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EXPLORING 3-TOTAL SUPER SUM CORDIAL LABELING FOR k-UNIFORM HYPERTREE

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Abstract

The idea of 3-total super sum cordial labeling for k-uniform hypertrees is presented and studied in this paper. We provide a detailed characterization of such labeling and demonstrate that 6-uniform hypertrees admit 3-total super sum cordial labeling for arbitrary numbers of hyperedges. The proposed labeling framework is supported through rigorous proofs and illustrative constructions, extending the known boundaries of cordial labeling in hypergraph theory.

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1. Introduction

As data formats and network science have advanced, conventional graph models are often insufficient for representing multi-way relationships. Hypergraphs, first introduced by Berge [1], generalize classical graphs by allowing hyperedges to connect any number of vertices. Labeling in hypergraphs—an extension of graph labeling—is a vibrant field of study encompassing graceful, magic, cordial, and other labeling types. For example, the concept of super edge-magic labeling, initially defined for graphs, was generalized to hypergraphs in [6]. Various other hypergraph labeling was explored in [2, 5]. Cordial labeling, proposed by Cahit [3], offers a relaxed framework compared to graceful or harmonious labelings and has found adaptations in hypergraphs [7, 11]. Sylwia Cichacz and colleagues [8] proved that every uniform hyperpath is k-cordial for every k. Additionally, the graceful labeling of k-uniform hypertrees has been studied in [4], where conditions for a hypertree to be graceful were established. Rinku Verma et al. [10,11] first proposed three total super sum cordial labeling for different graphs based on vertices and edges.

The notion of three total super sum cordial labeling for hypergraphs has not been investigated. This work bridges this gap, previously defined for graphs, to k-uniform hypertrees. By introducing the concept in Section 3, followed by foundational definitions in Section 2. Section 4 presents key lemmas, and Section 5 provides the main results proving the existence of this labeling in k-uniform hypertrees.

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2. Preliminaries

- ***Hypergraph***: *H*ypergraph is generalization of a graph, denoted by $H =< V, E >$, where E is subset of V, in which an edge can connect multiple numbers of vertices. In this case, edges are a subset of the set of vertices.
- ***k*-uniform Hypergraph:** Generalizing traditional graphs where edges link two vertices, a - ***k*** uniform hypergraph has every hyperedge linking exactly ***k*** vertices.
- ***Walk in a Hypergraph***: A walk is an alternate sequence consisting of vertices and hyperedges where each pair of successive vertices is bound together by the same hyperedge.
- ***Path in a Hypergraph***: A path is a walk in which all vertices and edges are distinct.
- ***Cycle in a Hypergraph***: A cycle is a walk where all vertices and edges are distinct, except for the starting and ending vertex, both are the same.
- ***Connected Hypergraph***: Any hypergraph with a path connecting each pair of vertices is said to be Connected.
- ***Hypertree***: A hypertree is an acyclic-connected hypergraph.

3. Definitions

Definition 3.1 ***3-Total Sum Cordial labeling for hypergraph***

Consider a hypergraph $H =< V, E >$ in which the hyperedges E are represented by a collection of vertices.

A 3-total sum cordial labeling of a hypergraph assigns labels from the set $\{0,1,2\}$ to both the vertices and hyperedges of H such that:

1. Each vertex $v \in V$ in H assigned a label $f(v)$ from the set $\{0,1,2\}$ respectively.
2. For each hyperedge $e \in E$, the label $f(e)$ is determined by the sum of the labels of the vertices contained in e, taken modulo 3. In other words,

   $f(e) = \left(\sum f(v)\right) \bmod 3 \; for \; v \in e.$
3. The labeling is cordial if the difference in frequencies of labels 0, 1, and 2 does not exceed 1. Mathematically $|F(i) - F(j)| \leq 1$, where $F(x)$ represents frequencies of labelled vertices and hyperedges assigned with $x \in \{0,1,2\}$ specifically, for each label.

A labeling satisfying conditions (1) and (2) is called 3-total sums labeling, while satisfying all three conditions defines 3-total sum cordial labeling.

Definition 3.2 3-*Total Super Sum Cordial labelling for hypergraph*

The 3-total sum cordial labeling for a hypergraph will be 3-total super sum cordial if, in addition to satisfying the given conditions, there exists a path in the hypergraph such that for each edge uv in the path, the labels of the vertices fulfil the condition: $|f(u) - f(v)| \leq 1$.

We are going to find 3-total super sum cordial labeling for $k-$uniform hypertree.

4. Structural Lemmas

We state and review the proof of key properties of k-uniform hypertrees used in our work:

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Lemma 4.1. *For two consecutive hyperedges in a hypertree, there exists exactly one vertex in common.*

Proof: If there is more than one vertex in common, a cycle is formed, which contradicts the definition of a hypertree.
If there is no vertex in the intersection, the hypergraph becomes disconnected, again contradicting the definition of a hypertree.

Lemma 4.2. *Each terminal hyperedge in a k-uniform hypertree contains $k-1$ vertices exclusive to itself.*

Proof: The first and last hyperedge contribute one vertex with the next and previous hyperedge respectively. Thus, the remaining $k-1$ vertices to complete the number of k-uniform belongs exclusive to the terminal vertices.

Lemma 4.3. *Each intermediate hyperedge contains $k-2$ vertices exclusive to itself.*

Proof: It is the trivial result as of lemma 1 and 2 with k vertices in each hyperedge the result is followed.

Lemma 4.4. *The order $|V|$ of k uniform hypertree is given by $(k-1)|E|+1$ where $|E|$ is the total number of hyperedges in hypergraph.*

Proof: In k-uniform hypertree, each hyperedge contains k vertices.
There is exactly one vertex in the intersection of each pair of consecutive hyperedges, contributing $(|E|-1)$ vertices in total.

Each of the two terminal hyperedges contributes $(k-1)$ exclusive vertices in itself.

Each one of the $(|E|-2)$ intermediate hyperedges contributes $k-2$ vertices exclusively.

Thus, the total number of vertices is:

$$2(k-1)+(|E|-1)+(|E|-2)(k-2)$$

Simplifying this expression yields: the total number of vertices i.e. order of k-uniform hypertree is given by $(k-1)|E|+1$.

In this paper we are going to consider k-uniform hypertree with m edges so as per the result of above lemmas it has following structure for 4 uniform hypertree with 4 hyperedges as shown in figure 1 for illustration, with total of 13 vertices.

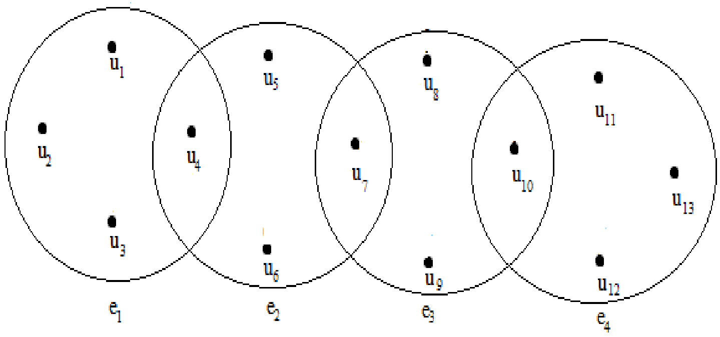

Figure 1: 4 Uniform Hypertree

5. Main Results

Theorem 5.1 A 3-total sum labeling exists for k-uniform hypertree.

Proof: Let m be the number of hyperedges in k-uniform hypertree with n vertices in it. Then using lemma 4.4, we have $n=(k-1)m+1$;

Based on vertex labeling strategies with different assignments at the intersections of hyperedges (labeled 0, 1, or 2), and using modulo 3 arithmetic, we establish balanced distributions of edge and vertex labels satisfying the cordiality condition.

Case 5.1.1 Vertex in intersection of hyperedges labeled '0'

The vertices are labeled as follows:

Assign

$f(u_1)=1;$

$f(u_n)=2;$

Define the following vertex labeling:

$$\left.\begin{array}{l} \text{for } 0\le i\le m-1, \\ f(u_{(k-1)i+p})=2 \quad : p \text{ even } 2,4,6,\ldots<k \\ f(u_{(k-1)i+q})=1 \quad : q \text{ odd } 3,5,7,\ldots<k \\ \text{for } 0\le i\le m-2, \\ f(u_{(k-1)i+k})=0 \end{array}\right\} \tag{5.1}$$

To satisfy 3-total sum labeling each $e\in E$ is labeled as

$f(e)=\left(\sum f(v)\right)\bmod 3 \; for \; v\in e.$

Now, If k is even, according to 3-total sum labeling

The first hyperedge label will be 1.

The intermediate $m-2$ hyperedge label will be 0.

The last hyperedge label will be 2.

If k is odd, according to 3-total sum labeling

The first hyperedge label will be 0.

The intermediate $m-2$ hyperedge label will be 2.

The last hyperedge edge label will be 1.

Case 5.1.2 Vertex in intersection of hyperedges labeled '1'

We label the vertices as follows:

Assign

$f(u_1)=0,$

$f(u_n)=2.$

Define the following vertex labeling:

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$$\left.\begin{array}{l} \text{for } 0 \le i \le m-1, \\ f(u_{(k-1)i+p}) = 2 \quad : p \text{ even } 2,4,6,\ldots < k \\ f(u_{(k-1)i+q}) = 0 \quad : q \text{ odd } 3,5,7,\ldots < k \\ \text{for } 0 \le i \le m-2, \\ f(u_{(k-1)i+k}) = 1 \end{array}\right\} \tag{5.2}$$

To satisfy 3-total sum labeling each $e \in E$ is labeled as

$f(e) = \left(\sum f(v)\right) \bmod 3 \; for \;\; v \in e.$

Now

If k is even, according to 3-total sum labeling

The first hyperedge label will be 0

The intermediate $m-2$ hyperedge label will be 1.

The last hyperedge label will be 2.

If k is odd, according to 3-total sum labeling

The first hyperedge label will be 2.

The intermediate $m-2$ hyperedge label will be 0.

The last hyperedge label will be 1.

Case 5.1.3 Vertex in intersection of hyperedges labeled '2'

We label the vertices as follows:

Assign

$f(u_1) = 1,$

$f(u_n) = 0.$

Define the following vertex labeling

$$\left.\begin{array}{l} \text{for } 0 \le i \le m-1, \\ f(u_{(k-1)i+p}) = 1 \quad : p \text{ even } 2,4,6,\ldots < k \\ f(u_{(k-1)i+q}) = 0 \quad : q \text{ odd } 3,5,7,\ldots < k \\ \text{for } 0 \le i \le m-2 \\ f(u_{(k-1)i+k}) = 2 \end{array}\right\} \tag{5.3}$$

Now

If k is even, according to 3-total sum labeling

The first hyperedge label will be 1.

The intermediate $m-2$ hyperedge label will be 2.

The last hyperedge label will be 0.

If k is odd, according to 3-total sum labeling

The first hyperedge label will be 2.

The intermediate $m-2$ hyperedge label will be 0.

The last hyperedge label will be 1. []

Thus, we get the frequencies of various labels {0, 1, 2} as shown in the following tables. Frequency tables demonstrate that cordiality is preserved.

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Table 1
Labeled '0' in the intersection

Vertex label a	Vertex label Frequency $F_v(a)$	Edge Label b	Edge label Frequency $F_e(b)$		Total Frequency		
			k : even	k : odd	$F(L)$	k : even	k : odd
0	$m-1$	0	$m-2$	1	0	$2m-3$	m
1	$\left(\left\lfloor \frac{k}{2} \right\rfloor -1\right)m+1$	1	1	1	1	$\left(\frac{k}{2}-1\right)m+2$	$\left(\left\lfloor \frac{k}{2} \right\rfloor -1\right)m+2$
2	$\left(\left\lceil \frac{k}{2} \right\rceil -1\right)m+1$	2	1	$m-2$	2	$\left(\frac{k}{2}-1\right)m+2$	$\left(\left\lceil \frac{k}{2} \right\rceil -1\right)m+m-1$

Table 2
Labeled '1' in the intersection

Vertex label a	Vertex label Frequency $F_v(a)$	Edge Label b	Edge label Frequency $F_e(b)$		Total Frequency		
			k : Even	k : odd	(L)	k : even	k : odd
0	$\left(\left\lceil \frac{k}{2} \right\rceil -1\right)m+1$	0	1	$m-2$	0	$\left(\frac{k}{2}-1\right)m+2$	$\left(\left\lceil \frac{k}{2} \right\rceil -1\right)m+m-1$
1	$m-1$	1	$m-2$	1	1	$2m-3$	m
2	$\left(\left\lfloor \frac{k}{2} \right\rfloor -1\right)m+1$	2	1	1	2	$\left(\frac{k}{2}-1\right)m+2$	$\left(\left\lfloor \frac{k}{2} \right\rfloor -1\right)m+2$

Table 3
Labeled '2' in the intersection

Vertex label a	Vertex label Frequency $F_v(a)$	Edge Label b	Edge label Frequency $F_e(b)$		Total Frequency		
			k : even	k : odd	$F(L)$	k : even	k : odd
0	$\left(\left\lceil \frac{k}{2} \right\rceil -1\right)m+1$	0	1	$m-2$	0	$\left(\frac{k}{2}-1\right)m+2$	$\left(\left\lceil \frac{k}{2} \right\rceil -1\right)m+m-1$
1	$\left(\left\lfloor \frac{k}{2} \right\rfloor -1\right)m+1$	1	1	1	1	$\left(\frac{k}{2}-1\right)m+2$	$\left(\left\lfloor \frac{k}{2} \right\rfloor -1\right)m+2$
2	$m-1$	2	$m-2$	1	2	$2m-3$	m

Theorem 5.2 *A k-* Uniform hypertree admits *3-total sum cordial labelling under the following conditions:*

(i) *If k is even:* $|6m-km| \leq 8$

(ii) *If k is odd:*

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|m|<=2, $\left|\left(\left\lceil \frac{k}{2} \right\rceil m - m\right)\right| \leq 2$ and $\left|\left(2m - \left\lfloor \frac{k}{2} \right\rfloor m\right)\right| \leq 1$.

Proof: Condition for the 3 total sum cordial labeling:

$| F(1) - F(0) | \leq 1$

$| F(2) - F(0) | \leq 1$

$| F(2) - F(1) | \leq 1$

(i)If k is even:

With lable '0' in intersection of hyperedge, we have

$| F(2) - F(1) | = 0$

Now,

$| F(2) - F(0) | = | F(1) - F(0) | \leq 1$

$\Rightarrow$

$\left| \left(\frac{k}{2} - 1\right) m + 2 - (2m - 3) \right| \leq 1$ (Using table 1)

$i.e. \left| \frac{km - 6m + 10}{2} \right| \leq 1$

$i.e. \left| 6m - km \right| \leq 8.$

This is the required condition for 3-total sum cordial labeling for m hyperedged k-uniform hypertree, with 0 in the intersection of hyperedges and k is even.

(ii)If k is odd:

With label '0' in intersection of hyperedge, we have

$| F(2) - F(1) | = | 2m - 3 |$ (using table 1)

thus

$| F(2) - F(1) | \leq 1$

$\Rightarrow$

$| 2m - 3 | \leq 1$

$\Rightarrow m \leq 2. \qquad (ii - a)$

Now,

$| F(2) - F(0) | \leq 1$

$\Rightarrow$

$\left| \left(\left\lceil \frac{k}{2} \right\rceil - 1 \right) m - 1 \right| \leq 1$ (using table 1)

$\left| \left(\left\lceil \frac{k}{2} \right\rceil m - m \right) \right| \leq 2 \qquad (ii - b)$

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$|F(1)-F(0)|\leq 1$

$\Rightarrow$ (using table 1)

$$\left|\left(2m-\left\lfloor\frac{k}{2}\right\rfloor m\right)\right|\leq 1 \qquad (ii-c)$$

For hypertree to be 3-total sum cordial when k is odd with m number of hyperedges the conditions *(ii-a, ii-b and ii-c)* should be satisfied simultaneously.
Following a similar method with labels '1' and '2' in the intersection of hyperedges, we get the same condition for these labeling to be 3-total sum cordial. []

With above mentioned conditions, when $'k:even'$ we find the existence of 3-total sum cordial labeling with $'m'$ number of hyperedges *k*- uniform hypertree, as shown in Table 4.

Table 4

m	k even	$\|6m-km\|\leq 8$
1	2,4,6,8,10,12,14	Satisfied
2	2,4,6,8,10,12	Satisfied
3	4, 6,8,10	Satisfied
4	4,6,8	Satisfied
5	6	Satisfied
6	6	Satisfied
7	6,8	Satisfied
8	8	Satisfied
≥ 9	6	Satisfied

When $'k'$ is odd, the only possibility with hypertree to be 3-total sum cordial labeling is with $m=1$ and $k\in\{1,3,5,7\}$.

Theorem 5.3 A *k*-uniform hypertree admits a 3-total super sum cordial labeling under the same conditions as in Theorem 5.2.
Proof: As explored it is noted, when this type of labeling is defined there exists a path in hypergraph, so that for every edge uv in the path, the condition $|f(u)-f(v)|\leq 1$ is satisfied thus as per definition 3.2 for such hypertree 3- total super sum cordial labeling exists.

Conclusion:

A comprehensive examination of hypertree structures has revealed the existence of 3-total super sum cordial labeling, which depends on the parameters k (uniformity) and m (number of hyperedges). Specifically, for the case where k is even, we observe that a 6-uniform hypertree satisfies the condition regardless of the number of edges. This breakthrough finding

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holds profound implications for addressing complex scheduling conundrums. By leveraging the properties of 6-uniform hypertrees, researchers and practitioners can develop innovative solutions for day-wise scheduling problems, optimizing resource allocation, and streamlining operations.

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