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DYNAMICAL ANALYSIS AND OPTIMAL CONTROL OF SEIR EPIDEMIC MODEL

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Abstract

Stability analysis and optimum control make up the SEIR (Susceptible, Exposed, Infected, and Recovered) epidemic model that we provide in this article. Both of these factors are critical for comprehending the pandemic. The data for the reproduction number R_0 is complete. By analysing the asymptotical stability, we may locate the disease-free and endemic equilibrium points. It has been shown that a limited framework does in fact exist, and that the answers are not negative. Within the context of the control issue, two primary control strategies are considered. The immunisation and treatment plans are as follows. We outline the essential requirements for the initial order so that you may have the maximum degree of control that is practically possible. Numerical simulations use the Runge-Kutta fourth order approach to ascertain a solution to the derived optimality issue.

KEYWORDS: SEIR epidemic model, Reproduction number, Vaccination, Optimal control, Treatment, Stability.

INTRODUCTION:

Epidemiological models are regularly used to explain the dynamics of contagion diseases in populations. Mathematical modeling has turned out to be a significant tool in analyzing the control and spread of infectious diseases. Daniel Bernoulli initiated formation of mathematical model in the area of epidemic. The main reason for formulating a epidemic disease model is to analyse the containing behavior of the endemic and to find out the feasible strategies to control it. The criteria in such epidemiological models are on the incidence rate at which people shift from the group of susceptible to the group of infectives. In many epidemic models, the bilinear incidence rate βSI and the general incidence rate $\frac{\beta SI}{N}$ are commonly used. The latter one seems more practical then the bilinear incidence rate, as it includes the change in behavior and crowding effect of the infective individuals and check the unboundedness of the contact rate by opting appropriate parameters.

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Many studies on SEIR models have come forward to analyze the role of optimal control, [1]. Kermack, W.O., McKendrick, A.G.: A contribution to the mathematical theory of epidemics. Proc. R. Soc. Lond. 115, 700–721 (1927). Stability Analysis of an Epidemic Model with Infected Immigrants and Optimal Vaccination, International Journal of Recent Technology and Engineering (IJRTE), ISSN: 2277-3878, Volume -8 Issue-2, July 2019, PP3071-3077. by M. Sridevi, B. Ravindra Reddy [2]. [3]. R. Srilatha and B. Ravindra Reddy, Stability Analysis of Three Species Model in Series Mutualism with Bionomic and Optimal Harvesting of Two Terminal Species, International Journal of Scientific and Innovative Mathematical Research (IJSIMR), Volume 2, Issue 12, December 2014, PP 1043-1052, ISSN 2347-307X (Print) & Issn 2347- 3142(Online). [4]. RANJITH KUMAR. G, LAKSHMINARAYANA.K, KARUNA BNR AND RAVINDRA REDDY.B, Optimal control an SIR Epidemic Model under Saturated Treatment Function, Bull. Cal. Math. Soc, 112, (1) 19- 30 (2020). [5]. Li, M.Y.: An Introduction to Mathematical Modeling of Infectious Diseases. Springer, Cham (2018) Liu et al. Journal of Inequalities and Applications 2024 2024:66 Page 36 of 37. [6]. Julien, A., Portet, S.: A simple model for COVID-19. Infect. Dis. Model. 5, 309–315 (2020) [7]. Awasthi, A.: A mathematical model for transmission dynamics of COVID-19 infection. Eur. Phys. J. Plus 138, 285 (2023). [8]. Xu, C., Yu, Y., Ren, G., Sun, Y., Si, X.: Stability analysis and optimal control of a fractional-order generalized SEIR model for the COVID-19 pandemic. Appl. Math. compute. 457, 128210 (2023). [9]. Nesteruk, I.: Simulations and predictions of COVID-19 pandemic with the use of SIR model. Innov. Biosyst. Bioeng. 4, 110–121 (2020). [10]. Cooper, I., Mondal, A., Antonopoulos, C.G.: A SIR model assumption for the spread of COVID-19 in different communities. Chaos Solitons Fractals 139, 110057 (2020). [11]. Annas, S., Pratama, M.I., Rifandi, M., Sanusi, W., Side, S.: Stability analysis and numerical simulation of SEIR model for pandemic COVID-19 spread in Indonesia. Chaos Solitons Fractals 139, 110072 (2020). [12]. Ahmed, N., Elsonbaty, A., Raza, A., Rfiq, M., Adel, W.: Numerical simulation and stability analysis of a novel reaction-diffusion COVID-19 model. Nonlinear Dyn. 106, 1293–1310 (2021).[13]. Biswas, S.K., Ahmed, N.U.: Mathematical modeling and optimal intervention of COVID-19 outbreak. Quant. Biol. 1, 84–92 (2021). [14]. Kouidere, A., EL Youssoufi, L., Ferjouchia, H., Balatif, O., Rachik, M.: Optimal control of mathematical modeling of the spread of the COVID pandemic with highlighting the negative impact of quarantine on diabetic's people with cost-effectiveness. [15]. Chaos Solitons Fractals 145, 110777 (2021) 15 Deressa, C.T., Duressa, G.F.: Modeling and optimal control analysis of transmission dynamics of COVID-19: the case of Ethiopia. Alex. Eng. J. 60, 719–732 (2021). [16]. Ahmed, M., Masood, M., Sarkar, M.: Bifurcation analysis and optimal control of discrete SIR model for COVID-19. Chaos Solitons Fractals 174, 113899 (2023).[17]. Guo, Y., Li, T.: Modelling the competitive transmission of the Omecron strain and Delta strain of COVID-19. J. Math. Anal. Appl. 526, 127283 (2023). [18]. Chen, Y., Cheng, J., Jiang, Y., Liu, K.: A time delay dynamic system with external source for the local outbreak of 2019-nCoV. Appl. Anal. 101, 146–157 (2022). [19]. Paul, S., Lorin, E.: Estimation of COVID-19 recovery and decease periods in Canada using delay model. Sci. Rep. 11, 23763 (2021). [20]. Liu, C., Loxton, R., Teo, K.L.: A computational method for solving time-delay optimal control problems with free terminal time. Syst. Control Lett. 72,

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This study intends at presenting an ideal control problem to the SEIR model rather than just restricting oneself to a specific model. To achieve this, a part of the total susceptible population is pressed into service as a control in our model, with percentage being a time dependent function. The stated aim is to diminish the rate of infection in those individuals who have been affected and those vulnerable to infection. It also aims to ensure recovery to the maximum populace affected by diseases.

The remaining part of this paper follows the structure given below.

The mathematical model is given in section 2. In section 3 is equilibrium points and stability, Boundedness in section 4, The Optimal Vaccination is section 5, in section 6 is Numerical Simulation, Conclusion is section 7, References is section 8.

2. Mathematical Model

The model can be represented by the following system of differential equations.

$$\frac{dS}{dt} = A - \beta_1 S\,E - \beta_2\,S\,I - dS$$

$$\frac{d\,E}{dt} = \beta_1 S\,E + \beta_2\,S\,I - \sigma E - (\gamma_1 + d)\,E \qquad (2.1)$$

$$\frac{d\,I}{dt} = \sigma E - \gamma_2 I - (\mu + d)\,I \qquad \frac{dR}{dt} = \gamma_1 E + \gamma_2 I - dR$$

With initial conditions S (0) ≥ 0, E (0) ≥ 0, I (0) ≥ 0, R≥ 0

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where t is the time of process; S (t), E (t), and I (t) are respectively, the susceptible, exposed, infected populations. A is the recruitment tare of the population, d is the natural death rate, β_1, β_2 are transmission rates for E and I, σ is the progression rate from E and I, γ_1, γ_2 are recovery rates for E and I, μ is the disease- induced mortality rate.

3.EQUILIBRIUM POINTS AND STABILITY

In this section, we will analyze the local stability of system (2.1) at the equilibrium points

For system (2.1), the disease – free and the endemic equilibrium points are defined as

$$E_0 = \left(\frac{A}{d}, 0, 0\right) \text{ and } E_1 = (S^*, E^*, I^*) \tag{3.1}$$

where

$$S^* = \frac{[\sigma(\gamma_2 + \mu + d) + (\gamma_1 + d)(\gamma_2 + \mu + d)]}{[\beta_1(\gamma_2 + \mu + d) + \beta_2 \sigma]}$$

$$E^* = \frac{(\gamma_2 + \mu + d)}{\sigma} I^* \tag{3.2}$$

$$I^* = \frac{(R_0 - 1)}{\frac{\beta_1 (\gamma_2 + \mu + d)}{d\,\sigma} + \frac{\beta_2}{d}}$$

Now, we consider the basic reproduction number R_0 which can be used to determine whether a disease is an endemic. To begin with, we rewrite system (2.1) as

F = $[\beta_1 S E + \beta_2 S I \; 0 \; 0]$ and V = $(\sigma E + (\gamma_1 + d) - \sigma E + (\gamma_1 + \mu + d) I)$

we, obtain the following Jacobian matrices F and V at the disease – free equilibrium point E_0

F = $[\beta_1 S_0 \; \beta_2 S_0 \; 0 \; 0]$ and V = $[\sigma + \gamma_1 + d \; 0 - \sigma \; \gamma_2 + \mu + d]$

Thus, R_0 can be defined as

$$R_0 = \text{F } V^{-1} = \frac{\beta_1 A}{d(\sigma + \gamma_1 + d)} + \frac{\beta_2 A \sigma}{d((\sigma + \gamma_1 + d)(\mu + \gamma_2 + d))} \tag{3.3}$$

Theorem1: The system (2.1) has

(I) Disease – free equilibrium, if $R_0 < 1$ and

(II) If $R_0 > 1$, then there is a unique positive equilibrium called the endemic equilibrium.

New we shall study the local stability of each equilibrium.

The Jacobian matrix of (1) at E_0 is

$$J_{E_0} = \left[-d \; -\beta_1 \frac{A}{d} - \beta_2 \frac{A}{d} \; 0 \; \beta_1 \frac{A}{d} - (\sigma + \gamma_1 + d) \; \beta_2 \frac{A}{d} \; 0 \; \sigma \; -(\mu + \gamma_2 + d)\right]$$

The characteristic equation of above matrix is

The roots are $\lambda_1 = -d$,

$$\lambda^2 + \left[-\beta_1 \frac{A}{d} + \sigma + \gamma_1 + d + \gamma_2 + \mu + d\right]\lambda + (\gamma_1 + d + \sigma)(\gamma_2 + \mu + d) - \beta_1 \frac{A}{d}(\gamma_2 + \mu + d) - \sigma\beta_2 \frac{A}{d} = 0$$

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$$(\gamma_1 + d + \sigma)(\gamma_2 + \mu + d) > \beta_1 \frac{A}{d}(\gamma_2 + \mu + d) + \sigma\beta_2 \frac{A}{d}$$

$$\frac{\beta_1 A}{d(\gamma_1 + \sigma + d)} + \frac{\sigma \beta_2 A}{d(\gamma_1 + \sigma + d)(\gamma_2 + \mu + d)} < 1$$

therefore $R_0 < 1$

clearly, if $R_0 < 1$ then the disease – free equilibrium is a locally asymptotically stable.

The Jacobian matrix of (2.1) at $E_1 = (S^*, E^*, I^*)$ is given by

$$J_{E_1} = [-(\beta_1 E + \beta_2 I + d) - \beta_1 S - \beta_2 S (\beta_1 E + \beta_2 I) \beta_1 S - (\sigma + \gamma_1 + d) \beta_2 S 0 \sigma - (\gamma_2 + \mu + d)] \quad (3.4)$$

The eigenvalues determined by the following characteristic equation:

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \quad (3.5)$$

$A_1 = \beta_1 E + \beta_2 I + 3d + \beta_1 S + \sigma + \mu + \gamma_1 + \gamma_2$ $A_2 = \beta_1 E\sigma + \beta_2 I\sigma - \beta_1 E(\gamma_2 + \mu + d) - \beta_2 I(\gamma_2 + \mu + d) - \beta_1 S(\gamma_2 + \mu + d) + (\gamma_1 + \sigma + d)(\gamma_2 + \mu + d)$

$A_3 = (\beta_1 E + \beta_2 I + d)[(\gamma_1 + \sigma + d)(\gamma_2 + \mu + d) - \beta_1 S(\gamma_2 + \mu + d) - \beta_2 S\sigma] + \beta_1 S(\gamma_2 + \mu + d)(\beta_1 E + \beta_2 I) + \beta_2 S(\beta_1 E + \beta_2 I)$

Recall that the positive endemic equilibrium point E_1 exists if and only if $R_0 > 1$. Thus, by the Routh- Hurwitz theorem (3.5). Hence the endemic equilibrium point E_1 is a locally asymptotically stable.

4. Boundedness

Theorem 1: The solutions of the system (2.1) are bounded in $\emptyset \in R_+^3$.

Proof: To prove this result. Let Z = S + E + I + R

$$\frac{dZ}{dt} = \frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt} \quad (4.1)$$

$$\frac{dZ}{dt} = A - dZ - \mu I \leq A - dZ$$

$$\frac{dZ}{dt} + dZ \leq A \quad (4.2)$$

By theory of differential inequalities, we obtain

$$Z = \frac{A}{d} + Ce^{-dt} \quad (4.3)$$

if t → ∞, we have $sup\ sup\ Z \leq \frac{A}{d}$. Hence all solutions of the system are bounded in the region.

$$\emptyset = \left\{(S.E.I.R) \in R_+^3;\ S(0) \geq 0, E(0) \geq 0, I(0) \geq 0, R \geq 0, S + E + I + R \leq \frac{A}{d}\right\} \quad (4.4)$$

5.The Optimal Vaccination

An ideal control problem is designed to minimize the objective functional

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$$J(u_1, u_2) = \int_0^T \left[I(t) + \frac{C_1}{2} u_1^2 + \frac{C_2}{2} u_2^2 \right] dt \quad (5.1)$$

Subject to

$$\frac{dS}{dt} = A - \beta_1 S E - \beta_2 S I - dS - u_1 S$$

$$\frac{dE}{dt} = \beta_1 S E + \beta_2 S I - \sigma E - (\gamma_1 + d) E \quad (5.2)$$

$$\frac{dI}{dt} = \sigma E - \gamma_2 I - (\mu + d) I - u_2 I \qquad \frac{dR}{dt} = \gamma_1 E + \gamma_2 I - dR + u_1 S + u_2 I$$

where

- u_1(t) represents preventive measures (e.g. Vaccination, isolation)
- u_2(t) represents treatment strategies (e.g. medication, hospitalization)
- I(t) represents the number of infections.
- C_1, C_2 are cost weights for controls u_1, u_2.

For its presence, a control system (5.2) with initial conditions is factored in. It is then presented as:

$$P_t = M P + N(P) \quad (5.3)$$

Where

$$P = [S\ E\ I\ R], \quad M = \begin{bmatrix} -(d+u_1) & 0 & 0 & 0 \\ 0 & -(\sigma+\gamma_1+d) & 0 & 0 \\ 0 & \sigma & \gamma_1 & -(\gamma_2+\mu+d+u_2) \\ 0 & \gamma_2+u_2 & -d & \end{bmatrix}$$

Equation (5.3) is surely a system with non-Linear bounded coefficient.

We set

$$Q(P) = M P + N(P) \quad (5.4)$$

Now,

$$N(P_1) - N(P_2) = [A - (\beta_1 S_1 E_1 + \beta_2 S_1 I_1) \quad \beta_1 S_1 E_1 + \beta_2 S_1 I_1 \quad 0 \quad 0]^T - [A - (\beta_1 S_2 E_2 + \beta_2 S_2 I_2) \quad \beta_1 S_2 E_2 + \beta_2 S_2 I_2 \quad 0 \quad 0]^T$$

$$|N(P_1) - N(P_2)| = \leq 2\beta_1 |(S_1 E_1 - S_2 E_2) + (S_1 I_1 - S_2 I_2)|$$

$$\leq \eta(|S_1 E_1 - S_2 E_2| + |S_1 I_1 - S_2 I_2|)$$

Where $\eta = \max(2\beta_1; 2\beta_2)$

Also, we obtain from attempting a solution the following equation:

$|Q(P_1) - Q(P_2)| \leq V|P_1 - P_2|$ Where V = max $(\eta . \|M\|) < \infty$.

Predictably, function Q is made Lipschitz continuous on a uniform scale. Based on the definition we started out with reference to control u_1, u_2 and the restriction on S, E, I, R≥ 0, it becomes easy to fix a solution to the model (5.3). Going back to optimal control problem (5.1)

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and (5.3), to identify an ideal solution, the following Lagrangian of the problem serves the model well:

L (S, E, I u_1, u_2) = I + $\frac{C_1}{2} u_1^2 + \frac{C_2}{2} u_2^2$

So as to analyze the minimum cost of the Lagrangian, we outline Hamiltonian

H(S,E,I,R,$u_1, u_2, \lambda_1, \lambda_2, \lambda_3, t$) = L(S,E,I u_1, u_2) + $\lambda_1(t)\frac{dS}{dt} + \lambda_2(t)\frac{dE}{dt} + \lambda_3(t)\frac{dI}{dt} + \lambda_4(t)\frac{dR}{dt}$ (5.5)

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$are adjoint (costate) variables.

THEOREM (1)

There occurs an optimal control u^* with J(u^*) = $J(u^*)$ subject to the initial conditions of the control system (5.3).

Proof: To prove the existence of an optimal control, the result in (Lukes 1982, birkhoff and Rota, 1989) is called in to service. It may be observed that the control and the state variables are nonnegative values. In order that the values arrived at in the problem are made minimum, one needs to satisfy the convexity of the objective function u.

The control space is defined as

$U_{ad} = \left\{\frac{u}{u}\ is\ measerble, 0 \le u \le u_{max} < \infty,\ t \in [0,T]\right\}$

The system which is optimal is bounded from establishing the compactness requisite for ensuring the existence of optimal control. Besides, the integrand in (5.1).

$I(t) + \frac{C_1}{2}u_1^2 + \frac{C_2}{2}u_2^2$ is convex on the control u. Also, we are able to without difficulty see that, there exist a constant $\rho > 1$, $\omega_1 and\ \omega_2$ are positive numbers where J(u). We conclude that an optimal control exists.

When arriving at the optimal solution, portraying's maximum principle comes to our rescue (Lenhartand Workman; Morton and Nancy, (Lukes, 1982 and Wang, 2006)).

so that the Hamiltonian

H (t, x, u, $\boldsymbol{\lambda}$) = f (t, x, u) + $\boldsymbol{\lambda}$g (t, x, u) (5.6)

If (x^*, u^*) is an optimal solution of an ideal control problem, then there exists a non- trivial vector function $\boldsymbol{\lambda}$(t) = $\lambda_1(t)\ \lambda_2(t), \lambda_3(t), \dots, \lambda_n(t)$ satisfying the following inequalities:

$\frac{dx}{dt} = \frac{\partial H(t,\ x^*,\ u^*,\ \lambda)}{\partial \lambda}$, 0 = $\frac{\partial H(t,\ x^*,\ u^*,\ \lambda)}{\partial u}$, $\frac{d\lambda}{dt} = -\frac{\partial H(t,\ x^*,\ u^*,\ \lambda)}{\partial x}$ (5.7)

It follows from the derivation above

$\{u^* = 0, \quad if\ \frac{\partial H}{\partial u} < 0\ 0 \le u^* \le u_{max,} \quad if\ \frac{\partial H}{\partial u} =$ 0 $u^* = u_{max,} \quad if\ \frac{\partial H}{\partial u} > 0$ (5.8)

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What remains is applying the necessary conditions to the Hamiltonian H (5.6).

THEOREM (2)

The control variable u^*, associated optimal state solutions S^*, E^*, $I^* and\ R^*$ of the (5.1) and (5.2). Then, there exist adjoint variables $\lambda_1, \lambda_2, \lambda_3$ that satisfy

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial S} = \lambda_1(\beta_1 E + \beta_2 I + d + u_1) - \lambda_2\beta_1 E - \lambda_3\beta_2 I$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial E} = \lambda_1\beta_1 S - \lambda_2(\beta_1 S - \sigma - \gamma_1 - d) + \lambda_3\sigma \qquad (5.9)$$

$$\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial I} = 1 + \lambda_1\beta_2 S - \lambda_2\beta_2 S + \lambda_3(\gamma_2 + \mu + d + u_2)$$

with boundary conditions

$$\lambda_i(T) = 0, \quad i = 1,2,3. \qquad (5.10)$$

In addition to this, the ideal control u^* is given by

$$u^* = \max\left(\min\left(\frac{(\lambda_1-\lambda_3)S^*}{\tau},\ u_{max}\right), 0\right) \qquad (5.11)$$

Proof: The Hamiltonian (5.5) is employed so as to determine the boundary conditions for the adjoint equations. By setting S = S^*, E = E^*, I = I^*, R = R^*, and differentiating the Hamiltonian w.r.t S, E, I, and R, respectively, equation (5.5) is obtained:

seeking recourse to optimality conditions, the following equations are achieved

$\frac{\partial H}{\partial u}\tau u^* - \lambda_1 S^* + \lambda_3 S^* = 0$, at u = u^*

which leads to $u^* = \frac{(\lambda_1-\lambda_3)S^*}{\tau}$

$u^* = \{0, \quad if \quad \frac{(\lambda_1-\lambda_3)S^*}{\tau} \le 0\ 0 \le u^* \le u_{max}, \quad if \quad 0 < \frac{(\lambda_1-\lambda_3)S^*}{\tau} <$
$u_{max}\ u_{max}, \quad if \quad \frac{(\lambda_1-\lambda_3)S^*}{\tau} \ge u_{max}$

Hence the optimal control is characterized like

$$u^* = \max\left(\min\left(\frac{(\lambda_1-\lambda_3)S^*}{\tau},\ u_{max}\right), 0\right)$$

So, the optimality system is given by

$$\frac{dS}{dt} = A - \beta_1 S\,E - \beta_2\,S\,I - dS - \max\left(\min\left(\frac{(\lambda_1-\lambda_3)S^*}{\tau},\ u_{max}\right), 0\right) S$$

$$\frac{d\,E}{dt} = \beta_1 S\,E + \beta_2\,S\,I - \sigma E - (\gamma_1 + d)\,E$$

$$\frac{d\,I}{dt} = \sigma E - \gamma_2 I - (\mu + d)\,I - \max\left(\min\left(\frac{(\lambda_1-\lambda_3)S^*}{\tau},\ u_{max}\right), 0\right) I$$

$$\frac{dR}{dt} = \gamma_1 E + \gamma_2 I - dR + max\left(min\left(\frac{(\lambda_1-\lambda_3)S^*}{\tau},\ u_{max}\right), 0\right) S + max\left(min\left(\frac{(\lambda_1-\lambda_3)S^*}{\tau},\ u_{max}\right), 0\right) I$$

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$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial S} = \lambda_1(\beta_1 E + \beta_2 I + d + max\,(min\,(\frac{(\lambda_1-\lambda_3)S^*}{\tau},\ u_{max}\,),0\,)) - \lambda_2\beta_1 E - \lambda_3\beta_2 I$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial E} = \lambda_1\beta_1 S - \lambda_2(\beta_1 S - \sigma - \gamma_1 - d) + \lambda_3\sigma$$

$$\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial I} = 1 + \lambda_1\beta_2 S - \lambda_2\beta_2 S + \lambda_3(\gamma_2 + \mu + d + max\,(min\,(\frac{(\lambda_1-\lambda_3)S^*}{\tau},\ u_{max}\,),0\,))$$

(5.12)

6. Numerical Simulation

To demonstrate the theoretical result obtained in this paper,

we will give some numerical simulations. we consider the hypothetical set of parameter values as the following.

Example1: Consider the following set of parameter values

A = 5; $\beta_1 = 0.1$; $\beta_2 = 0.09$; $\gamma_1 = 0.9$; $\gamma_2 = 0.5$; d = 0.9; $\sigma = 0.7$;

$\mu = 0.8$; then the system (2.1) has an infection – free equilibrium $E_0$with $R_0 = 0.4494 < 1$, which is locally asymptotically stable. (Fig 1)

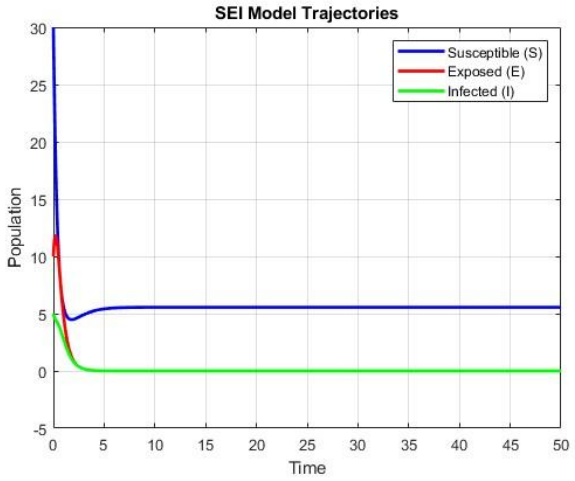

Fig 1 Trajectories of system (2.1) at infection – free equilibrium E_0

Example2: For the following set of parameter values

A = 5; $\beta_1 = 0.05$; $\beta_2 = 0.09$; $\gamma_1 = 0.9$; $\gamma_2 = 0.15$; d = 0.002; $\sigma = 0.01$;

$\mu = 0.8$; then the system (2.1) has an endemic equilibrium $E_1$with $R_0 = 373.4059 > 1$,

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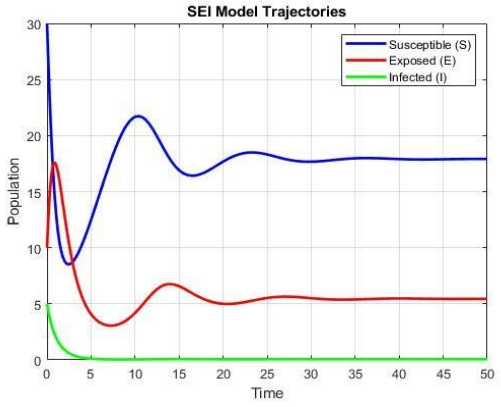
which is asymptotically stable. (Fig 2,3)



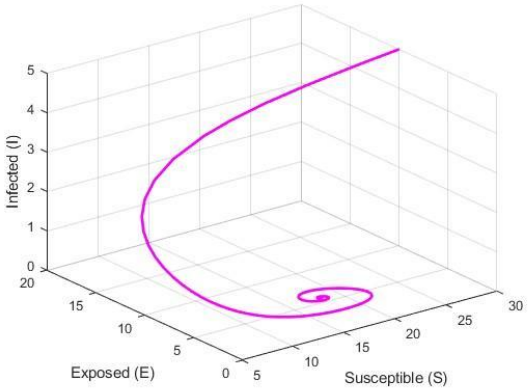


Fig2 Fig3

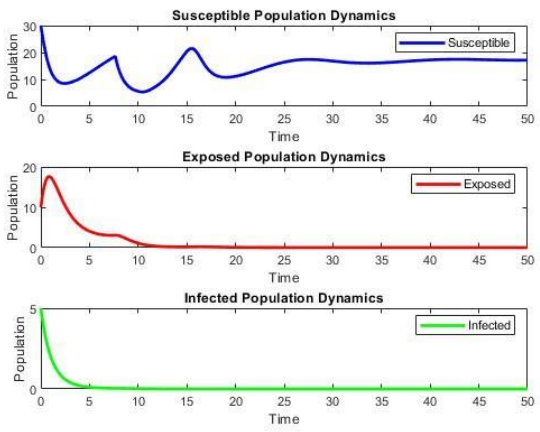
Fig (2,3). Trajectories of system (2.1) at endemic equilibrium E_1

Example3. A plot of Susceptible, Exposed, Infected and Recovered individuals with control is made by considering the following set of parameters as

A = 5; $\beta_1 = 0.05$; $\beta_2 = 0.09$; $\gamma_1 = 0.9$; $\gamma_2 = 0.15$; d = 0.002; $\sigma = 0.01$;

$\mu = 0.8$; $c_1 = 0.01$; $c_2 = 0.2$;

To find a solution to optimality system (5.12), Runge-Kutta fourth order procedure is very helpful alongside forward backward method. Six ordinary differential equations based on the state and adjoint equations, together with the two controls, come in handy when solving the equation. The process commences with an educated guess for the adjoint variables and following that. Runge- Kutta fourth order procedure in time is called into play. The transversally conditions compel solving the adjoint equations through backward Range- Kutta fourth order procedure. Figures (4,5) illustrate this clearly.



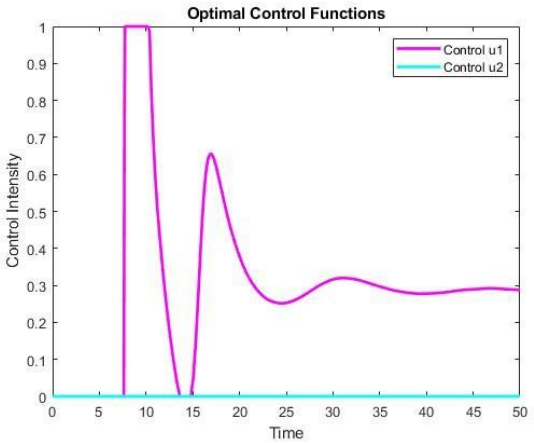


Fig4 Fig5

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Fig 4 & Fig5 represent the optimal control

7. Conclusion

In this article, we analysed the dynamics of the SEIR epidemic model and effective strategies for management. In our study, we identified disease-free and endemic equilibrium points by looking at the basic reproduction number, which is r0. The treatment group and the vaccine group each have their own control. Moreover, two controls are already in place. To find out which control problem worked best, the research model was used. The number of susceptible and diseased people could go down and the number of people who get well might go up if the vaccine campaign is done well.

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