AV-S_b-MS with $\|\varsigma\| \geq 1$, , suppose $\Gamma,\Omega: \mathcal{L}^3 \to \mathcal{L}$ be two mappings satisfying for all $\ell,\varkappa,\mathfrak{z},æ,œ,\mathfrak{s} \in \mathcal{L}$

$$4\varsigma^2\varrho_\varsigma(\Gamma(\ell,\varkappa,\mathfrak{z}),\Gamma(\ell,\varkappa,\mathfrak{z}),\Omega(æ,œ,\mathfrak{s})) \preccurlyeq \frac{a[\varrho_\varsigma(\ell,\ell,æ) + \varrho_\varsigma(\varkappa,\varkappa,œ) + \varrho_\varsigma(\mathfrak{z},\mathfrak{z},\mathfrak{s})]}{3}$$
$$+\frac{a[\varrho_\varsigma(\ell,\ell,\Gamma(\ell,\varkappa,\mathfrak{z}))\,.\,\varrho_\varsigma(æ,æ,\Gamma(\ell,\varkappa,\mathfrak{z}))+\varrho_\varsigma(æ,æ,\Omega(æ,œ,\mathfrak{s}))\,.\,\varrho_\varsigma(\ell,\ell,\Omega(æ,œ,\mathfrak{s}))]}{1+\varrho_\varsigma(\ell,\ell,\Omega(æ,œ,\mathfrak{s}))+\varrho_\varsigma(æ,æ,\Gamma(\ell,\varkappa,\mathfrak{z}))} \quad \text{.......(3.1.1)} \quad \text{Where}$$

a$\in \mathcal{A}_+$ in which $\|a\| < \frac{1}{2}$ .Then $\Gamma$and Ω have a unique common tripled fixed point in $\mathcal{L}$.

Proof: Let $æ_0,œ_0,\mathfrak{s}_0 \in \mathcal{L}$ be arbitrary, we construct the sequences $\{æ_p\}$, $\{œ_p\}$, $\{\mathfrak{s}_p\}$ in $\mathcal{L}$ as

$\Gamma(æ_{2p},œ_{2p},\mathfrak{s}_{2p})= æ_{2p+1}$, $\Gamma(œ_{2p},\mathfrak{s}_{2p},æ_{2p,})= œ_{2p+1}$, $\Gamma(\mathfrak{s}_{2p},æ_{2p},œ_{2p})= \mathfrak{s}_{2p+1}$

$\Omega(æ_{2p+1},œ_{2p+1},\mathfrak{s}_{2p+1})= æ_{2p+2}$, $\Omega(œ_{2p+1},\mathfrak{s}_{2p+1},æ_{2p+1,})= œ_{2p+2}$,

$\Omega(\mathfrak{s}_{2p+1},æ_{2p+1},œ_{2p+1})= \mathfrak{s}_{2p+2}$ where p= 0,1,2 …………

To establish the existence of a common tripled fixed point for the mappings Γ,Ω in $\mathcal{L}$,suppose that $\varrho_\varsigma(æ_{2p,},æ_{2p,},æ_{2p+1}) \succ 0_\mathcal{A}$, $\varrho_\varsigma(œ_{2p,},œ_{2p,},œ_{2p+1}) \succ 0_\mathcal{A}$ and $\varrho_\varsigma(\mathfrak{s}_{2p,},\mathfrak{s}_{2p,},\mathfrak{s}_{2p+1}) \succ 0_\mathcal{A}$ $\forall$ p. If this assumption fails, then there exists some p such that $æ_{2p} = æ_{2p+1}$, $œ_{2p} = œ_{2p+1}$ and $\mathfrak{s}_{2p} = \mathfrak{s}_{2p+1}$ .Consequently , $(æ_{2p},œ_{2p},\mathfrak{s}_{2p})$ serves as a common tripled fixed point of Γ,Ω and hence completing the proof . By using (3.1.1), for each p$\in N$,we have

$$\varrho_\varsigma(æ_{2p+1,},æ_{2p+1,},æ_{2p+2}) = \varrho_\varsigma(\Gamma(æ_{2p},œ_{2p},\mathfrak{s}_{2p}),\Gamma(æ_{2p},œ_{2p},\mathfrak{s}_{2p}),\Omega(æ_{2p+1},œ_{2p+1},\mathfrak{s}_{2p+1}))$$
$$\leq 4\varsigma^2\varrho_\varsigma(\Gamma(æ_{2p},œ_{2p},\mathfrak{s}_{2p}),\Gamma(æ_{2p},œ_{2p},\mathfrak{s}_{2p}),\Omega(æ_{2p+1},œ_{2p+1},\mathfrak{s}_{2p+1}))$$
$$\preccurlyeq \frac{a\{\varrho_\varsigma(æ_{2p},æ_{2p},æ_{2p+1}) + \varrho_\varsigma(œ_{2p},œ_{2p},œ_{2p+1}) + \varrho_\varsigma(\mathfrak{s}_{2p},\mathfrak{s}_{2p},\mathfrak{s}_{2p+1})\}}{3}$$
$$+\frac{a\left\{\begin{array}{c}\varrho_\varsigma(æ_{2p},æ_{2p},\Gamma(æ_{2p},œ_{2p},\mathfrak{s}_{2p}))\,.\,\varrho_\varsigma(æ_{2p+1},æ_{2p+1},\Gamma(æ_{2p},œ_{2p},\mathfrak{s}_{2p}))\\ +\ \varrho_\varsigma(æ_{2p+1},æ_{2p+1},\Omega(æ_{2p+1},œ_{2p+1},\mathfrak{s}_{2p+1}))\,.\,\varrho_\varsigma(æ_{2p},æ_{2p},\Omega(æ_{2p+1},œ_{2p+1},\mathfrak{s}_{2p+1}))\end{array}\right\}}{1+\varrho_\varsigma(æ_{2p},æ_{2p},\Omega(æ_{2p+1},œ_{2p+1},\mathfrak{s}_{2p+1}))+\varrho_\varsigma(æ_{2p},æ_{2p},\Gamma(æ_{2p},œ_{2p},\mathfrak{s}_{2p}))}$$
$$\preccurlyeq \frac{a[\varrho_\varsigma(æ_{2p},æ_{2p},æ_{2p+1}) + \varrho_\varsigma(œ_{2p},œ_{2p},œ_{2p+1}) + \varrho_\varsigma(\mathfrak{s}_{2p},\mathfrak{s}_{2p},\mathfrak{s}_{2p+1})]}{3}$$

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$$+\frac{a[\varrho_\varsigma(æ_{2p},æ_{2p},æ_{2p+1})\cdot\varrho_\varsigma(æ_{2p+1},æ_{2p+1},æ_{2p+1})+\varrho_\varsigma(æ_{2p+1},æ_{2p+1},æ_{2p+2})\cdot\varrho_\varsigma(æ_{2p},æ_{2p},æ_{2p+2})]}{1+\varrho_\varsigma(æ_{2p},æ_{2p},æ_{2p+2})+\varrho_\varsigma(æ_{2p},æ_{2p},æ_{2p+1})}$$

$$\preccurlyeq\frac{a[\varrho_\varsigma(æ_{2p},æ_{2p},æ_{2p+1})+\varrho_\varsigma(œ_{2p},œ_{2p},œ_{2p+1})+\varrho_\varsigma(ꞩ_{2p},ꞩ_{2p},ꞩ_{2p+1})]}{3}$$

$$+\ a\varrho_\varsigma(æ_{2p+1},æ_{2p+1},æ_{2p+2})$$

Which implies that

$$\varrho_\varsigma(æ_{2p+1},æ_{2p+1},æ_{2p+2})\preccurlyeq\frac{a(1_{\mathcal{A}}-a)^{-1}}{3}\left\{\begin{matrix}\varrho_\varsigma(æ_{2p},æ_{2p},æ_{2p+1})+\varrho_\varsigma(œ_{2p},œ_{2p},œ_{2p+1})\\+\varrho_\varsigma(ꞩ_{2p},ꞩ_{2p},ꞩ_{2p+1})\end{matrix}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (3.1.2)$$

Similarly, we can prove that

$$\varrho_\varsigma(œ_{2p+1},œ_{2p+1},œ_{2p+2})\preccurlyeq\frac{a(1_{\mathcal{A}}-a)^{-1}}{3}\left\{\begin{matrix}\varrho_\varsigma(æ_{2p},æ_{2p},æ_{2p+1})+\varrho_\varsigma(œ_{2p},œ_{2p},œ_{2p+1})\\+\varrho_\varsigma(ꞩ_{2p},ꞩ_{2p},ꞩ_{2p+1})\end{matrix}\right\}\ \ldots\ldots\ldots\ldots\ldots\ (3.1.3)$$

Also

$$\varrho_\varsigma(ꞩ_{2p+1},ꞩ_{2p+1},ꞩ_{2p+2})\preccurlyeq\frac{a(1_{\mathcal{A}}-a)^{-1}}{3}\left\{\begin{matrix}\varrho_\varsigma(æ_{2p},æ_{2p},æ_{2p+1})+\varrho_\varsigma(œ_{2p},œ_{2p},œ_{2p+1})\\+\varrho_\varsigma(ꞩ_{2p},ꞩ_{2p},ꞩ_{2p+1})\end{matrix}\right\}\ldots\ldots\ldots\ldots\ (3.1.4)$$

Adding (3.1.2), (3.1.3) and (3.1.4), we get

$$\varrho_\varsigma(æ_{2p+1},æ_{2p+1},æ_{2p+2})+\varrho_\varsigma(œ_{2p+1},œ_{2p+1},œ_{2p+2})+\varrho_\varsigma(ꞩ_{2p+1},ꞩ_{2p+1},ꞩ_{2p+2})$$
$$\preccurlyeq a(1_{\mathcal{A}}-a)^{-1}\left\{\begin{matrix}\varrho_\varsigma(æ_{2p},æ_{2p},æ_{2p+1})+\varrho_\varsigma(œ_{2p},œ_{2p},œ_{2p+1})\\+\varrho_\varsigma(ꞩ_{2p},ꞩ_{2p},ꞩ_{2p+1})\end{matrix}\right\}$$

Which implies that

$$\left\|\varrho_\varsigma(æ_{2p+1},æ_{2p+1},æ_{2p+2})+\varrho_\varsigma(œ_{2p+1},œ_{2p+1},œ_{2p+2})+\varrho_\varsigma(ꞩ_{2p+1},ꞩ_{2p+1},ꞩ_{2p+2})\right\|$$
$$\leq\|a(1_{\mathcal{A}}-a)^{-1}\|\left\|\begin{matrix}\varrho_\varsigma(æ_{2p},æ_{2p},æ_{2p+1})+\varrho_\varsigma(œ_{2p},œ_{2p},œ_{2p+1})\\+\varrho_\varsigma(ꞩ_{2p},ꞩ_{2p},ꞩ_{2p+1})\end{matrix}\right\|$$

Since b = $a(1_{\mathcal{A}}-a)^{-1}$ and by using Lemma 2.5, we have a$\in\mathcal{A}'_+$ with $\|a\|<\frac{1}{2}$ one has $(1_{\mathcal{A}}-a)^{-1}\in\mathcal{A}'_+$ and $a(1_{\mathcal{A}}-a)^{-1}\in\mathcal{A}'_+$ with $\|a(1_{\mathcal{A}}-a)^{-1}\|<1$ so that $\|b\|<1$.

Let $\Delta_{2p+1}=\varrho_\varsigma(æ_{2p+1},æ_{2p+1},æ_{2p+2})+\varrho_\varsigma(œ_{2p+1},œ_{2p+1},œ_{2p+2})+\varrho_\varsigma(ꞩ_{2p+1},ꞩ_{2p+1},ꞩ_{2p+2})$ then we have

$$\|\Delta_{2p+1}\|\leq\|b\|\|\Delta_{2p}\|\leq\|b\|^2\|\Delta_{2p-1}\|\leq\ldots\ldots\ldots\ldots\leq\|b\|^{2p+1}\|\Delta_0\|$$

It follows that as p$\rightarrow\infty$, we get $\Delta_p\rightarrow 0_{\mathfrak{A}}$. If $\Delta_0=0_{\mathcal{A}}$, then from (S_{b_2}) of Definition 2.1, we know $(æ_0,œ_0,ꞩ_0)$ is a tripled fixed point of Γ,Ω. Now letting $\Delta_0\succ 0_{\mathcal{A}}$, for any p$\in N$ and $l\in N$, using condition (S_{b_3}) of definition 2.1,

$$\varrho_\varsigma(æ_{p+l},æ_{p+l},æ_p)\preccurlyeq\varsigma\left(\begin{matrix}\varrho_\varsigma(æ_{p+l},æ_{p+l},æ_{p+l-1})+\varrho_\varsigma(æ_{p+l},æ_{p+l},æ_{p+l-1})\\+\varrho_\varsigma(æ_p,æ_p,æ_{p+l-1})\end{matrix}\right)$$
$$\preccurlyeq 2\varsigma\ \varrho_\varsigma(æ_{p+l},æ_{p+l},æ_{p+l-1})+\varsigma\ \varrho_\varsigma(æ_p,æ_p,æ_{p+l-1})$$
$$\preccurlyeq 2\varsigma\ \varrho_\varsigma(æ_{p+l},æ_{p+l},æ_{p+l-1})+2\varsigma^2\varsigma\ \varrho_\varsigma(æ_{p+l-1},æ_{p+l-1},æ_{p+l-2})$$
$$+\varsigma^2\varsigma\ \varrho_\varsigma(æ_{p+l-2},æ_{p+l-2},æ_p)$$

……………………………………………………………………………………………

$$\preccurlyeq 2\varsigma\ \varrho_\varsigma(æ_{p+l},æ_{p+l},æ_{p+l-1})+2\varsigma^2\varsigma\ \varrho_\varsigma(æ_{p+l-1},æ_{p+l-1},æ_{p+l-2})$$
$$+2\varsigma^3\varsigma\ \varrho_\varsigma(æ_{p+l-2},æ_{p+l-2},æ_{p+l-3})+\ldots\ldots\ldots\ldots\ldots\ldots+2\varsigma^l\varsigma\ \varrho_\varsigma(æ_{p+1},æ_{p+1},æ_p)$$

Similarly,

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$\varrho_\varsigma(\oe_{p+l},\oe_{p+l},\oe_p) \preccurlyeq 2\varsigma\ \varrho_\varsigma(\oe_{p+l},\oe_{p+l},\oe_{p+l-1}) + 2\varsigma^2\varsigma\ \varrho_\varsigma(\oe_{p+l-1},\oe_{p+l-1},\oe_{p+l-2})$
$+2\varsigma^3\varsigma\ \varrho_\varsigma(\oe_{p+l-2},\oe_{p+l-2},\oe_{p+l-3}) + \dots\dots + 2\varsigma^l\varsigma\ \varrho_\varsigma(\oe_{p+1},\oe_{p+1},\oe_p)$
And
$\varrho_\varsigma(\mathfrak{s}_{p+l},\mathfrak{s}_{p+l},\mathfrak{s}_p) \preccurlyeq 2\varsigma\ \varrho_\varsigma(\mathfrak{s}_{p+l},\mathfrak{s}_{p+l},\mathfrak{s}_{p+l-1}) + 2\varsigma^2\varsigma\ \varrho_\varsigma(\mathfrak{s}_{p+l-1},\mathfrak{s}_{p+l-1},\mathfrak{s}_{p+l-2})$
$+2\varsigma^3\varsigma\ \varrho_\varsigma(\mathfrak{s}_{p+l-2},\mathfrak{s}_{p+l-2},\mathfrak{s}_{p+l-3}) + \dots\dots + 2\varsigma^l\varsigma\ \varrho_\varsigma(\mathfrak{s}_{p+1},\mathfrak{s}_{p+1},\mathfrak{s}_p)$
Consequently,
$\varrho_\varsigma(\ae_{p+l},\ae_{p+l},\ae_p) + \varrho_\varsigma(\oe_{p+l},\oe_{p+l},\oe_p) + \varrho_\varsigma(\mathfrak{s}_{p+l},\mathfrak{s}_{p+l},\mathfrak{s}_p)$
$\preccurlyeq 2\varsigma\,\Delta_{p+l-1} + 2\varsigma^2\Delta_{p+l-2} + +2\varsigma^3\Delta_{p+l-3} + \dots\dots + 2\varsigma^l\Delta_p$
$\preccurlyeq 2\varsigma\, b^{p+l-1}\Delta_0 + 2\varsigma^2\, b^{p+l-2}\Delta_0 + +2\varsigma^3\, b^{p+l-3}\Delta_0 + \dots\dots + 2\varsigma^l\, b^p\Delta_0$
$\preccurlyeq 2\sum_{m=p}^{m=p+l-1}\varsigma^{l-m+p}\, b^{\frac{m}{2}}\Delta_0^{\frac{1}{2}}\Delta_0^{\frac{1}{2}}b^{\frac{m}{2}}$
$\preccurlyeq 2\sum_{m=p}^{m=p+l-1}\varsigma^{l-m+p}\left(b^{\frac{m}{2}}\Delta_0^{\frac{1}{2}}\right)^*\left(b^{\frac{m}{2}}\Delta_0^{\frac{1}{2}}\right)$
$=2\sum_{m=p}^{m=p+l-1}\varsigma^{l-m+p}\left|b^{\frac{m}{2}}\Delta_0^{\frac{1}{2}}\right|^2 \le 2\left\|\sum_{m=p}^{m=p+l-1}\varsigma^{l-m+p}\left|b^{\frac{m}{2}}\Delta_0^{\frac{1}{2}}\right|^2\right\| 1_{\mathcal{A}}$
$\le 2\|\Delta_0\|\sum_{m=p}^{m=p+l-1}\|b\|^m\|\varsigma^{l-m+p}\|\, 1_{\mathcal{A}}$
$\le 2\|\Delta_0\|\frac{\|\varsigma\|^{l+1}\|b\|^m}{\|\varsigma\|-\|b\|} 1_{\mathcal{A}} \to 0$ as $p\to\infty$.
This implies that $\{\ae_p\}$, $\{\oe_p\}$ *and* $\{\mathfrak{s}_p\}$ are Cauchy sequences in $(\mathcal{L},\mathcal{A},\varrho_\varsigma)$ with respect to $\mathcal{A}$. Since $\mathcal{L}$ is complete, there exist $\mathfrak{x},\mathfrak{y},\mathfrak{z}\in\mathcal{L}$ such that $\ae_p\to\mathfrak{x}$, $\oe_p\to\mathfrak{y}$, $\mathfrak{s}_p\to\mathfrak{z}$ as $p\to\infty$. Now we show that $\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z})=\mathfrak{x}$, $\Gamma(\mathfrak{y},\mathfrak{z},\mathfrak{x})=\mathfrak{y}$, $\Gamma(\mathfrak{z},\mathfrak{x},\mathfrak{y})=\mathfrak{z}$. we suppose on contrary that $\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z})\neq\mathfrak{x}$, $\Gamma(\mathfrak{y},\mathfrak{z},\mathfrak{x})\neq\mathfrak{y}$, $\Gamma(\mathfrak{z},\mathfrak{x},\mathfrak{y})\neq\mathfrak{z}$, by Lemma (2.6), we have
$\frac{1}{2\varsigma}\ \varrho_\varsigma(\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\mathfrak{x}) \preccurlyeq \lim_{p\to\infty}\inf\ \varrho_\varsigma(\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\ae_{2p+2})$
$=\lim_{p\to\infty}\sup\ \varrho_\varsigma(\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\Omega(\ae_{2p+1},\oe_{2p+1},\mathfrak{s}_{2p+1}))$

$$\preccurlyeq \lim_{p\to\infty}\sup\frac{1}{4\varsigma^2}\left\{\begin{array}{c}\dfrac{a\{\varrho_\varsigma(\mathfrak{x},\mathfrak{x},\ae_{2p+1})+\varrho_\varsigma(\mathfrak{y},\mathfrak{y},\oe_{2p+1})+\varrho_\varsigma(\mathfrak{z},\mathfrak{z},\mathfrak{s}_{2p+1})\}}{3}\\ +\dfrac{a\left\{\begin{array}{c}\varrho_\varsigma(\mathfrak{x},\mathfrak{x},\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}))\ .\ \varrho_\varsigma(\ae_{2p+1},\ae_{2p+1},\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}))\\ +\ \varrho_\varsigma(\ae_{2p+1},\ae_{2p+1},\Omega(\ae_{2p+1},\oe_{2p+1},\mathfrak{s}_{2p+1})).\ \varrho_\varsigma(\mathfrak{x},\mathfrak{x},\Omega(\ae_{2p+1},\oe_{2p+1},\mathfrak{s}_{2p+1}))\end{array}\right\}}{1+\varrho_\varsigma(\mathfrak{x},\mathfrak{x},\Omega(\ae_{2p+1},\oe_{2p+1},\mathfrak{s}_{2p+1}))+\varrho_\varsigma(\ae_{2p+1},\ae_{2p+1},\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}))}\end{array}\right\}$$

$$\preccurlyeq \lim_{p\to\infty}\sup\frac{1}{4\varsigma^2}\left\{\begin{array}{c}\dfrac{a\{\varrho_\varsigma(\mathfrak{x},\mathfrak{x},\ae_{2p+1})+\varrho_\varsigma(\mathfrak{y},\mathfrak{y},\oe_{2p+1})+\varrho_\varsigma(\mathfrak{z},\mathfrak{z},\mathfrak{s}_{2p+1})\}}{3}\\ +\dfrac{a\left\{\begin{array}{c}\varrho_\varsigma(\mathfrak{x},\mathfrak{x},\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}))\ .\ \varrho_\varsigma(\ae_{2p+1},\ae_{2p+1},\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}))\\ +\ \varrho_\varsigma(\ae_{2p+1},\ae_{2p+1},\ae_{2p+2}).\ \varrho_\varsigma(\mathfrak{x},\mathfrak{x},\ae_{2p+2})\end{array}\right\}}{1+\varrho_\varsigma(\mathfrak{x},\mathfrak{x},\ae_{2p+2})+\varrho_\varsigma(\ae_{2p+1},\ae_{2p+1},\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}))}\end{array}\right\}$$

$\preccurlyeq \frac{1}{4\varsigma^2}2\varsigma a\varrho_\varsigma(\mathfrak{x},\mathfrak{x},\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z})) \preccurlyeq \frac{1}{2\varsigma}\varrho_\varsigma(\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\mathfrak{x})$
It follows that,

$$\|\varrho_\varsigma(\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\mathfrak{x})\| \preccurlyeq \|a\|\|\varrho_\varsigma(\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\mathfrak{x})\|$$

Since $\|a\|<\frac{1}{2}$, which implies that $\|\varrho_\varsigma(\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\mathfrak{x})\|=0$ and hence $\varrho_\varsigma(\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\mathfrak{x})=0_{\mathcal{A}}$, imply that $\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z})=\mathfrak{x}$. Similarly we can prove that $\Gamma(\mathfrak{y},\mathfrak{z},\mathfrak{x})=\mathfrak{y}$ and $\Gamma(\mathfrak{z},\mathfrak{x},\mathfrak{y})=\mathfrak{z}$. So, we have proved $(\mathfrak{x},\mathfrak{y},\mathfrak{z})$ is common tripled fixed point (CTFP) of Γ,Ω. Now we show Γ and Ω have unique common tripled fixed point. For this assume $(\mathfrak{x}',\mathfrak{y}',\mathfrak{z}')\in\mathcal{L}$ is another common tripled fixed point of Γ and Ω. Then
$0_{\mathcal{A}} \preccurlyeq \varrho_\varsigma(\mathfrak{x},\mathfrak{x},\mathfrak{x}') = \varrho_\varsigma(\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\Omega(\mathfrak{x}',\mathfrak{y}',\mathfrak{z}'))$
$\preccurlyeq 4\varsigma^2\varrho_\varsigma(\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}),\Omega(\mathfrak{x}',\mathfrak{y}',\mathfrak{z}'))$

$\preccurlyeq \frac{a\{\varrho_\varsigma(\mathfrak{x}, \mathfrak{x}, \mathfrak{x}') + \varrho_\varsigma(\mathfrak{y}, \mathfrak{y}, \mathfrak{y}') + \varrho_\varsigma(\mathfrak{z}, \mathfrak{z}, \mathfrak{z}')\}}{3}$
$+ \frac{a\left\{\begin{array}{c}\varrho_\varsigma(\mathfrak{x}, \mathfrak{x}, \Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z})) \cdot \varrho_\varsigma(\mathfrak{x}', \mathfrak{x}', \Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z})) \\ + \varrho_\varsigma(\mathfrak{x}', \mathfrak{x}', \Omega(\mathfrak{x}',\mathfrak{y}',\mathfrak{z}')). \varrho_\varsigma(\mathfrak{x}, \mathfrak{x}, \Omega(\mathfrak{x}',\mathfrak{y}',\mathfrak{z}'))\end{array}\right\}}{1 + \varrho_\varsigma(\mathfrak{x}, \mathfrak{x}, \Omega(\mathfrak{x}',\mathfrak{y}',\mathfrak{z}')) + \varrho_\varsigma(\mathfrak{x}', \mathfrak{x}', \Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}))}$
$\preccurlyeq \frac{a\{\varrho_\varsigma(\mathfrak{x}, \mathfrak{x}, \mathfrak{x}') + \varrho_\varsigma(\mathfrak{y}, \mathfrak{y}, \mathfrak{y}') + \varrho_\varsigma(\mathfrak{z}, \mathfrak{z}, \mathfrak{z}')\}}{3}$
Similarly,
$0_{\mathcal{A}} \preccurlyeq \varrho_\varsigma(\mathfrak{y}, \mathfrak{y}, \mathfrak{y}') \preccurlyeq \frac{a\{\varrho_\varsigma(\mathfrak{x}, \mathfrak{x}, \mathfrak{x}') + \varrho_\varsigma(\mathfrak{y}, \mathfrak{y}, \mathfrak{y}') + \varrho_\varsigma(\mathfrak{z}, \mathfrak{z}, \mathfrak{z}')\}}{3}$
And
$0_{\mathcal{A}} \preccurlyeq \varrho_\varsigma(\mathfrak{z}, \mathfrak{z}, \mathfrak{z}') \preccurlyeq \frac{a\{\varrho_\varsigma(\mathfrak{x}, \mathfrak{x}, \mathfrak{x}') + \varrho_\varsigma(\mathfrak{y}, \mathfrak{y}, \mathfrak{y}') + \varrho_\varsigma(\mathfrak{z}, \mathfrak{z}, \mathfrak{z}')\}}{3}$
Therefore,
$\|\varrho_\varsigma(\mathfrak{x}, \mathfrak{x}, \mathfrak{x}') + \varrho_\varsigma(\mathfrak{y}, \mathfrak{y}, \mathfrak{y}') + \varrho_\varsigma(\mathfrak{z}, \mathfrak{z}, \mathfrak{z}')\| < \|a\| \|\varrho_\varsigma(\mathfrak{x}, \mathfrak{x}, \mathfrak{x}') + \varrho_\varsigma(\mathfrak{y}, \mathfrak{y}, \mathfrak{y}') + \varrho_\varsigma(\mathfrak{z}, \mathfrak{z}, \mathfrak{z}')\|$
Since $\|a\| < \frac{1}{2}$, it is incongruous. Consequently, $\mathfrak{x} = \mathfrak{x}'$, $\mathfrak{y} = \mathfrak{y}'$, $\mathfrak{z} = \mathfrak{z}'$
Therefore, UCTFP of Γ and Ω is $(\mathfrak{x}, \mathfrak{y}, \mathfrak{z})$.

Theorem 3.1.2. Let $(\mathcal{L}, \mathcal{A}, \varrho_\varsigma)$ be a complete C^*-AV-S_b-MS with $\|\varsigma\| \geq 1$ .Suppose that the mapping $\Gamma: \mathcal{L}^3 \to \mathcal{L}$ satisfies for all $\ell, \varkappa, \mathfrak{z}, æ, œ, \mathfrak{s} \in \mathcal{L}$,

$$\varrho_\varsigma(\Gamma(\ell,\varkappa,\mathfrak{z}), \Gamma(\ell,\varkappa,\mathfrak{z}), \Gamma(æ,œ,\mathfrak{s})) \preccurlyeq a\left\{\begin{array}{c}\frac{[\varrho_\varsigma(\ell,\ell,æ)+\varrho_\varsigma(\varkappa,\varkappa,œ)+\varrho_\varsigma(\mathfrak{z},\mathfrak{z},\mathfrak{s})]}{3} \\ + \frac{\varrho_\varsigma(\Gamma(\ell,\varkappa,\mathfrak{z}), \Gamma(\ell,\varkappa,\mathfrak{z}), æ), \varrho_\varsigma(\Gamma(æ,œ,\mathfrak{s}), \Gamma(æ,œ,\mathfrak{s}), \ell)}{1+\varrho_\varsigma(\ell,\ell,æ)+\varrho_\varsigma(\varkappa,\varkappa,œ)+\varrho_\varsigma(\mathfrak{z},\mathfrak{z},\mathfrak{s})}\end{array}\right\} a^* \quad ..(3.1.5)$$

Where a$\in \mathcal{A}_+$ in which $\|a\| < \frac{1}{\sqrt{2}}$.Then Γ has a unique tripled fixed point in $\mathcal{L}$.
Proof: Let $æ_0, œ_0, \mathfrak{s}_0 \in \mathcal{L}$ be arbitrary, we construct the sequences $\{æ_p\}$, $\{œ_p\}$, $\{\mathfrak{s}_p\}$ in $\mathcal{L}$ as
$\Gamma(æ_p, œ_p, \mathfrak{s}_p) = æ_{p+1}$, $\Gamma(œ_p, \mathfrak{s}_p, æ_p) = œ_{p+1}$, $\Gamma(\mathfrak{s}_p, æ_p, œ_p) = \mathfrak{s}_{p+1}$ where p= 0,1,2….
In C^*-algebra, if a, b $\in \mathcal{A}$ and a$\preccurlyeq b$, then for any $j \in \mathcal{A}_+$ both j^*aj *and* j^*bj are positive elements and $j^*aj \preccurlyeq j^*bj$. By using (5.1.5), for each p$\in N$,we have
$\varrho_\varsigma(æ_p, æ_p, æ_{p+1}) = \varrho_\varsigma(\Gamma(æ_{p-1}, œ_{p-1}, \mathfrak{s}_{p-1}), \Gamma(æ_{p-1}, œ_{p-1}, \mathfrak{s}_{p-1}), \Gamma(æ_p, œ_p, \mathfrak{s}_p))$

$$\preccurlyeq a\left\{\begin{array}{c}\frac{\varrho_\varsigma(æ_{p-1},æ_{p-1},æ_p)+\varrho_\varsigma(œ_{p-1},œ_{p-1},œ_p)+\varrho_\varsigma(\mathfrak{s}_{p-1},\mathfrak{s}_{p-1},\mathfrak{s}_p)}{3} \\ + \frac{\varrho_\varsigma(\Gamma(æ_{p-1},æ_{p-1},\mathfrak{s}_{p-1}), \Gamma(\ell,\varkappa,\mathfrak{z}),æ), \varrho_\varsigma(\Gamma(æ_p,œ_p,\mathfrak{s}_p), \Gamma(æ_p,œ_p,\mathfrak{s}_p), æ_{p-1})}{1+\varrho_\varsigma(æ_{p-1},æ_{p-1},æ_p)+\varrho_\varsigma(œ_{p-1},œ_{p-1},œ_p)+\varrho_\varsigma(\mathfrak{s}_{p-1},\mathfrak{s}_{p-1},\mathfrak{s}_p)}\end{array}\right\} a^*$$

$$\preccurlyeq a\left\{\frac{\varrho_\varsigma(æ_{p-1},æ_{p-1},æ_p)+\varrho_\varsigma(œ_{p-1},œ_{p-1},œ_p)+\varrho_\varsigma(\mathfrak{s}_{p-1},\mathfrak{s}_{p-1},\mathfrak{s}_p)}{3}\right\} a^* \quad …(3.1.6)$$

Similarly, we can prove that
$\varrho_\varsigma(œ_p, œ_p, œ_{p+1}) \preccurlyeq a\left\{\frac{\varrho_\varsigma(æ_{p-1},æ_{p-1},æ_p)+\varrho_\varsigma(œ_{p-1},œ_{p-1},œ_p)+\varrho_\varsigma(\mathfrak{s}_{p-1},\mathfrak{s}_{p-1},\mathfrak{s}_p)}{3}\right\} a^*$
…. (3.1.7)
Also
$\varrho_\varsigma(\mathfrak{s}_p, \mathfrak{s}, \mathfrak{s}_{p+1}) \preccurlyeq a\left\{\frac{\varrho_\varsigma(æ_{p-1},æ_{p-1},æ_p)+\varrho_\varsigma(œ_{p-1},œ_{p-1},œ_p)+\varrho_\varsigma(\mathfrak{s}_{p-1},\mathfrak{s}_{p-1},\mathfrak{s}_p)}{3}\right\} a^*$ ……(3.1.8)
Adding (3.1.6), (3.1.7) and (3.1.8), we get
$\varrho_\varsigma(æ_p, æ_p, æ_{p+1}) + \varrho_\varsigma(œ_p, œ_p, œ_{p+1}) + \varrho_\varsigma(\mathfrak{s}_p, \mathfrak{s}, \mathfrak{s}_{p+1})$
$\preccurlyeq a\{\varrho_\varsigma(æ_{p-1}, æ_{p-1}, æ_p) + \varrho_\varsigma(œ_{p-1}, œ_{p-1}, œ_p) + \varrho_\varsigma(\mathfrak{s}_{p-1}, \mathfrak{s}_{p-1}, \mathfrak{s}_p)\} a^*$
Let $\Delta_p = \varrho_\varsigma(æ_p, æ_p, æ_{p+1}) + \varrho_\varsigma(œ_p, œ_p, œ_{p+1}) + \varrho_\varsigma(\mathfrak{s}_p, \mathfrak{s}, \mathfrak{s}_{p+1})$ then we have
$\Delta_p \preccurlyeq a\Delta_{p-1}a^* \preccurlyeq a^2\Delta_{p-1}(a^*)^2 \preccurlyeq \ldots\ldots\ldots\ldots a^p\Delta_{p-1}(a^*)^p$
Now letting $\Delta_0 \succ 0_{\mathcal{A}}$, for any p$\in N$ and $l \in N$, using condition (S_{b_3}) of definition 2.1 similar to the concept of Cauchy Sequences in Theorem 3.1.1, then consequently, we get that

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$\varrho_\varsigma(æ_p, æ_p, æ_{p+1}) + \varrho_\varsigma(œ_p, œ_p, œ_{p+1}) + \varrho_\varsigma(\mathfrak{s}_p, \mathfrak{s}, \mathfrak{s}_{p+1})$
$\preccurlyeq 2\varsigma\, \Delta_{p+l-1} + 2\varsigma^2\Delta_{p+l-2} + + \mathbf{2}\varsigma^3\Delta_{p+l-3} + \ldots\ldots\ldots\ldots + \mathbf{2}\varsigma^l\Delta_p$
$\preccurlyeq 2\varsigma \quad a^{p+l-1}\Delta_0(a^*)^{p+l-1} + \quad 2\varsigma^2\, a^{p+l-2}\Delta_0(a^*)^{p+l-2} + \mathbf{2}\varsigma^3\, a^{p+l-3}\Delta_0(a^*)^{p+l-3} + \ldots\ldots\ldots\ldots +$
$\mathbf{2}\varsigma^l\, a^p\Delta_0(a^*)^p$
$\preccurlyeq 2\sum_{m=p}^{m=p+l-1} \varsigma^{l-m+p}\, a^m {\Delta_0}^{\frac{1}{2}}{\Delta_0}^{\frac{1}{2}}(a^*)^m$
$\preccurlyeq 2\sum_{m=p}^{m=p+l-1} \varsigma^{l-m+p}\left(a^m{\Delta_0}^{\frac{1}{2}}\right)\left(a^m{\Delta_0}^{\frac{1}{2}}\right)^*$
$=2\sum_{m=p}^{m=p+l-1} \varsigma^{l-m+p}\left|a^m{\Delta_0}^{\frac{1}{2}}\right|^2 \leq 2\left\|\sum_{m=p}^{m=p+l-1} \varsigma^{l-m+p}\left|a^m{\Delta_0}^{\frac{1}{2}}\right|^2\right\| 1_{\mathcal{A}}$
$\leq 2\|\Delta_0\|\sum_{m=p}^{m=p+l-1}\|a\|^{2m}\|\varsigma^{l-m+p}\|\, 1_{\mathcal{A}}$
$\leq 2\|\Delta_0\|\frac{\|\varsigma\|^{l+1}\|a\|^{2p}}{\|\varsigma\|-\|a\|}1_{\mathcal{A}} \to 0$ as $p \to \infty$.

This implies that $\{æ_p\}$, $\{œ_p\}$ *and* $\{\mathfrak{s}_p\}$ are Cauchy sequences in $(\mathcal{L}, \mathcal{A}, \varrho_\varsigma)$ with respect to $\mathcal{A}$ .Since $\mathcal{L}$ is complete, there exist $\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^* \in \mathcal{L}$ such that $æ_p \to \mathfrak{x}^*$, $œ_p \to \mathfrak{y}^*$, $\mathfrak{s}_p \to \mathfrak{z}^*$ as $p \to \infty$. Thus, from Lemma (2.6), we have

$\frac{1}{2\varsigma}\, \varrho_\varsigma(\Gamma(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^*), \Gamma(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^*), \mathfrak{x}^*) \preccurlyeq \lim_{p\to\infty} inf\, \varrho_\varsigma(\Gamma(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^*), \Gamma(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^*), æ_{p+1})$
$= \lim_{p\to\infty} inf\, \varrho_\varsigma((\Gamma(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^*), \Gamma(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^*), \Gamma(æ_p, œ_p, \mathfrak{s}_p))$

$$\preccurlyeq \limsup_{p\to\infty} \mathbf{a}\left\{\begin{array}{c}\frac{a\{\varrho_\varsigma(\mathfrak{x}^*, \mathfrak{x}^*, æ_p) + \varrho_\varsigma(\mathfrak{y}^*, \mathfrak{y}^*, œ_p) + \varrho_\varsigma(\mathfrak{z}^*, \mathfrak{z}^*, \mathfrak{s}_p)\}}{3} \\ + \frac{\{\varrho_\varsigma(\Gamma(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^*), \Gamma(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^*), æ_p).\ \varrho_\varsigma(\Gamma(æ_p, œ_p, \mathfrak{s}_p), (\Gamma(æ_p, œ_p, \mathfrak{s}_p), \mathfrak{x}^*)\}}{1 + \varrho_\varsigma(\mathfrak{x}^*, \mathfrak{x}^*, æ_p) + \varrho_\varsigma(\mathfrak{y}^*, \mathfrak{y}^*, œ_p) + \varrho_\varsigma(\mathfrak{z}^*, \mathfrak{z}^*, \mathfrak{s}_p)}\end{array}\right\} a^* = \mathbf{0}_{\mathcal{A}}$$

Hence, we obtain that $\varrho_\varsigma(\Gamma(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^*), \Gamma(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^*), \mathfrak{x}^*) = \mathbf{0}_{\mathcal{A}}$ .imply that $\Gamma(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^*) = \mathfrak{x}^*$. similarly, we can prove $\Gamma(\mathfrak{y}^*, \mathfrak{z}^*, \mathfrak{x}^*) = \mathfrak{y}^*$ and $\Gamma(\mathfrak{z}^*, \mathfrak{x}^*, \mathfrak{y}^*) = \mathfrak{z}^*$. So, we have proved $(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^*)$ is tripled fixed point (TFP) of Γ .Now $(\mathfrak{x}', \mathfrak{y}', \mathfrak{z}') \in \mathcal{L}$ is another common tripled fixed point of Γ and Ω .Then

$0_{\mathcal{A}} \preccurlyeq \varrho_\varsigma(\mathfrak{x}^*, \mathfrak{x}^*, \mathfrak{x}') = \varrho_\varsigma(\Gamma(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^*), \Gamma(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^*), \Omega(\mathfrak{x}', \mathfrak{y}', \mathfrak{z}'))$

$$\preccurlyeq \mathrm{a}\left\{\begin{array}{c}\frac{\varrho_\varsigma(\mathfrak{x}^*, \mathfrak{x}^*, \mathfrak{x}') + \varrho_\varsigma(\mathfrak{y}^*, \mathfrak{y}^*, \mathfrak{y}') + \varrho_\varsigma(\mathfrak{z}^*, \mathfrak{z}^*, \mathfrak{z}')}{3} \\ + \frac{\varrho_\varsigma(\Gamma(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^*), \Gamma(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^*), \mathfrak{x}').\varrho_\varsigma(\Gamma(\mathfrak{x}', \mathfrak{y}', \mathfrak{z}'), \Gamma(\mathfrak{x}', \mathfrak{y}', \mathfrak{z}'), \mathfrak{x}^*)}{1 + \varrho_\varsigma(\mathfrak{x}^*, \mathfrak{x}^*, \mathfrak{x}') + \varrho_\varsigma(\mathfrak{y}^*, \mathfrak{y}^*, \mathfrak{y}') + \varrho_\varsigma(\mathfrak{z}^*, \mathfrak{z}^*, \mathfrak{z}')}\end{array}\right\} a^*$$

$\preccurlyeq \frac{a\{\varrho_\varsigma(\mathfrak{x}^*, \mathfrak{x}^*, \mathfrak{x}') + \varrho_\varsigma(\mathfrak{y}^*, \mathfrak{y}^*, \mathfrak{y}') + \varrho_\varsigma(\mathfrak{z}^*, \mathfrak{z}^*, \mathfrak{z}')\}a^*}{3} + \mathrm{a}\varrho_\varsigma(\mathfrak{x}^*, \mathfrak{x}^*, \mathfrak{x}')a^*$

Similarly,

$0_{\mathcal{A}} \preccurlyeq \varrho_\varsigma(\mathfrak{y}^*, \mathfrak{y}^*, \mathfrak{y}') \preccurlyeq \frac{a\{\varrho_\varsigma(\mathfrak{x}^*, \mathfrak{x}^*, \mathfrak{x}') + \varrho_\varsigma(\mathfrak{y}^*, \mathfrak{y}^*, \mathfrak{y}') + \varrho_\varsigma(\mathfrak{z}^*, \mathfrak{z}^*, \mathfrak{z}')\}}{3} + \mathrm{a}\varrho_\varsigma(\mathfrak{y}^*, \mathfrak{y}^*, \mathfrak{y}')a^*$

And

$0_{\mathcal{A}} \preccurlyeq \varrho_\varsigma(\mathfrak{z}^*, \mathfrak{z}^*, \mathfrak{z}') \preccurlyeq \frac{a\{\varrho_\varsigma(\mathfrak{x}^*, \mathfrak{x}^*, \mathfrak{x}') + \varrho_\varsigma(\mathfrak{y}^*, \mathfrak{y}^*, \mathfrak{y}') + \varrho_\varsigma(\mathfrak{z}^*, \mathfrak{z}^*, \mathfrak{z}')\}}{3} + \mathrm{a}\varrho_\varsigma(\mathfrak{z}^*, \mathfrak{z}^*, \mathfrak{z}')a^*$

Therefore,

$0 \leq \|\varrho_\varsigma(\mathfrak{x}^*, \mathfrak{x}^*, \mathfrak{x}') + \varrho_\varsigma(\mathfrak{y}^*, \mathfrak{y}^*, \mathfrak{y}') + \varrho_\varsigma(\mathfrak{z}^*, \mathfrak{z}^*, \mathfrak{z}')\|$
$\leq \|a\|^2(1 - \|a\|^2)^{-1}\|\varrho_\varsigma(\mathfrak{x}^*, \mathfrak{x}^*, \mathfrak{x}') + \varrho_\varsigma(\mathfrak{y}^*, \mathfrak{y}^*, \mathfrak{y}') + \varrho_\varsigma(\mathfrak{z}^*, \mathfrak{z}^*, \mathfrak{z}')\|$

Since $\|a\| < \frac{1}{\sqrt{2}}$, so that $(1 - \|a\|^2)^{-1} < 1$, it is incongruous. Consequently, $\mathfrak{x}^* = \mathfrak{x}'$, $\mathfrak{y}^* = \mathfrak{y}'$, $\mathfrak{z}^* = \mathfrak{z}'$

Therefore, UCTFP of Γ is $(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^*)$.

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Theorem 3.1.3: Let $(\mathcal{L},\mathcal{A},\varrho_\varsigma)$ be a complete C^*-AV-S_b-MS with $\|\varsigma\| \geq 1$ .Suppose that the mapping Γ: $\mathcal{L}^3 \to \mathcal{L}$ satisfies for all $\ell,\varkappa,\mathfrak{z},æ,œ,\mathfrak{s} \in \mathcal{L}$, $\varrho_\varsigma(\Gamma(\ell,\varkappa,\mathfrak{z}),\Gamma(\ell,\varkappa,\mathfrak{z}),\Gamma(æ,œ,\mathfrak{s})) \preccurlyeq$

$$a\left\{\begin{array}{c}\varrho_\varsigma(\Gamma(\ell,\varkappa,\mathfrak{z}),\ \Gamma(\ell,\varkappa,\mathfrak{z}),\ell)\\ +\varrho_\varsigma(\ \Gamma(æ,œ,\mathfrak{s}),\Gamma(æ,œ,\mathfrak{s}),æ\ )\\ \dfrac{\varrho_\varsigma(\Gamma(\ell,\varkappa,\mathfrak{z}),\ \Gamma(\ell,\varkappa,\mathfrak{z}),æ).\varrho_\varsigma(\ \Gamma(æ,œ,\mathfrak{s}),\Gamma(æ,œ,\mathfrak{s}),\ell\ )}{1+\left\{\begin{array}{c}\varrho_\varsigma(\Gamma(\ell,\varkappa,\mathfrak{z}),\ \Gamma(\ell,\varkappa,\mathfrak{z}),æ)\\ +\varrho_\varsigma(\ \Gamma(æ,œ,\mathfrak{s}),\Gamma(æ,œ,\mathfrak{s}),\ell\ )\\ +\varrho_\varsigma(\ell,\ell,æ)+\varrho_\varsigma(\varkappa,\varkappa,œ)\\ +\varrho_\varsigma(\mathfrak{z},\mathfrak{z},\mathfrak{s})\end{array}\right\}}\end{array}\right\} \ldots\ldots\ldots\ldots(3.1.9)$$

Where a$\in \mathcal{A}_+$ in which $\|a\| < \frac{1}{2}$.Then Γ has a unique tripled fixed point in $\mathcal{L}$.

Proof: Let $æ_0, œ_0, \mathfrak{s}_0 \in \mathcal{L}$ be arbitrary, we construct the sequences $\{æ_p\}$, $\{œ_p\}, \{\mathfrak{s}_p\}$ in $\mathcal{L}$ as $\Gamma\ (æ_p, œ_p, \mathfrak{s}_p)= æ_{p+1}$, $\Gamma\ (œ_p, \mathfrak{s}_p, æ_p)= œ_{p+1}$, $\Gamma\ (\mathfrak{s}_p, æ_p, œ_p)= \mathfrak{s}_{p+1}$ where p= 0,1,2….

From (5.1.9), for each p$\in N$,we have

$$\varrho_\varsigma(æ_p, æ_p, æ_{p+1})= \varrho_\varsigma(\Gamma\ (æ_{p-1}, œ_{p-1}, \mathfrak{s}_{p-1}), \Gamma\ (æ_{p-1}, œ_{p-1}, \mathfrak{s}_{p-1}), \Gamma\ (æ_p, œ_p, \mathfrak{s}_p))$$
$$\leq 4\varsigma^2 \varrho_\varsigma(\Gamma\ (æ_{p-1}, œ_{p-1}, \mathfrak{s}_{p-1}), \Gamma\ (æ_{p-1}, œ_{p-1}, \mathfrak{s}_{p-1}), \Gamma\ (æ_p, œ_p, \mathfrak{s}_p))$$

$$\preccurlyeq a\left\{\begin{array}{c}\varrho_\varsigma(\Gamma(æ_{p-1},œ_{p-1},\mathfrak{s}_{p-1}),\ \Gamma(æ_{p-1},œ_{p-1},\mathfrak{s}_{p-1}),æ_{p-1})\\ +\varrho_\varsigma(\ \Gamma(æ_p,œ_p,\mathfrak{s}_p),\Gamma(æ_p,œ_p,\mathfrak{s}_p),æ_p\ )\\ +\dfrac{\varrho_\varsigma(\Gamma(æ_{p-1},œ_{p-1},\mathfrak{s}_{p-1}),\ \Gamma(æ_{p-1},œ_{p-1},\mathfrak{s}_{p-1}),æ_p).\varrho_\varsigma(\ \Gamma(æ_p,œ_p,\mathfrak{s}_p),\Gamma(æ_p,œ_p,\mathfrak{s}_p),æ_{p-1}\ )}{1+\left\{\begin{array}{c}\varrho_\varsigma(\Gamma(æ_{p-1},œ_{p-1},\mathfrak{s}_{p-1}),\ \Gamma(æ_{p-1},œ_{p-1},\mathfrak{s}_{p-1}),æ_p)\\ +\varrho_\varsigma(\ \Gamma(æ_p,œ_p,\mathfrak{s}_p),\Gamma(æ_p,œ_p,\mathfrak{s}_p),æ_{p-1}\ )\\ +\varrho_\varsigma(æ_{p-1},æ_{p-1},æ_p)+\varrho_\varsigma(œ_{p-1},œ_{p-1},œ_p)\\ +\varrho_\varsigma(\mathfrak{s}_{p-1},\mathfrak{s}_{p-1},\mathfrak{s}_p\ )\end{array}\right\}}\end{array}\right\}$$

$$\preccurlyeq a\big(\varrho_\varsigma(æ_p, æ_p, æ_{p-1}) + \varrho_\varsigma(æ_{p+1}, æ_{p+1}, æ_p)\big)$$

Thus, $\varrho_\varsigma(æ_{p+1}, æ_{p+1}, æ_p) \leq a(1_{\mathcal{A}} - a\ )^{-1}\varrho_\varsigma(æ_p, æ_p, æ_{p-1})$

Since b= $a(1_{\mathcal{A}} - a\ )^{-1}$ and by using Lemma 2.5, we have a$\in \mathcal{A}'_+$ with $\|a\| < \frac{1}{2}$ one has $(1_{\mathcal{A}} - a\ )^{-1} \in \mathcal{A}'_+$ and $a(1_{\mathcal{A}} - a\ )^{-1} \in \mathcal{A}'_+$ with$\|a(1_{\mathcal{A}} - a\ )^{-1}\| < 1$ so that $\|b\| < 1$. Hence $\varrho_\varsigma(æ_{p+1}, æ_{p+1}, æ_p) \leq$ b $\varrho_\varsigma(æ_p, æ_p, æ_{p-1})$ .Similarly ,we have $\varrho_\varsigma(œ_{p+1}, œ_{p+1}, œ_p) \leq$ b $\varrho_\varsigma(œ_p, œ_p, œ_{p-1})$ and $\varrho_\varsigma(\mathfrak{s}_{p+1}, \mathfrak{s}_{p+1}, \mathfrak{s}_p) \leq$ b $\varrho_\varsigma(\mathfrak{s}_p, \mathfrak{s}_p, \mathfrak{s}_{p-1})$.

Which implies that $\{æ_p\}$, $\{œ_p\}$ *and* $\{\mathfrak{s}_p\}$ are Cauchy sequences in $(\mathcal{L},\mathcal{A},\varrho_\varsigma)$ with respect to $\mathcal{A}$, therefore by completeness of $\mathcal{L}$ there exist $\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{z}^* \in \mathcal{L}$ such that $\lim_{p\to\infty} æ_p=\mathfrak{x}^*$, $\lim_{p\to\infty} œ_p=\mathfrak{y}^*$, $\lim_{p\to\infty} \mathfrak{s}_p=\mathfrak{z}^*$.

$$\frac{1}{2\varsigma}\ \varrho_\varsigma(\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\mathfrak{x}^*\ ) \preccurlyeq \lim_{p\to\infty} inf\ \varrho_\varsigma(\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),æ_{p+1}\ )$$
$$=\lim_{p\to\infty} inf\, \varrho_\varsigma((\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(æ_p, œ_p,\mathfrak{s}_p)\ )$$

$$\preccurlyeq \mathbf{\limsup_{p\to\infty}\frac{1}{4\varsigma^2}\ a}\left\{\begin{array}{c}\varrho_\varsigma(\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\mathfrak{x}^*)\\ +\varrho_\varsigma(\ \Gamma(æ_p,œ_p,\mathfrak{s}_p),\Gamma(æ_p,œ_p,\mathfrak{s}_p),æ_p\ )\\ +\dfrac{\varrho_\varsigma(\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),æ_p).\varrho_\varsigma(\ \Gamma(æ_p,œ_p,\mathfrak{s}_p),\Gamma(æ_p,œ_p,\mathfrak{s}_p),\mathfrak{x}^*\ )}{1+\left\{\begin{array}{c}\varrho_\varsigma(\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),æ_p)\\ +\varrho_\varsigma(\ \Gamma(æ_p,œ_p,\mathfrak{s}_p),\Gamma(æ_p,œ_p,\mathfrak{s}_p),\mathfrak{x}^*\ )\\ +\varrho_\varsigma(\mathfrak{x}^*,\mathfrak{x}^*,æ_p)+\varrho_\varsigma(\mathfrak{y}^*,\mathfrak{y}^*,œ_p)\\ +\varrho_\varsigma(\mathfrak{z}^*,\mathfrak{z}^*,\mathfrak{s}_p\ )\end{array}\right\}}\end{array}\right\}$$

$$\preccurlyeq \frac{1}{2\varsigma}a\varrho_\varsigma(\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*)\ ,\mathfrak{x}^*)$$

It follows that

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$\left\|\varrho_\varsigma(\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\mathfrak{x}^*)\right\| \leq \|a\|\left\|\varrho_\varsigma(\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\mathfrak{x}^*)\right\|$

Since $\|a\| < \frac{1}{2}$,which implies that $\left\|\varrho_\varsigma(\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\mathfrak{x}^*)\right\| = 0$ and hence $\varrho_\varsigma(\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\mathfrak{x}^*)=0_{\mathcal{A}}$,imply that $\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*) = \mathfrak{x}^*$ .Similarly we can prove that $\Gamma(\mathfrak{y}^*,\mathfrak{z}^*,\mathfrak{x}^*) = \mathfrak{y}^*$ and $\Gamma(\mathfrak{z}^*,\mathfrak{x}^*,\mathfrak{y}^*) = \mathfrak{z}^*$ .Hence $(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*)$ is the tripled fixed point of Γ .

Now if $(\mathfrak{x}',\mathfrak{y}',\mathfrak{z}') \in \mathcal{L}$ is another common tripled fixed point of Γ .Then by applying Eq. (3.1.9)

$0_{\mathcal{A}} \preccurlyeq \varrho_\varsigma(\mathfrak{x}^*,\mathfrak{x}^*,\mathfrak{x}') = \varrho_\varsigma(\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(\mathfrak{x}',\mathfrak{y}',\mathfrak{z}'))$

$\preccurlyeq \mathbf{4}\varsigma^{\mathbf{2}}\varrho_\varsigma(\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(\mathfrak{x}',\mathfrak{y}',\mathfrak{z}'))$

$$\preccurlyeq a\left\{\begin{array}{c}\varrho_\varsigma(\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\mathfrak{x}^*)\\ +\varrho_\varsigma(\Gamma(\mathfrak{x}',\mathfrak{y}',\mathfrak{z}'),\Gamma(\mathfrak{x}',\mathfrak{y}',\mathfrak{z}'),\mathfrak{x}')\\ +\dfrac{\varrho_\varsigma(\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\mathfrak{x}').\varrho_\varsigma(\Gamma(\mathfrak{x}',\mathfrak{y}',\mathfrak{z}'),\Gamma(\mathfrak{x}',\mathfrak{y}',\mathfrak{z}'),\mathfrak{x}^*)}{1+\left\{\begin{array}{c}\varrho_\varsigma(\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\Gamma(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*),\mathfrak{x}')\\ +\varrho_\varsigma(\Gamma(\mathfrak{x}',\mathfrak{y}',\mathfrak{z}'),\Gamma(\mathfrak{x}',\mathfrak{y}',\mathfrak{z}'),\mathfrak{x}^*)\\ +\varrho_\varsigma(\mathfrak{x}^*,\mathfrak{x}^*,\mathfrak{x}')+\varrho_\varsigma(\mathfrak{y}^*,\mathfrak{y}^*,\mathfrak{y}')\\ +\varrho_\varsigma(\mathfrak{z}^*,\mathfrak{z}^*,\mathfrak{z}')\end{array}\right\}}\end{array}\right\}$$

$\preccurlyeq a\varrho_\varsigma(\mathfrak{x}^*,\mathfrak{x}^*,\mathfrak{x}')$

It follows that $0 \leq \left\|\varrho_\varsigma(\mathfrak{x}^*,\mathfrak{x}^*,\mathfrak{x}')\right\| \leq \|a\|\left\|\varrho_\varsigma(\mathfrak{x}^*,\mathfrak{x}^*,\mathfrak{x}')\right\|$. since $\|a\| < \frac{1}{2}$, which implies that $\left\|\varrho_\varsigma(\mathfrak{x}^*,\mathfrak{x}^*,\mathfrak{x}')\right\| = 0$ and hence $\mathfrak{x}^* = \mathfrak{x}'$ .similarly , we prove $\mathfrak{y}^* = \mathfrak{y}'$ and $\mathfrak{z}^* = \mathfrak{z}'$. Therefore, the UTFP of Γ is $(\mathfrak{x}^*,\mathfrak{y}^*,\mathfrak{z}^*)$.

Corollary 3.1.4: Let $(\mathcal{L},\mathcal{A},\varrho_\varsigma)$ be a complete C^*-AV-S_b-MS with $\|\varsigma\| \geq 1$ .Suppose that the mapping $\Gamma: \mathcal{L}^3 \to \mathcal{L}$ satisfies for all $\ell, \varkappa, \mathfrak{z}, æ, œ, \mathfrak{s} \in \mathcal{L}$,

$\varrho_\varsigma(\Gamma(\ell,\varkappa,\mathfrak{z}),\Gamma(\ell,\varkappa,\mathfrak{z}),\Gamma(æ,œ,\mathfrak{s})) \preccurlyeq a\left\{\frac{\varrho_\varsigma(\ell,\ell,æ)+\varrho_\varsigma(\varkappa,\varkappa,œ)+\varrho_\varsigma(\mathfrak{z},\mathfrak{z},\mathfrak{s})}{3}\right\}a^*$

Where $a \in \mathcal{A}_+$ in which $\|a\| < 1$ .Then Γ has a unique tripled fixed point in $\mathcal{L}$.

Proof: Following similar steps as in the proof of Theorem (3.1.2), we obtain the proof of our corollary.

Example 3.1: Let $\mathcal{L} = \mathbb{R}$ and $\mathcal{A}=M_2(C)$ be a all 2x2 complex matrices whose norm is $\|\mathcal{A}\|$=max $\{|\lambda_1|,|\lambda_2|,|\lambda_3|,|\lambda_4|\}$ where $\lambda_i's$ are the Eigen values of $\mathcal{A}$ .Then ,clearly $(\mathcal{L},\mathcal{A},\varrho_\varsigma)$ is a complete C^*-AV-S_b-MS with $\|\varsigma\| \geq 2$ whenever $\varrho_\varsigma: \mathcal{L}^3 \to \mathcal{A}$ be as $\varrho_\varsigma(p,q,r)=(p^2+q^2+r^2)$A where A=$\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix} \in \mathcal{A}_+$ self -adjoint . we define mappings Γ , Ω : $\mathcal{L}^3 \to \mathcal{L}$ as follows $\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}) = \frac{\mathfrak{x}}{8}+\frac{\mathfrak{y}}{10}+\frac{\mathfrak{z}}{20}$, $\Omega(æ,œ,\mathfrak{s}) = \frac{æ}{10}+\frac{œ}{50}+\frac{\mathfrak{s}}{40}$ for all $\mathfrak{x},\mathfrak{y},\mathfrak{z},æ,œ,\mathfrak{s} \in \mathcal{L}$. By taking a$\in \mathcal{A}$ with $\|a\| < \frac{1}{2}$, it can be easily verified that inequality (3.1.1) holds for all $\mathfrak{x},\mathfrak{y},\mathfrak{z},æ,œ,\mathfrak{s} \in \mathcal{L}$.

Hence, all the conditions of Theorem 5.1.1 are satisfied. Therefore, the mappings Γ and Ω admit a unique common tripled fixed point (UCTFP), namely $(\mathfrak{x},\mathfrak{x},\mathfrak{x})=(0_{\mathcal{A}},0_{\mathcal{A}},0_{\mathcal{A}})$ in $\mathcal{L}$.

Example 3.2: Let $\mathcal{L} = \mathbb{R}$ and $\mathcal{A}=\mathbb{R}$ so $\mathcal{A}_+ = [0,\infty)$. Define $\varrho_\varsigma: \mathcal{L}^3 \to \mathcal{A}$ as $\varrho_\varsigma(p,q,r)=$ $|p-q|+|q-r|$ for all p, q, r$\in \mathbb{R}$ then, clearly $(\mathcal{L},\mathcal{A},\varrho_\varsigma)$ is a complete C^*-AV-S_b-MS with $\varsigma = 1$ so $\|\varsigma\| = 1 \geq 1$ .Let $\Gamma: \mathcal{L}^3 \to \mathcal{L}$ as follows $\Gamma(\mathfrak{x},\mathfrak{y},\mathfrak{z}) = \frac{\mathfrak{x}+\mathfrak{y}+\mathfrak{z}}{6}$ for all $\mathfrak{x},\mathfrak{y},\mathfrak{z} \in \mathcal{L}$ .By taking a=$\frac{1}{2}$ $<\frac{1}{\sqrt{2}}$, it can easily verify that inequality (3.1.5) holds for all $\mathfrak{x},\mathfrak{y},\mathfrak{z} \in \mathcal{L}$ .Hence , all the conditions of Theorem 5.1.2 are satisfied. Thus, Γ has a unique tripled fixed point in $\mathbb{R}$, which is $(\mathfrak{x},\mathfrak{x},\mathfrak{x})=(0,0,0)$.

3.2. Application to Integral Equations

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As an application to Corollary 3.1.4, we examine the presence of a singular solution to an integral equation in this section.

Theorem 3.1.5: Let us examine the following integral equation.

$æ(t) = \int_{\varepsilon} (\mathcal{S}(r,t,s) + \mathcal{T}(r,t,s))$ (𝔣 (t, s, $æ(s)$) + 𝔤 (t, s, $æ(s)$) + 𝔥 (r, t, $æ(s)$))ds +k(t) ∀ r , s , $t \in \varepsilon$ .... (5.1.10)

where ϵ is a Lebesgue measurable set and m $(\varepsilon)<\infty$. Let $\varepsilon = [0,1]$ *and* denote $\mathcal{L} = L^{\infty}(\varepsilon)$ be the space of essentially bounded measurable functions on ε. we consider the following assumptions:

(i_0) 𝔣 , 𝔤, 𝔥 : $\varepsilon \times \varepsilon \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{S}, \mathcal{T}$: $\varepsilon \times \varepsilon \times \varepsilon \to \mathbb{R}$ are integrable, and k$\in L^{\infty}(\varepsilon)$.

(i_1) there is $\theta \in (0,1)$ such that $i, j \in \varepsilon$ and $æ(i), æ(j) \in \mathbb{R}$,

$$0 \le \left|\mathfrak{f}\left(\mathrm{i,j},æ(j)\right) - \mathfrak{f}\left(\mathrm{i,j},\varkappa(j)\right)\right| \le \left(\frac{2^{1-p\theta}}{3}\right)^{\frac{1}{p}} |æ(j) - \varkappa(j)|\,,$$

$$-\left(\frac{2^{1-p\theta}}{3}\right)^{\frac{1}{p}} |æ(j) - \varkappa(j)| \le \left|\mathfrak{g}\left(\mathrm{i,j},æ(j)\right) - \mathfrak{g}\left(\mathrm{i,j},\varkappa(j)\right)\right| \le 0,$$

$$0 \le \left|\mathfrak{h}\left(\mathrm{i,j},æ(j)\right) - \mathfrak{h}\left(\mathrm{i,j},\varkappa(j)\right)\right| \le \left(\frac{2^{1-p\theta}}{3}\right)^{\frac{1}{p}} |æ(j) - \varkappa(j)|\,.$$

(i_2) $sup_{t,s\in\varepsilon} \int_{\varepsilon} (\mathcal{S}(r,t,s) + \mathcal{T}(r,t,s))ds\le 1$.

Thus, the integral equation admits a unique solution in $L^{\infty}(\varepsilon)$.

Proof: Let $\varepsilon = [0,1]$ *and* Let $\mathcal{L} = L^{\infty}(\varepsilon)$ denote the space of essentially bounded measurable functions on ε and $\mathcal{H} = L^2(\varepsilon)$ .The set of bounded linear operators on Hilbert space $\mathcal{H}$ denoted by L $(\mathcal{H})$. we equip $\mathcal{L}$ with $\varrho_{\varsigma}: \mathcal{L}^3 \to \mathrm{L}(\mathcal{H})$,which is ascertained by $\varrho_{\varsigma}(æ, \mathfrak{B}, \boldsymbol{\alpha}) = M_{(|æ-\boldsymbol{\alpha}|+|\mathfrak{B}-\boldsymbol{\alpha}|)^p}$,where $M_{(|æ-\boldsymbol{\alpha}|+|\mathfrak{B}-\boldsymbol{\alpha}|)^p}$ is the multiplication operator on $\mathcal{H}$ ascertained by $M_{\phi}(\propto) = \phi \propto$, $\propto\in \mathcal{H}$ .Therefore , $(\mathcal{L}, \mathrm{L}(\mathcal{H}), \varrho_{\varsigma})$ is a complete C^*-AV-S_bMS with ς $=2^{2(p-1)}$ where p=2>1.

Define $\Gamma: \mathcal{L} \times \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ be as ∀ r , s , $t \in \varepsilon$, $æ, \varkappa, \mathfrak{B} \in \mathcal{L}$,

$$\Gamma(æ, \varkappa, \mathfrak{B}) = \int_{\varepsilon} (\mathcal{S}(r,t,s) + \mathcal{T}(r,t,s)) \left(\mathfrak{f}\,(\mathrm{t,s},æ(s)) + \mathfrak{g}\,(\mathrm{t,s},\varkappa(s)) + \mathfrak{h}\,(\mathrm{r,t},\mathfrak{B}(s))\right)\mathrm{d}s + \mathrm{k}(t)$$

We have $\varrho_{\varsigma}(\Gamma(\ell,\varkappa,\mathfrak{z}), \Gamma(\ell,\varkappa,\mathfrak{z}), \Gamma(æ,œ,\mathfrak{B})) = M_{(2|\Gamma(\ell,\varkappa,\mathfrak{z})-\Gamma(æ,œ,\mathfrak{B})|)^p}$

Let us first evaluate the following expression:

$$2^p|\Gamma(\ell,\varkappa,\mathfrak{z}) - \Gamma(æ,œ,\mathfrak{B})(t)|^p = 2^p \left|\int_{\varepsilon} (\mathcal{S}(r,t,s) + \mathcal{T}(r,t,s)) \left(\mathfrak{f}\left(\mathrm{t,s},\ell(s)\right) + \mathfrak{g}\left(\mathrm{t,s},\varkappa(s)\right) + \mathfrak{h}\left(\mathrm{r,t},\mathfrak{z}(s)\right)\right) - \int_{\varepsilon} (\mathcal{S}(r,t,s) + \mathcal{T}(r,t,s)) \left(\mathfrak{f}\,(\mathrm{t,s},æ(s)) + \mathfrak{g}\,(\mathrm{t,s},œ(s)) + \mathfrak{h}\,(\mathrm{r,t},\mathfrak{B}(s))\right)\mathrm{d}s\right|^p$$

$$= 2^p \left|\int_{\varepsilon} (\mathcal{S}(r,t,s) + \mathcal{T}(r,t,s)) \begin{pmatrix} \mathfrak{f}\left(\mathrm{t,s},\ell(s)\right) - \mathfrak{f}\left(\mathrm{t,s},æ(s)\right) \\ +\mathfrak{g}\left(\mathrm{t,s},\varkappa(s)\right) - \mathfrak{g}\left(\mathrm{t,s},œ(s)\right) \\ +\mathfrak{h}\left(\mathrm{r,t},\mathfrak{z}(s)\right) - \mathfrak{h}\,(\mathrm{r,t},\mathfrak{B}(s)) \end{pmatrix} ds\right|^p$$

$$\le 2^p 2^{p-1} \left(\int_{\varepsilon} \left(\frac{2^{1-p\theta}}{3}\right)^{\frac{1}{p}} |(\mathcal{S}(r,t,s) + \mathcal{T}(r,t,s))|\, ds\right)^p \begin{bmatrix} |\ell(s) - æ(s)|^p \\ +|\varkappa(s) - œ(s)|^p \\ +|\mathfrak{z}(s) - \mathfrak{B}(s)|^p \end{bmatrix}$$

$$\le \frac{\theta}{3} \left(sup_{t,s\in\varepsilon} \int_{\varepsilon} \left(\frac{2^{1-p\theta}}{3}\right)^{\frac{1}{p}} |(\mathcal{S}(r,t,s) + \mathcal{T}(r,t,s))|\, ds\right)^p \begin{bmatrix} (2\|\ell(s) - æ(s)\|_{\infty})^p \\ +(2\|\varkappa(s) - œ(s)\|_{\infty})^p \\ +(2\|\mathfrak{z}(s) - \mathfrak{B}(s)\|_{\infty})^p \end{bmatrix}$$

$$\le \frac{\theta}{3} \begin{bmatrix} (2\|\ell(s) - æ(s)\|_{\infty})^p \\ +(2\|\varkappa(s) - œ(s)\|_{\infty})^p \\ +(2\|\mathfrak{z}(s) - \mathfrak{B}(s)\|_{\infty})^p \end{bmatrix}$$

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Therefore,
$\| \varrho_\varsigma(\Gamma(\ell,\varkappa,\mathfrak{z}) ,\Gamma(\ell,\varkappa,\mathfrak{z}),\Gamma(æ,œ,\mathfrak{B}))\| = sup_{\|\phi\|=1}\langle M_{(2|\Gamma(\ell,\varkappa,\mathfrak{z})-\Gamma(æ,œ,\mathfrak{B})|)^P}\phi ,\phi\rangle$
$= sup_{\|\phi\|=1}\langle 2^p M_{(|\Gamma(\ell,\varkappa,\mathfrak{z})-\Gamma(æ,œ,\mathfrak{B})|)^P}\phi ,\phi\rangle$
$= sup_{\|\phi\|=1}\int_\varepsilon \ (2^p\left||\Gamma(\ell,\varkappa,\mathfrak{z})-\Gamma(æ,œ,\mathfrak{B})(t)|\right|^p)\phi(t)\overline{\phi(t)}\ \mathrm{dt}$
$\leq \quad sup_{\|\phi\|=1}\int_\varepsilon \ |\phi(t)|^2\,dt\frac{\theta}{3}[(2\|\ell(s)-æ(s)\|_\infty)^p+(2\|\varkappa(s)-œ(s)\|_\infty)^p+(2\|\mathfrak{z}(s)-\mathfrak{B}(s)\|_\infty)^p]$
$\leq \frac{\theta}{3}\left[\|\varrho_\varsigma(\ell,\ell,æ)\|+\|\varrho_\varsigma(\varkappa\ ,\varkappa,œ)\|+\|\varrho_\varsigma(\mathfrak{z},\mathfrak{z},\mathfrak{B})\|\right]$
$\leq \frac{\|a\|^2}{3}\left[\|\varrho_\varsigma(\ell,\ell,æ)\|+\|\varrho_\varsigma(\varkappa\ ,\varkappa,œ)\|+\|\varrho_\varsigma(\mathfrak{z},\mathfrak{z},\mathfrak{B})\|\right]$.

By setting a= $\begin{bmatrix}\frac{1}{3} & 0\\ 0 & \frac{1}{2}\end{bmatrix}$ $=\sqrt{\theta 1}_{\ L(\mathcal{H})}$, then a$\in$ L($\mathcal{H}$) and $\|a\| = \sqrt{\theta} = \frac{1}{\sqrt{2}} < 1$.
Hence, applying our Corollary 3.1.4, we get the desired result.

3.3. Application to Homotopy

We now offer research on the possibility of a singular homotopy theory solution.

Theorem 3.1.6: If $(\mathcal{L},\mathcal{A},\varrho_\varsigma)$ is a complete C^*-AV-S_bMS, Suppose $\mathfrak{U}$ and $\overline{\mathfrak{U}}$ are open and closed subsets of $\mathcal{L}$,respectively , such that $\mathfrak{U}\subseteq\overline{\mathfrak{U}}$ .Let $\mathfrak{L}_b$:$\overline{\mathfrak{U}}^3\times[0,1]\ \longrightarrow\mathcal{L}$ be an homotopy operator meeting the requirements listed below.

(τ_0) $æ\neq\mathfrak{L}_b(æ,\varkappa,\mathfrak{b},s)$, $\varkappa\neq\mathfrak{L}_b(\varkappa,\mathfrak{b},æ,s)$and $\mathfrak{B}\neq\mathfrak{L}_b(\mathfrak{b},æ,\varkappa,s)$ for each $æ,\varkappa,\mathfrak{b}\in\partial\mathfrak{U}$ and s$\in$ [0,1] (here $\partial\mathfrak{U}$ is boundary of $\mathfrak{U}$ in $\mathcal{L}$)

(τ_1) there exist $\ell,\varkappa,\mathfrak{z},æ,œ,\mathfrak{B}\in\overline{\mathfrak{U}}$, s$\in$ [0,1]and a$\in\mathcal{A}$ with $\|a\|<1$ such that
$\varrho_\varsigma(\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(æ,œ,\mathfrak{b},s))\preccurlyeq\ \mathrm{a}\left[\frac{\varrho_\varsigma(\ell\ ,\ell\ ,æ\ )+\varrho_\varsigma(\varkappa\ ,\varkappa\ ,œ\ )+\varrho_\varsigma(\mathfrak{z}\ ,\mathfrak{z}\ ,\mathfrak{b}\ )}{3}\right]a^*$

(τ_2) $\exists\ M_b\ \succcurlyeq 0_\mathcal{A}$ such that $\varrho_\varsigma(\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},t))\leq\|M_b\||s-t|$ for every $\ell,\varkappa,\mathfrak{z}\in\overline{\mathfrak{U}}$ and s, t $\in$ [0,1] ;

Then $\mathfrak{L}_b(.,s)$ has a TFP for some s $\in$ [0,1] $\Leftrightarrow\mathfrak{L}_b(.,t)$ has a TFP point for some t $\in$ [0,1] .

Proof: From (τ_2) it follows that $\mathfrak{L}_b$ is continuous in the second variable. From (τ_1) it follows that $\mathfrak{L}_b$ is continuous in the first variable. Consequently, we observe that:
$\varrho_\varsigma\left(\mathfrak{L}_b(æ,\varkappa,\mathfrak{b},s),\mathfrak{L}_b(æ,\varkappa,\mathfrak{b},s),\mathfrak{L}_b(æ_p,\varkappa_p,\mathfrak{b}_p,s)\right)\preccurlyeq\ \mathrm{a}\left[\frac{\varrho_\varsigma(æ\ ,æ\ ,æ_p\ )+\varrho_\varsigma(\varkappa\ ,\varkappa\ ,\varkappa_p\ )+\varrho_\varsigma(\mathfrak{b}\ ,\mathfrak{b}\ ,\mathfrak{b}_p\ )}{3}\right]a^*\ \longrightarrow 0_\mathcal{A}$
as p$\longrightarrow\infty$

Whenever $æ_p\ \longrightarrow\ æ$ and $\varkappa_p\ \longrightarrow\ \varkappa$,then $\varrho_\varsigma\left(\mathfrak{L}_b(æ,\varkappa,\mathfrak{b},s),\mathfrak{L}_b(æ,\varkappa,\mathfrak{b},s),\mathfrak{L}_b(æ_p,\varkappa_p,\mathfrak{b}_p,s)\right)\longrightarrow 0_\mathcal{A}$ as p$\longrightarrow\infty$. Therefore, $\varrho_\varsigma\left(\mathfrak{L}_b(æ,\varkappa,\mathfrak{b},s),\mathfrak{L}_b(æ,\varkappa,\mathfrak{b},s),\mathfrak{L}_b(æ_p,\varkappa_p,\mathfrak{b}_p,s)\right)=0_\mathcal{A}$. Now
$\varrho_\varsigma\left(\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(æ,œ,\mathfrak{b},t)\right)$
$\preccurlyeq\quad 2\quad\varsigma\varrho_\varsigma(\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(æ,œ,\mathfrak{b},s))+\varsigma\varrho_\varsigma(\mathfrak{L}_b(æ,œ,\mathfrak{b},t)\ ,\mathfrak{L}_b(æ,œ,\mathfrak{b},t),\mathfrak{L}_b(æ,œ,\mathfrak{b},s))$
$\preccurlyeq 2\ \varsigma\varrho_\varsigma(\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(æ,œ,\mathfrak{b},t))+\varsigma\|M_b\||s-t|$
$\preccurlyeq\ \mathrm{a}\left[\frac{\varrho_\varsigma(\ell\ ,\ell\ ,æ\ )+\varrho_\varsigma(\varkappa\ ,\varkappa\ ,œ\ )+\varrho_\varsigma(\mathfrak{z}\ ,\mathfrak{z}\ ,\mathfrak{b}\ )}{3}\right]a^*+\varsigma\|M_b\||s-t|\longrightarrow 0_\mathcal{A}$ as $(\ell,\varkappa,\mathfrak{z},s)\longrightarrow(æ,œ,\mathfrak{b},t)$.

Hence is a continuous on $\overline{\mathfrak{U}}^3\times[0,1]$. Also
$\varrho_\varsigma(\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(æ,œ,\mathfrak{b},s))\preccurlyeq\ \mathrm{a}\left[\frac{\varrho_\varsigma(\ell\ ,\ell\ ,æ\ )+\varrho_\varsigma(\varkappa\ ,\varkappa\ ,œ\ )+\varrho_\varsigma(\mathfrak{z}\ ,\mathfrak{z}\ ,\mathfrak{b}\ )}{3}\right]a^*$
if $\ell\ \neq æ$ **,** $\varkappa\neq œ$ **,** $\mathfrak{z}\neq\mathfrak{B}$.

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$$\begin{pmatrix} \varrho_\varsigma(\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(æ,œ,\mathfrak{b},s)) \\ +\varrho_\varsigma(\mathfrak{L}_b(\varkappa,\mathfrak{z},\ell,s),\mathfrak{L}_b(\varkappa,\mathfrak{z},\ell,s),\mathfrak{L}_b(œ,\mathfrak{b},æ,s)) \\ +\varrho_\varsigma(\mathfrak{L}_b(\mathfrak{z},\ell,\varkappa,s),\mathfrak{L}_b(\mathfrak{z},\ell,\varkappa,s),\mathfrak{L}_b(\mathfrak{b},æ,œ,s)) \end{pmatrix} \preccurlyeq a \begin{pmatrix} \varrho_\varsigma(\ell,\ell,æ) \\ +\varrho_\varsigma(\varkappa,\varkappa,œ) \\ \varrho_\varsigma(\mathfrak{z},\mathfrak{z},\mathfrak{b}) \end{pmatrix} a^*$$

Now consider the set

$$\mathfrak{B} = \begin{Bmatrix} s\in[0,1]: \ell=\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s), \varkappa=\mathfrak{L}_b(\varkappa,\mathfrak{z},\ell,s), \mathfrak{z}=\mathfrak{L}_b(\mathfrak{z},\ell,\varkappa,s) \\ for\ some\ \ell,\varkappa\in\mathfrak{U} \end{Bmatrix}$$

Suppose s is a limit point of $\mathfrak{B}$ .Then there exists a $\{s_p\}$ in $\mathfrak{B}$ such that $s_p \longrightarrow s$ .Then there exists a sequence $\{\ell_p\},\{\varkappa_p\},\{\mathfrak{z}_p\}\in \mathcal{L}$ such that $\ell_p=\mathfrak{L}_b(\ell_p,\varkappa_p,\mathfrak{z}_p,s_p)$, $\varkappa_p=\mathfrak{L}_b(\varkappa_p,\mathfrak{z}_p,\ell_p,s_p)$ and $\mathfrak{z}_p=\mathfrak{L}_b(\mathfrak{z}_p,\ell_p,\varkappa_p,s_p)$ .Now we show that $\{\ell_p\},\{\varkappa_p\}$ and $\{\mathfrak{z}_p\}$ are ϱ_ς-Cauchy sequences in $(\mathcal{L},\mathfrak{A},\varrho_\varsigma)$. Suppose that $\{\ell_p\},\{\varkappa_p\},\{\mathfrak{z}_p\}$ are not ϱ_ς-Cauchy sequences with respect $\mathfrak{A}$ .So there exists $\varepsilon > 0_{\mathfrak{A}}$ and monotonically increasing of natural numbers $\{q_z\}$ and $\{p_z\}$ such that $p_z > q_z$.

$\varrho_\varsigma(\ell_{q_z},\ell_{q_z},\ell_{p_z}) \succcurlyeq \epsilon \quad \varrho_\varsigma(\varkappa_{q_z},\varkappa_{q_z},\varkappa_{p_z}) \succcurlyeq \epsilon \quad \varrho_\varsigma(\mathfrak{z}_{q_z},\mathfrak{z}_{q_z},\mathfrak{z}_{p_z}) \succcurlyeq \epsilon$ .... (3.1,11)

and

$\varrho_\varsigma(\ell_{q_z},\ell_{q_z},\ell_{p_{z-1}}) \prec \epsilon \ , \varrho_\varsigma(\varkappa_{q_z},\varkappa_{q_z},\varkappa_{p_{z-1}}) \prec \epsilon \quad \varrho_\varsigma(\mathfrak{z}_{q_z},\mathfrak{z}_{q_z},\mathfrak{z}_{p_z}) \prec \epsilon$ ...... (3.1.12)

*F*rom (3.1,11) and (3.1.12), we have

$$\epsilon \preccurlyeq \varrho_\varsigma(\ell_{q_z},\ell_{q_z},\ell_{p_z})$$
$$\preccurlyeq 2k\varrho_\varsigma(\ell_{q_z},\ell_{q_z},\ell_{q_{z+1}}) + k\varrho_\varsigma(\ell_{q_{z+1}},\ell_{q_{z+1}},\ell_{p_z}).$$

Letting z⟶ ∞ we have that

$\frac{\epsilon}{k} \preccurlyeq \lim_{z\to\infty} \varrho_\varsigma(\ell_{q_{z+1}},\ell_{q_{z+1}},\ell_{p_z})$ .... (3.1.13)

Choose $\in=\frac{\delta}{12\|M_b\|\|\varsigma\|}>0$ such that $|s-s_0|<\in$ and $\ell\in\overline{\varrho_\varsigma(\ell_0,\frac{\delta}{6\|\varsigma\|\|a\|^2})}$, $\ell\neq\ell_0$, $\varkappa\in\overline{\varrho_\varsigma(\varkappa_0,\frac{\delta}{6\|\varsigma\|\|a\|^2})}$, $\varkappa\neq\varkappa_0$, $\mathfrak{z}\in\overline{\varrho_\varsigma(\mathfrak{z}_0,\frac{\delta}{6\|\varsigma\|\|a\|^2})}$ $\mathfrak{z}\neq\mathfrak{z}_0$ then

$$\varrho_\varsigma(\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s_0),\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s_0),\mathfrak{L}_b(\ell_0,\varkappa_0,\mathfrak{z}_0,s_0)) \preccurlyeq a\left[\frac{\varrho_\varsigma(\ell,\ell,\ell_0)+\varrho_\varsigma(\varkappa,\varkappa,\varkappa_0)+\varrho_\varsigma(\mathfrak{z},\mathfrak{z},\mathfrak{z}_0)}{3}\right]a^*$$

Therefore,

$$\begin{pmatrix} \|\varrho_\varsigma(\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s_0),\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s_0),\mathfrak{L}_b(\ell_0,\varkappa_0,\mathfrak{z}_0,s_0))\| \\ +\|\varrho_\varsigma(\mathfrak{L}_b(\varkappa,\mathfrak{z},\ell,s_0),\mathfrak{L}_b(\varkappa,\mathfrak{z},\ell,s_0),\mathfrak{L}_b(\varkappa_0,\mathfrak{z}_0,\ell_0,s_0))\| \\ +\|\varrho_\varsigma(\mathfrak{L}_b(\mathfrak{z},\ell,\varkappa,s_0),\mathfrak{L}_b(\mathfrak{z},\ell,\varkappa,s_0),\mathfrak{L}_b(\mathfrak{z}_0,\varkappa_0,\ell_0,s_0))\| \end{pmatrix} < \|a\|^2 \begin{pmatrix} \|\varrho_\varsigma(\ell,\ell,\ell_0)\| \\ +\|\varrho_\varsigma(\varkappa,\varkappa,\varkappa_0)\| \\ +\|\varrho_\varsigma(\mathfrak{z},\mathfrak{z},\mathfrak{z}_0)\| \end{pmatrix}$$
$$< \frac{\delta}{2\|\varsigma\|}$$

But

$$\varrho_\varsigma(\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(\ell_0,\varkappa_0,\mathfrak{z}_0,s_0))$$
$$\preccurlyeq 2\varsigma\varrho_\varsigma(\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s_0)) + \varsigma\varrho_\varsigma(\mathfrak{L}_b(\ell_0,\varkappa_0,\mathfrak{z}_0,s_0),\mathfrak{L}_b(\ell_0,\varkappa_0,\mathfrak{z}_0,s_0),\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s_0))$$
$$\preccurlyeq 2\varsigma\|M_b\||s-s_0| + \varsigma\varrho_\varsigma(\mathfrak{L}_b(\ell_0,\varkappa_0,\mathfrak{z}_0,s_0),\mathfrak{L}_b(\ell_0,\varkappa_0,\mathfrak{z}_0,s_0),\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s_0))$$

Therefore,

$$\begin{pmatrix} \|\varrho_\varsigma(\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(\ell_0,\varkappa_0,\mathfrak{z}_0,s_0))\| \\ +\|\varrho_\varsigma(\mathfrak{L}_b(\varkappa,\mathfrak{z},\ell,s),\mathfrak{L}_b(\varkappa,\mathfrak{z},\ell,s),\mathfrak{L}_b(\varkappa_0,\mathfrak{z}_0,\ell_0,s_0))\| \\ +\|\varrho_\varsigma(\mathfrak{L}_b(\mathfrak{z},\ell,\varkappa,s),\mathfrak{L}_b(\mathfrak{z},\ell,\varkappa,s),\mathfrak{L}_b(\mathfrak{z}_0,\varkappa_0,\ell_0,s_0))\| \end{pmatrix} \leq 6\|\varsigma\|\|M_b\||s-s_0| + \frac{\|\varsigma\|\delta}{\|2\varsigma\|}$$
$$< 6\|\varsigma\|\|M_b\|\epsilon + \frac{\delta}{2}$$
$$< 6\|\varsigma\|\|M_b\|\frac{\delta}{12\|M_b\|\|\varsigma\|} + \frac{\delta}{2}$$
$$= \delta$$

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Hence

$$\begin{pmatrix} \|\varrho_\varsigma(\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\ell_0)\| \\ +\|\varrho_\varsigma(\mathfrak{L}_b(\varkappa,\mathfrak{z},\ell,s),\mathfrak{L}_b(\varkappa,\mathfrak{z},\ell,s),\varkappa_0)\| \\ +\|\varrho_\varsigma(\mathfrak{L}_b(\mathfrak{z},\ell,\varkappa,s),\mathfrak{L}_b(\mathfrak{z},\ell,\varkappa,s),\mathfrak{z}_0)\| \end{pmatrix} \leq$$

$$\begin{pmatrix} \|\varrho_\varsigma(\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s),\mathfrak{L}_b(\ell_0,\varkappa_0,\mathfrak{z}_0,s_0))\| \\ +\|\varrho_\varsigma(\mathfrak{L}_b(\varkappa,\mathfrak{z},\ell,s),\mathfrak{L}_b(\varkappa,\mathfrak{z},\ell,s),\mathfrak{L}_b(\varkappa_0,\mathfrak{z}_0,\ell_0,s_0))\| \\ +\|\varrho_\varsigma(\mathfrak{L}_b(\mathfrak{z},\ell,\varkappa,s),\mathfrak{L}_b(\mathfrak{z},\ell,\varkappa,s),\mathfrak{L}_b(\mathfrak{z}_0,\varkappa_0,\ell_0,s_0))\| \end{pmatrix} < \delta$$

Therefore, $\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s) \in \overline{\varrho_\varsigma(\ell_0,\frac{\delta}{6\|\varsigma\|\|a\|^2})}$, $\mathfrak{L}_b(\varkappa,\mathfrak{z},\ell,s) \in \overline{\varrho_\varsigma(\varkappa_0,\frac{\delta}{6\|\varsigma\|\|a\|^2})}$ and $\mathfrak{L}_b(\mathfrak{z},\ell,\varkappa,s) \in \overline{\varrho_\varsigma(\mathfrak{z}_0,\frac{\delta}{6\|\varsigma\|\|a\|^2})}$. Thus, for any s ,with $|s-s_0|<\epsilon$ and s$\in[0,1]$,it follows that $\Gamma:\overline{\varrho_\varsigma(\mathfrak{z}_0,\frac{\delta}{6\|\varsigma\|\|a\|^2})}^3 \longrightarrow \overline{\varrho_\varsigma(\ell_0,\frac{\delta}{6\|\varsigma\|\|a\|^2})}$ defined by $\Gamma(\ell,\varkappa,\mathfrak{z})=\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s)$ satisfies all the hypothesis of the theorem(5.1.6) .Hence Γ has a tripled fixed point . i.e. $\Gamma(\ell,\varkappa,\mathfrak{z})=\ell$ for some $\ell \in \overline{\varrho_\varsigma(\ell_0,\frac{\delta}{6\|\varsigma\|\|a\|^2})} \subseteq \mathfrak{U}$, $\Gamma(\varkappa,\mathfrak{z},\ell)=\varkappa$ for some $\varkappa \in \overline{\varrho_\varsigma(\varkappa_0,\frac{\delta}{6\|\varsigma\|\|a\|^2})} \subseteq \mathfrak{U}$ and $\Gamma(\mathfrak{z},\ell,\varkappa)=\mathfrak{z}$ for some $\mathfrak{z} \in \overline{\varrho_\varsigma(\mathfrak{z}_0,\frac{\delta}{6\|\varsigma\|\|a\|^2})} \subseteq \mathfrak{U}$ therefore , $\Gamma(\ell,\varkappa,\mathfrak{z})=\mathfrak{L}_b(\ell,\varkappa,\mathfrak{z},s)=\ell$,
$\Gamma(\varkappa,\mathfrak{z},\ell)=\mathfrak{L}_b(\varkappa,\mathfrak{z},\ell,s)=\varkappa$ and $\Gamma(\mathfrak{z},\ell,\varkappa)=\mathfrak{L}_b(\mathfrak{z},\ell,\varkappa,s)=\mathfrak{z}$ hence $s\in\mathfrak{B}$ .Thus $|s-s_0|<\epsilon \Rightarrow s\in\mathfrak{B}$. But

$$\lim_{z\to\infty}\varrho_\varsigma(\ell_{q_z},\ell_{q_z},\ell_{p_z})$$
$$=\lim_{z\to\infty}\left(\varrho_\varsigma\left(\mathfrak{L}_b(\ell_{q_{z+1}},\varkappa_{q_{z+1}},\mathfrak{z}_{q_{z+1}},s_{q_{z+1}}),\mathfrak{L}_b(\ell_{q_{z+1}},\varkappa_{q_{z+1}},\mathfrak{z}_{q_{z+1}},s_{q_{z+1}}),\mathfrak{L}_b(\ell_{pz},\varkappa_{p_z},\mathfrak{z}_{p_z},s_{p_z})\right)\right)$$
$$\preccurlyeq a\left[\frac{\varrho_\varsigma(\ell_{q_{z+1}},\ell_{q_{z+1}},\ell_{pz})+\varrho_\varsigma(\varkappa_{q_{z+1}},\varkappa_{q_{z+1}},\varkappa_{p_z})+\varrho_\varsigma(\mathfrak{z}_{q_{z+1}},\mathfrak{z}_{q_{z+1}},\mathfrak{z}_{p_z})}{3}\right]a^*$$

It follows that $\lim_{z\to\infty}(1-\|a\|^2)\begin{pmatrix}\|\varrho_\varsigma(\ell_{q_{z+1}},\ell_{q_{z+1}},\ell_{pz})\|+\|\varrho_\varsigma(\varkappa_{q_{z+1}},\varkappa_{q_{z+1}},\varkappa_{p_z})\| \\ +\|\varrho_\varsigma(\mathfrak{z}_{q_{z+1}},\mathfrak{z}_{q_{z+1}},\mathfrak{z}_{p_z})\|\end{pmatrix}\leq 0$

Thus $\lim_{z\to\infty}\varrho_\varsigma(\ell_{q_{z+1}},\ell_{q_{z+1}},\ell_{pz})=0_{\mathfrak{A}}$. $\lim_{z\to\infty}\varrho_\varsigma(\varkappa_{q_{z+1}},\varkappa_{q_{z+1}},\varkappa_{p_z})=0_{\mathfrak{A}}$ and $\lim_{z\to\infty}\varrho_\varsigma(\mathfrak{z}_{q_{z+1}},\mathfrak{z}_{q_{z+1}},\mathfrak{z}_{p_z})=0_{\mathfrak{A}}$. Hence from (5.1.13), we have $\epsilon\preccurlyeq 0_{\mathcal{A}}$ which is a contradiction. Hence $\{\ell_p\}$, $\{\varkappa_p\}$ and $\{\mathfrak{z}_p\}$ are a C^*-AV-S_bCS in C^*-AV-S_bMS $(\mathcal{L},\mathcal{A},\varrho_\varsigma)$ and by the completeness of $(\mathcal{L},\mathcal{A},\varrho_\varsigma)$,there exist æ, œ $\in\mathfrak{U}$ with
$\lim_{p\to\infty}\ell_p=$ æ , $\lim_{p\to\infty}\varkappa_p=$ œ and $\lim_{p\to\infty}\mathfrak{z}_p=\mathfrak{b}$.
Suppose $s_p\to s$ .then $(\ell_p,\varkappa_p,\mathfrak{z}_p,s_p)\to($ æ, œ, $\mathfrak{b},s)$ .Since $\mathfrak{L}_b$ is continuous so that $\mathfrak{L}_b(\ell_p,\varkappa_p,\mathfrak{z}_p,s_p)\to\mathfrak{L}_b($ æ, œ, $\mathfrak{b},s)$, $\mathfrak{L}_b(\varkappa_p,\mathfrak{z}_p,\ell_p,s_p)\to\mathfrak{L}_b($ œ, $\mathfrak{b}$, æ, $s)$ and $\mathfrak{L}_b(\mathfrak{z}_p,\ell_p,\varkappa_p,s_p)\to\mathfrak{L}_b(\mathfrak{b}$, æ, œ, $s)$ .But $\mathfrak{L}_b(\ell_p,\varkappa_p,\mathfrak{z}_p,s_p)=\ell_p\to$ æ , $\mathfrak{L}_b(\varkappa_p,\mathfrak{z}_p,\ell_p,s_p)=\varkappa_p\to$ œ and $\mathfrak{L}_b(\mathfrak{z}_p,\ell_p,\varkappa_p,s_p)=\mathfrak{z}_p\to\mathfrak{b}$ .Therefore ,we have $\mathfrak{L}_b($ æ, œ, $\mathfrak{b},s)=$ æ , $\mathfrak{L}_b($ œ, $\mathfrak{b}$, æ, $s)=$ œ and $\mathfrak{L}_b(\mathfrak{b}$, æ, œ, $s)=\mathfrak{b}$ .Hence $\mathfrak{B}$ is closed .
Now we show that $\mathfrak{B}$ is open. Let $\mathfrak{B}$ be s_0 . Then $\ell_0,\varkappa_0,\mathfrak{z}_0$ exists in $\mathfrak{U}$ such that $\ell_0=\mathfrak{L}_b(\ell_0,\varkappa_0,\mathfrak{z}_0,s_0)$, $\varkappa_0=\mathfrak{L}_b(\varkappa_0,\mathfrak{z}_0,\ell_0,s_0)$ *and* $\mathfrak{z}_0=\mathfrak{L}_b(\mathfrak{z}_0,\ell_0,\varkappa_0,s_0)$. Because $\mathfrak{U}$ is open, $\delta>0$ must exist for $\varrho_\varsigma(\ell,\ell,\ell_0)\preccurlyeq\delta$, $\varrho_\varsigma(\varkappa,\varkappa,\varkappa_0)\preccurlyeq\delta$ and $\varrho_\varsigma(\mathfrak{z},\mathfrak{z},\mathfrak{z}_0)\preccurlyeq\delta$ implies that $\ell,\varkappa,\mathfrak{z}\in\mathfrak{U}$. Choose ϵ such that $0<\epsilon<\frac{\delta}{12\|k\|\|M_b\|}$,then s_0 is an interior point of $\mathfrak{B}$ .Hence $\mathfrak{B}$ is open. Consequently $\mathfrak{B}$ is both closed and open. Therefore, either $\mathfrak{B}=\emptyset$ or $\mathfrak{B}=[0,1]$ .Now Suppose $\mathfrak{L}_b(.;s)$ has a tripled fixed point for some $s\in[0,1]$,then$\mathfrak{B}\neq\emptyset$ *so that* $\mathfrak{B}=[0,1]$. *Therefore* $\mathfrak{L}_b(.;t)$ has a tripled fixed point for some $s\in[0,1]$.

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3.4. CONCLUSION

In this study, we derive applications to homotopy theory and nonlinear integral equations by employing tripled fixed-point theorems for a pair of mappings satisfying certain rational contractive type within the frame work of complete C^*-AV-S_bMS .

3.5. CONFLICT OF INTERESTS

The authors declare that there is no conflict of interest.

REFERENCES:

[1] S. Banach, 1922, Sur les operations dans les ensembles abstraits et leur application aux equations integrals, Fundam.Math.,1922, pp. 133-181.
[2] P. Naresh1, G. Upender Reddy, B. Srinuvasa Rao, Existence suzuki type fixed point results in A_b-metric spaces with application, Int. J. Anal. Appl. (2022), 20:67
[3] B. S. Rao, L. V. Kumar, C. Maheswari, S. M. Shaw, On certain quadruple fixed points of integral type contraction mappings in G-metric spaces and its applications, J. Sci. Res. 14 (2), 501-512, 2022.
[4] D. Ram Prasad, G.N.V. Kishore, Hseyin Isk, B. Srinuvasa Rao, and G. Adi Lakshmi, C^* -algebra valued fuzzy soft metric spaces and results for hybrid pair of mappings, Axioms 2019, 8, 0099; doi:10.3390/axioms8030099.
[5] D. Ram Prasad, G.N.V. Kishore, B. Srinuvasa Rao, G. Adhi Lakshmi, Results in partially ordered bipolar metric spaces via (α, φ)-contractive condition, International Journal of Recent Technology and Engineering (IJRTE), Volume-8, Issue-1S3, 2019.
[6] S. Sedghi, A. Gholidahneh, K. P. R. Rao, Common fixed point of two R-weakly commuting mappings in S_b-metric spaces, Math. Sci. Lett. 6, No. 3(2017), 249-253.
[7] Z. H. Ma, L.N. Jiang, H.K. Sun, C^*-algebra-valued metric spaces and related fixed point Theo rems, Fixed Point Theory Appl. 2014,2014, 206.
[8] Z. H. Ma, L.N. Jiang, C-algebra-valued b-metric spaces and related fixed- point theorems, Fixed Point Theory Appl. 2015, 2015, 222.
[9] S.S. Razavi, H.P. Masiha, Common fixed -point theorems in C^*-algebra valued b-metric spaces with applications to integral equations, Fixed Point Theory 2019, 20, 649662.
[10] S. Sedghi, Gholidahneh, T. Dosenovic, J. Esfahani and S. Radenkovic, Common fixed point of four maps in S_b-metric spaces, Journal of Linear and Topological Algebra, Vol.5(2), (2016), 93- 104.
[11] N. Souayah, N. Mlaiki A fixed point theorems in S_b-metric spaces, Journal of mathematics and computer science, 16(2016), 131-139.
[12] Souayah Nizar, A fixed point in partial S_b-metric spaces, An. St. univ ovidius constante, vol24(3),2016, 315-362, doi: 10.1515/auom-2016-0062.
[13] K.P.R. Sastry, K.K.M. Sarma, P. Krishna Kumari, Fixed point theorems for (ψ, φ, λ)-contractions in S_b-metric spaces, International Journal of Mathematics Trends and Tech nology (IJMTT) Volume 56(1), 2018, PP: 28-38.
[14] Y. Rohen, T. Dosenovic, S. Radenovic, A Note on the Paper A Fixed Point Theorems in S_b-Metric Spaces, Filo mat. 31 (2017), 3335-3346.
[15] Duduka Venkatesh, V. Naga Raju, some fixed-point outcomes in S_b-metric spaces using (ϕ, ψ)-generalized weakly contractive maps in S_b-metric spaces, Global Journal of Pure and Applied Mathematics, Volume 18, Number 2(2022), pp. 753-770.
[16] Rohen y, Dose Nović T, Randanovic S, A note on a paper A fixed -point theorems in S_b-metric spaces, Filomat 31(11), 3335-3346(2017).

Volume 38 No. 3s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

[17] K. P. R. Rao, G. N.V. Kishore, Sk. Sadik, Unique common coupled fixed- point theorem for four maps in S_b-metric spaces, Journal of Linear and Topological Algebra, Vol. 06, No. 01, 2017, 29- 43.
[18] Sh. Sedghi, A. Gholidahneh, T. Dosenovic, J. Esfahani, S. Radenovic, Common fixed point of four maps in S_b-metric spaces, Journal of Linear and Topological Algebra Vol. 05, No. 02, 2016, 93- 104.
[19] Kishore GNV, Rao KPR, Panthi D, Srinuvsa Rao B, Satyanaraya S, some applications via fixed point results in partially ordered S_b-metric spaces, Fixed point theory and applications (2017), 2017;10, doi: 10.1186/s13663-017-0603-2.
[20] S.S. Razavi, H.P. Masiha, C^*-Algebra-Valued S_b-Metric Spaces and Applications to Integral Equations, Faculty of Mathematics, K.N. Toosi University of Technology, Tehran, Iran, 2023, in progress.
[21] S. S. Razavi, H. P. Masiha, Manuel De La Sen,Applications in Integral Equations through Common Results in C-Algebra-Valued S_bMetric Spaces, Axioms 2023,12, 413.
[22] G.J. Murphy, C^*-algebras and operator theory, Academic Press: London, UK, 1990.
[23] Q. L. Xin, L. N. Jiang, Z. H. Ma, Common fixed -point theorems in C^*-algebra-valued metric spaces, Journal of Nonlinear Science and Applications, 9(2016), 4617-4627
[24] Berinde,V., Borcut,M., Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Anal,.74(15), Pp.4889-4897,2011.

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