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# Generalized wave equations for the Bessel operator, with the Morse potential on the real line and on the hyperbolic spaces

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**Abstract**

In this paper we give explicit formula for the solution of the generalized wave equation associated to the Bessel

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operator, as an application of our results we solve explicitly the generalized wave equations with the Morse potential and on the hyperbolic spaces.

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**Key Words and Phrases:** Generalized wave equation, Bessel operator, Morse potential, Laplace Beltrami operator, Hyperbolic space.

# 1 Introduction

These equations and their solutions have applications in various fields like mathematical physics signal processing and the study of the special functions The aim of this research paper is to solve explicitly the generalized wave equation associated to the Bessel operator.

$$L_a = x^2 \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - a^2 x^2, \tag{1}$$

The Bessel differential operator is very important in analysis and its applications, the Poisson and heat semigroups associated to the Bessel operator have been studied by Adam et al.[2]. For the explicit formulas of the generalized Poisson equation associated to the Bessel operator see Adam et al.[3]. An explicit formulas of the wave equation with the Morse potential and on the hyperbolic plane are given in Abdelhaye et al.[1].

In this work we solve explicitly the following generalized wave equation.

$$\begin{cases} (w)\, L_a u(t,x) = B_\nu u(t,x), (t,x) \in \mathbb{R}_+^2 \\ (a)\, \lim_{t\to 0} t^{-\nu} u(t,x) = 0, \\ (b)\, \lim_{t\to 0} t^{-\nu} u_t(t,x) = u_0(x), u_0 \in C_0^\infty(\mathbb{R}^+). \end{cases} \tag{2}$$

where $L_a$ is the Bessel differential operator given in (1) and

$$B_\nu = \frac{\partial^2}{\partial t^2} + 2\nu \frac{\partial}{\partial t} + \nu^2, \tag{3}$$

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with $a$ and $\nu$ are real parameters.
This paper is organized as follows, in Sect. 2, we give the solutions of the generalized wave equation associated to the Bessel operator, in closed form. Section 3 is devoted to the solution of the generalized Cauchy problem for the generalized wave equation for the Bessel operator, in section 4 the case of Morse Potential is considered. Finally, in Sect. 5, the explicit solution of the generalized Cauchy problem for the generalized wave equation on the hyperbolic space is obtained.

## 2 Solutions of the generalized wave equation associated to the Bessel operator

In this section we solve explicitly the generalized wave equation $(w)$ in the problem (2), and for this we need the following Lemma

**Lemma 0.1.** *Let $z = 2xx' \cosh t - x^2 - x'^2$, with $x, x', t \in \mathbb{R}^+$, set $u(z) = \phi(x, x', t)$, then the following formulas hold:*

- *i)* $\frac{\partial \phi}{\partial x} = (2x' \cosh t - 2x)\frac{\partial f}{\partial z}$, $\frac{\partial^2 \phi}{\partial x^2} = (2x' \cosh t - 2x)^2 \frac{\partial^2 f}{\partial z^2} - 2\frac{\partial f}{\partial z}$,
- *ii)*$\frac{\partial \phi}{\partial t} = 2xx' \sinh t \frac{\partial f}{\partial z}$, $\frac{\partial^2 \phi}{\partial t^2} = (2xx' \sinh t)^2 \frac{\partial^2 f}{\partial z^2} + 2xx' \cosh t \frac{\partial f}{\partial z}$,
- *iii) Let $A = x - x'e^t$, then we have*
  $A^{\nu+2}[L^a - B_\nu]A^{-\nu}W(t, x, x') =$
  $$nx^2A^2w_{xx} - A^2W_{tt} + Ax(-2\nu + A)W_x - 2\nu AxW_t - a^2x^2A^2W. \tag{4}$$

*Proof.* The parts i) and ii) are simples, to see iii), we have
$A^{\nu+2}L^aA^{-\nu}W(t, x, x') =$

$$x^2A^2w_{xx} + (-2\nu Ax^2 + xA^2)W_x + \left(\nu(\nu + 1)x^2 - \nu xA - a^2x^2A^2\right)W, \tag{5}$$

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and
$A^{\nu+2}B_\nu A^{-\nu}W(t,x,x') =$

$$A^2W_{tt}+2\nu AxW_t+\left(\nu x'e^tA+\nu(\nu+1)x'^2e^{2t}+2\nu^2x'e^tA+\nu^2A^2\right)W. \tag{6}$$

Using the above two formulas we can write:
$A^{\nu+2}[L^a - B_\nu]A^{-\nu}W(t,x,x') =$

$$x^2A^2w_{xx} - A^2W_{tt} + Ax(-2\nu + A)W_x - 2\nu AxW_t - a^2x^2A^2W. \tag{7}$$

and the proof of the Lemma is finished. □

**Proposition 1.** *The general solution of the generalized wave equation* $(w)$ *in the problem* (2) *is given by:*

$$u_\nu(t,x,x') = a_\nu u^1_\nu(t,x,x') + b_\nu u^2_\nu(t,x,x') \tag{8}$$

*with*

$$u^1_\nu(t,x,x') = \left(\frac{x - x'e^{-t}}{x - x'e^t}\right)^{\frac{\nu}{2}} J_\nu(|a|\sqrt{2xx'\cosh t - x^2 - x'^2}) \tag{9}$$

*and*

$$u^2_\nu(t,x,x') = \left(\frac{x - x'e^{-t}}{x - x'e^t}\right)^{\frac{\nu}{2}} Y_\nu(|a|\sqrt{2xx'\cosh t - x^2 - x'^2}) \tag{10}$$

*where* $a, b \in \mathbb{C}$ *and* $J_\nu, Y_\nu$ *are Bessel functions of the first and second kind respectively.*

*Proof.* Setting $u(t,x,x') = A^{-\nu}W_\nu(t,x,x')$ in the generalized wave equation $(w)$ in the problem (2) and making use of the part $iii)$ of the above Lemma we obtain
$[L^a - B_\nu]A^{-\nu}W(t,x,x') = 0.$
or equivalently:

$$x^2A^2w_{xx} - A^2W_{tt} + Ax(-2\nu + A)W_x - 2\nu AxW_t - a^2x^2A^2W = 0. \tag{11}$$

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Setting $z = 2xx' \cosh t - x^2 - x'^2$ in the equation (11) in view of the parts i) and ii) of the Lemma 0.1, we obtain

$$\{z\frac{\partial^2}{\partial z^2} + (1-\nu)\frac{\partial}{\partial z} + \frac{a^2}{4}\}e_\nu = 0 \tag{12}$$

or

$$\{z^2\frac{\partial^2}{\partial z^2} + (1-\nu)z\frac{\partial}{\partial z} + \frac{a^2}{4}z\}e_\nu. = 0 \tag{13}$$

The equation (13) is a Lommel differential equation,(see magnus et al.[11] p. 77.)

$$x^2\frac{\partial^2 u}{\partial x^2} + (1-2\alpha)x\frac{\partial u}{\partial x} + (\beta\gamma x^\gamma)^2 u + (\alpha^2 - \nu^2\gamma^2)u = 0, \tag{14}$$

has two independent solutions: $x^\alpha J_\nu(\beta x^\gamma)$ and $x^\alpha Y_\nu(\beta x^\gamma)$,
where $J_\nu$ and $Y_\nu$ are the Bessel functions of the first and second kind respectively, with $\alpha = \frac{\nu}{2}$, $\beta = |a|$, and $\gamma = 1/2$.
We obtain the following solutions for the equation (12):
$v_\nu^1(z) = z^{\nu/2}J_\nu(|a|z^{1/2})$ and $v_\nu^2 = z^{\nu/2}Y_\nu(|a|z^{1/2})$,
The solutions of the generalized wave equation $(w)$ in (2) are
$u_\nu^1(z) = (-1)^{\nu/2}A^{-\nu}z^{\nu/2}J_\nu(|a|z^{1/2})$,
$u_\nu^2 = (-1)^{\nu/2}A^{-\nu}z^{\nu/2}Y_\nu(|a|z^{1/2})$,
and using the fact that

$$z = -(x - x'e^t)(x - x'e^{-t}), \tag{15}$$

we have:

$$u_\nu^1(t,x,x') = \left(\frac{x - x'e^{-t}}{x - x'e^t}\right)^{\frac{\nu}{2}} J_\nu(|a|\sqrt{2xx'\cosh t - x^2 - x'^2}) \tag{16}$$

and

$$u_\nu^2(t,x,x') = \left(\frac{x - x'e^{-t}}{x - x'e^t}\right)^{\frac{\nu}{2}} Y_\nu(|a|\sqrt{2xx'\cosh t - x^2 - x'^2}), \tag{17}$$

and the proof of the Proposition is finished. □

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# 3 Cauchy problem for the generalized wave equation

In this section we give the explicit solution to the generalized Cauchy problem (2), for this we prove the following Lemmas:

**Lemma 0.2.** *Let $z = 2xx'\cosh(t) - x^2 - x'^2$ as in the Lemma 0.1, then we have:*
*i)* $z = 4xx'\sinh(t/2) - (x-x')^2 = -(x - x'e^t)(x - x'e^{-t})$.
*ii)* $z \geq 0$ *if and only if* $xe^{-t} < x' < xe^t$.
*iii) Setting* $x = e^X$ *and* $x' = e^{X'}$, *then we have:*
$Z(t,X,X') = 4e^{X+X'}\left(\sinh^2(\frac{t}{2}) - \sinh^2(\frac{X'-X}{2})\right)$
*iv) Setting*

$$X' = X - 2\arg sh(s\sinh(\frac{t}{2})), \tag{18}$$

*then we have:*

$$Z(X,s,t) = 4e^{2X-2\arg\sinh(s\sinh t/2)}\sinh^2\frac{t}{2}(1-s^2). \tag{19}$$

The proof of this lemma is simple and in consequence is left to the reader.

**Lemma 0.3.** *Let $Z$ given by the formula (19) then we have*
$iii) Z(t) \sim_0 t^2 e^{2X}(1-s^2)$
$iv) Z'(t) \sim_0 2te^{2X}(1-s^2)$
$v) z^{-\nu/2}J_\nu(|a|z^{\nu/2}) \sim_0 \frac{2^{-\nu}}{\Gamma(1+\nu)}|a|^\nu$
$vi) \frac{\partial}{\partial t} z^{\nu/2}J_\nu(|a|z^{1/2}) = \frac{|a|^\nu}{2^{\nu-1}\Gamma(\nu)}e^{2\nu X}(1-s^2)^\nu t^{2\nu-1}$.
*Let* $G(s,t,X,U_0) = \frac{U_0(X-2\arg\sinh(s\sinh(t/2)))}{e^{\nu(X-2\arg\sinh(s\sinh(t/2)))}\sqrt{1+s^2\sinh^2(t/2)}}$,
*with* $U_0 \in C_0^\infty(\mathbb{R}^+)$ *then we have*
$vii) G(s,t,X,U) \sim_0 \frac{U_0(X)}{e^{\nu X}}$
$viii) \frac{\partial}{\partial t} G(s,t,X,U) \sim_0 \frac{-sU_0'(X)+\nu U_0(X)}{e^{\nu X}}$.

International Journal of Applied Mathematics
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*Proof.* The parts i), ii), iii) and iv) are simples. To see v) we use essentially the formulas see Lebedev [10] p.134

$$J_\alpha(z) \sim_0 \frac{2^{-\alpha}}{\Gamma(1+\alpha)} z^\alpha, \tag{20}$$

and the formula (19). For the proof of vi) we recall the formula see Magnus [11] p. 67

$$\left(\frac{d}{xdx}\right)^m [x^{-\nu} J_\nu(x)] = (-1)^m x^{-\nu-m} J_{\nu+m}(x), \tag{21}$$

to obtain

$$\frac{d}{dx}[x^{-\nu} J_\nu(|a|x)] = -|a| x^{-\nu} J_{\nu+1}(x) \tag{22}$$

and this formula gives

$$\frac{d}{dt} z^{-\nu/2} J_\nu(|a|z^{1/2}) = -|a| z^{-\nu/2} J_{\nu+1}(|a|z^{1/2}) \frac{\partial z^{1/2}}{\partial t} \tag{23}$$

or

$\frac{\partial}{\partial t} z^{\nu/2} J_\nu(|a|z^{1/2}) = \frac{1}{2}|a| z^{(\nu-1)/2} J_{\nu-1}(|a|z^{1/2}) \frac{dz}{dt}$

and finally we use iv) to have the result of vi). The parts vii) and viii) are evident, and the proof of the Lemma is finished. □

**Theorem 1.** *For $a \in \mathbb{R}^\star$ and $-1 < \nu < 1$, the wave problem (2) has the unique solution given by:*

$$u(t,x) = \int_{xe^{-t}}^{xe^t} w_a^\nu(t,x,x') u_0(x') \frac{dx'}{x'^{1+\nu}}, \tag{24}$$

*with*

$$w_a^\nu(t,x,x') = c_\nu \left(\frac{x - x'e^{-t}}{x - x'e^t}\right)^{\frac{\nu}{2}} J_\nu(|a|\sqrt{2xx'\cosh t - x^2 - x'^2}), \tag{25}$$

*and*

$$c_\nu = \frac{(-1)^{\nu/2}\Gamma(\nu+1)}{2|a|^\nu}, \tag{26}$$

*and $J_\nu$ is the Bessel function of the first kind.*

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*Proof.* Setting $x = e^X$ and $x' = e^{X'}$, in the integral (24), we obtain:

$$U(t,X) = \int_{X-t<X'<X+t} W_a^\nu(t,X,X')U_0(X')\frac{dX'}{e^{\nu X'}}, \qquad (27)$$

with
$W_a^\nu(t,X,X') = c_\nu \left(\frac{e^X - e^{X'-t}}{e^X - e^{X'+t}}\right)^{\nu/2} \times$

$$J_\nu\left(2|a|e^{(X+X')/2}\sqrt{\sinh^2 t/2 - \sinh^2\frac{(X-X')}{2}}\right),$$

where $c_\nu$ is as in (26) and $J_\nu$ are the Bessel function of the first kind.

Setting $X' = X - 2\arg sh(s\sinh(\frac{t}{2}))$ in the formula (27), we obtain:

$$U(t,X) = \int_{-1}^{1} w_a^\nu(s,t,X)ds, \qquad (28)$$

with
$w_a^\nu(s,t,X) =$

$$(-1)^{\nu/2}c_\nu A^{-\nu}z^{\nu/2}J_\nu(|a|z^{1/2})G(s,t,X,U)2\sinh(t/2), \qquad (29)$$

with $A, Z$ and $G$ are as in the Lemma 0.2.

Using the Lemma 0.3, we have

$$U(t,X) \sim_0 (-1)^{-\nu/2}c_\nu\frac{|a|^\nu}{2^\nu\Gamma(\nu+1)}t^{\nu+1}U_0(X)\int_{-1}^{1}(1+s)^\nu ds,$$

and

$$t^{-\nu}U(t,X) \sim_0 (-1)^{-\nu/2}c_\nu\frac{|a|^\nu}{2^\nu\Gamma(\nu+1)}tU_0(X)\int_{-1}^{1}(1+s)^\nu ds.$$

and it is clear that the first condition $(\alpha)$ in the problem (2) is satisfied.

International Journal of Applied Mathematics
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To prove the second condition $(\beta)$ in the problem (2), we have:

$$\begin{aligned}\frac{d}{dt}\big[w_a^\nu(s,t,X)\big] &= (-1)^{\nu/2}c_\nu\Big[\frac{d}{dt}\big(A^{-\nu}\big)\, z^{\nu/2}J_\nu(|a|z^{1/2})G(s,t,X,U)\,2\sinh(t/2)\\ &\quad + A^{-\nu}\,\frac{d}{dt}\big(z^{\nu/2}J_\nu(|a|z^{1/2})\big)\,G(s,t,X,U)\,2\sinh(t/2)\\ &\quad + A^{-\nu}z^{\nu/2}J_\nu(|a|z^{1/2})\frac{d}{dt}(G(s,t,X,U))\,2\sinh(t/2)\\ &\quad + A^{-\nu}z^{\nu/2}J_\nu(|a|z^{1/2})G(s,t,X,U)\,\frac{d}{dt}(2\sinh(t/2))\Big]\\ &= (-1)^{\nu/2}c_\nu\,\{C_1+C_2+C_3+C_4\}.\end{aligned}$$

and using the Lemma 0.3 we obtain

$$C_1 \sim_0 (-1)^{-\nu+1}\frac{|a|^\nu}{2^\nu\Gamma(\nu)}t^\nu U_0(X)(1+s)^\nu,$$
$$C_2 \sim_0 (-1)^{-\nu}\frac{|a|^\nu}{2^{\nu-1}\Gamma(\nu)}t^\nu U_0(X)(1+s)^\nu,$$
$$C_3 \sim_0 o(t^{\nu+1}),$$
$$C_4 \sim_0 (-1)^{-\nu}\frac{|a|^\nu}{2^\nu\Gamma(\nu+1)}t^\nu U_0(X)(1+s)^\nu,$$

$$C_1+C_2+C_3+C_4$$
$$\sim_0$$
$$(-1)^{-\nu}(-\frac{|a|^\nu}{2^\nu\Gamma(\nu)}+\frac{|a|^\nu}{2^{\nu-1}\Gamma(\nu)}+\frac{|a|^\nu}{2^\nu\Gamma(\nu+1)})t^\nu U_0(X)(1+s)^\nu$$
$$(-1)^{-\nu}\frac{|a|^\nu}{2^\nu\Gamma(\nu)}t^\nu U_0(X)(1+s)^\nu[$$

$$\begin{aligned}\frac{d}{dt}[w_\nu(s,t,X)] &= (-1)^{-\nu/2}c_\nu\frac{|a|^\nu}{2^\nu\Gamma(\nu)}(1+1/\nu)t^\nu U_0(X)(1+s)^\nu\\ &= (-1)^{-\nu/2}c_\nu\frac{|a|^\nu}{2^\nu\Gamma(\nu)}(1+1/\nu)t^\nu U_0(X)(1+s)^\nu\\ &= (-1)^{-\nu/2}c_\nu\frac{|a|^\nu}{2^\nu\Gamma(\nu)}[(\nu+1)/\nu]t^\nu U_0(X)(1+s)^\nu.\end{aligned}$$

International Journal of Applied Mathematics
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$$\lim_{t\longrightarrow 0} t^{-\nu}U_t(t,X)$$
$$\lim_{t\longrightarrow 0} t^{-\nu}U_t(t,X) =$$
$$(-1)^{-\nu/2}c_\nu \frac{|a|^\nu}{\Gamma(\nu)2^\nu}[(\nu+1)/\nu]U_0(X)\int_{-1}^{1}(1+s)^\nu ds$$
$$= (-1)^{-\nu/2}c_\nu \frac{|a|^\nu}{2^\nu\Gamma(\nu)}[(\nu+1)/\nu]\frac{2^{\nu+1}}{\nu+1}U_0(X)$$
$$= c_\nu \frac{(-1)^{-\nu/2}2|a|^\nu}{\Gamma(\nu+1)}U_0(X) = U_0(X).$$

The proof of the theorem is finished. □

# 4 Generalized wave equation with Morse potential

The importance of the Schrodinger operator with the Morse potential in both theory and application of mathematics and physics may be found in literature (see for example Fidiani E. [4], Hassanabadi H. and Zare S., [5], and Ikeda N., Matsumoto H. [6]). Note that the purely vibrational levels of diatomic molecules with angular momentum l = 0 have been described by the Morse potential since 1929, see P. Morse [13].

This section is devoted to the following generalized wave problems

$$\begin{cases} (w)'\, M_a U(t,X) = B_\nu U(t,X), (t,X) \in \mathbb{R}^2_+ \\ (a)'\lim_{t\to 0} t^{-\nu}U(t,X) = 0, \\ (b)'\lim_{t\to 0} t^{-\nu}U_t(t,X) = U_0(X), U_0 \in C_0^\infty(\mathbb{R}^+). \end{cases} \tag{30}$$

associated to the second order differential operator of the Morse type:

$$M^a = \frac{\partial^2}{\partial X^2} - a^2 \exp 2X. \tag{31}$$

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**Theorem 2.** *For a real number the problem* (30) *has the unique solution given by*

$$U(t,X) = \int_{|X-X'|<t} W_a^{\nu}(t,X,X')U_0(X')\frac{dX'}{e^{\nu X'}} \tag{32}$$

*with*

$W_a^{\nu}(t,X,X') = c_\nu \left(\frac{e^X - e^{X'-t}}{e^X - e^{X'+t}}\right)^{\nu/2} \times$

$$J_\nu\left(2|a|e^{(X+X')/2}\sqrt{\sinh^2 t/2 - \sinh^2\frac{(X-X')}{2}}\right), \tag{33}$$

*where $c_\nu$ is as in* (26) *and $J_\nu$ are the Bessel function of the first kind*

*Proof.* By change of variables $x = e^X$ the problem (30)is transformed in the problem (2). □

# 5 Generalized wave equation on the hyperbolic space

The wave equation on the hyperbolic space is solved explicitly at the first times by P. D, Lax et R. S, Phillips [9], see also A. Intissar and M. V. Ould Moustapha [7] and [8]. Here we give the solution of the generalized wave equation on the hyperbolic space, using the solution of the generalized wave equation associated to the Bessel operator given in theorem 1.

In this paper The hyperbolic space is modeled by the upper half space $\mathbb{H}^n = \{\, w = (x_1, x_2, \ldots, x_{n-1}, x_n) = (x, x_n) \in \mathbb{R}^n \;:\; x_n > 0 \,\}.$, with the hyperbolic metric

$$ds^2 = \frac{dx_1^2 + dx_2^2 + ... + dx_{n-1}^2 + dx_n^2}{x_n^2}. \tag{34}$$

Note that the metric $ds$ is invariant with respect to the group

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$G = SO(n,1)$. The associated hyperbolic volume form $d\mu(w)$ is given by:

$$d\mu(w) = \frac{dx_1 dx_2 ... dx_{n-1} dx_n}{x_n^n}, \tag{35}$$

and the hyperbolic distance $\rho(w, w')$ between two points $w$ and $w'$ in the hyperbolic space $\mathbb{H}^n$ is given by:

$$\cosh^2(\rho(w, w')/2) = \frac{|x - x'|^2 + x_n^2 + x_n'^2}{2x_n x_n'}, \tag{36}$$

with the Laplace-Beltrami operator

$$\mathcal{L}_n = x_n^2 \Delta_{n-1} + (2-n)\frac{\partial}{\partial x_n} + ((n-1)/2)^2, \tag{37}$$

and

$$\Delta_{n-1} = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}. \tag{38}$$

The main objective of this section is to solve the following generalized wave problems

$$\begin{cases} (w)'' \, \mathcal{L}_n \Phi(t, w) = B_\nu \Phi(t, w), (t, w) \in \mathbb{R}_+ \times H^n \\ (a)'' \lim_{t \to 0} t^{-\nu} \Phi(t, w) = 0, \\ (b)'' \lim_{t \to 0} t^{-\nu} \Phi_t(t, w) = \Phi_0(w), \Phi_0 \in C_0^\infty(H^n). \end{cases} \tag{39}$$

associated to the Laplace-Beltrami operator $\mathcal{L}_n$.
Now we recall some fact about the Fourier transform:

For $f \in L^1(\mathbb{R}^n)$, the Fourier and inverse Fourier transforms of $f$ are given by

$$(\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} f(x) \exp(-i\xi.x) dx. \tag{40}$$

$$(\mathcal{F}^{-1}f)(x) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} f(\xi) \exp(i\xi.x) d\xi. \tag{41}$$

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**Lemma 2.1.** *Let $L^a$ be the Bessel operator in* (1), *then for a real $\alpha$ we have*

- *i)* $x^{-\alpha}L^a x^\alpha = x^2\frac{\partial^2}{\partial x^2} + (2\alpha+1)x\frac{\partial}{\partial x} + \alpha^2 - a^2x^2.$
- *ii)For radial function $f \in L^1(\mathbb{R})$*

$$(\mathcal{F}^{-1}f)(x) = |x|^{1-\frac{n}{2}} \int_0^{+\infty} f(r) J_{\frac{n}{2}-1}(r\,|x|) r^{\frac{n}{2}} dr. \tag{42}$$

- *iii)* $\mathcal{F}(f(x)g(x))(\xi) = (2\pi)^{-n}(\mathcal{F}f * \mathcal{F}g)(\xi).$
- *iv)we have the following formula intertwining the Laplace Beltrami operator $\mathcal{L}_n$ on the hyperbolic space $\mathbb{H}^n$ and the Bessel operator $L^{|\xi|}$*

$$\mathcal{F}\left[x_n^{(n-1)/2}\mathcal{L}_n x_n^{(1-n)/2}\Phi\right](\xi) = L^{|\xi|}\mathcal{F}\Phi(\xi). \tag{43}$$

*Proof.* the first part $i$) is simple, for the second part $ii$) see [16] $p.226$, and the part $iii$) is given in [15] $p.34$.
A simple computation, using the part $i$) of the Lemma and some Fourier transform properties, gives the part $iv$). □

**Lemma 2.2.**
- *i) Set* $\int_0^{+\infty} J_\mu(at)J_\nu(bt)t^{\mu-\nu+1}dt = I(a,b,\rho,\mu,\nu)$ *then we have:*
$I(a,b,\rho,\mu,\nu) = \frac{2^{\mu-\nu+1}a^\mu b^{-\nu}}{\Gamma((\nu-\mu)}(b^2-a^2)^{\nu-\mu-1}, b > a.$
$I(a,b,\rho,\mu,\nu) = 0, b < a.$
- *ii) Let $w^\nu_{|\xi|}(t,x_n,x'_n)$ be the wave kernel for the generalized wave equation for the Bessel operator given by* (25), *then we have*
$\mathcal{F}^{-1}\left[w^\nu_{|\xi|}(t,x_n,x'_n)\right](x) =$
$K_\nu(x_n - x'_n e^t)^{-\nu}\left(2x_n x'_n\cosh t - x_n^2 - x'^2_n - |x|^2\right)_+^{\nu-(n-1)/2},$
*with*
$K^n_\nu = (-1)^\nu\frac{2^{(n-3)/2-\nu}\Gamma(\nu+1)}{\Gamma(\nu-(n-3)/2)}.$

International Journal of Applied Mathematics
Volume 38 No. 12s, 2025
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*Proof.* For i) see Magnus et al. [11] p.99. To prove the part ii), recall that the function $w^{\nu}_{|\xi|}(t, x_n, x'_n)$ is radial in $\xi = (\xi_1, \xi_2, ..., \xi_{n-1})$, using the part ii) of the Lemma 2.1 we can write:
$\mathcal{F}^{-1}\left[w^{\nu}_{|\xi|}(t, x_n, x'_n)\right](x) =$
$|x|^{(3-n)/2} \int_0^{\infty} w^{\nu}_r(t, x_n, x'_n) J_{(n-3)/2}(r|x|) r^{(n-1)/2} dr.$
$= c_\nu |x|^{(3-n)/2} (-1)^{\nu/2} A^{-\nu} z^{\nu/2} \times$
$\int_0^{+\infty} J_\nu(rz^{1/2}) J_{(n-3)/2}(r|x|) r^{(n-1)/2-\nu} dr =$
with $A(t)$ and $z$ are as in the 0.1. Using the first part $i$), with $\nu = \nu$, $\mu = (n-3)/2$, $a = |x - x'|$, and $b = z^{1/2}$, we obtain

$\mathcal{F}^{-1}\left[w^{\nu}_{|\xi|}(t, x_n, x'_n)\right](x) =$
$c_\nu (-1)^{\nu/2} |x|^{(3-n)/2} z^{\nu/2} |x|^{(n-3)/2} z^{-\nu/2} \frac{2^{(n-1)/2-\nu}}{\Gamma(\nu-(n-3)/2)}$
$A^{-\nu} \left(2x_n x'_n \cosh t - x_n^2 - x'^2_n - |x|^2\right)^{\nu-(n-1)/2}.$
$= K^n_\nu A^{-\nu} \left(2x_n x'_n \cosh t - x_n^2 - x'^2_n - |x|^2\right)^{\nu-(n-1)/2}.$
and the proof of the Lemma is finished. □

**Corollary 3.** *The Cauchy problem for the generalized wave equation on the hyperbolic space, $\mathbb{H}^n$, has the unique solution given by*

$$\Phi(t, w) = \int_{H^n} w^{\nu}_n(t, w, w') \Phi_0(w') d\mu(w'), \tag{44}$$

*with*
$w^{\nu}_n(t, w, w') = K^n_\nu \frac{(x_n x'_n)^\nu}{(x_n - x'_n e^t)^\nu} \left(\cosh t - \cosh \rho(w, w')\right)^{\nu-(n-1)/2}$
*with*
$C^n_\nu = \frac{\Gamma(\nu)}{2\pi^{(n-1)/2}\Gamma(\nu-(n-3)/2)}.$

*Proof.* In view of the part iv) of the Lemma 2.1, the wave problem on the hyperbolic space (39) is transformed into the Cauchy problem for the generalized wave equation (2), with
$u(t, x_n) = \mathcal{F}\left[x_n^{(1-n)/2} \Phi(t, x, x_n)\right](\xi)$ and $u_0(x_n) = x_n^{(1-n)/2} \mathcal{F}[\Phi_0(x, x_n)](\xi)$.
that is:

International Journal of Applied Mathematics
Volume 38 No. 12s, 2025
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$$\mathcal{F}\left[x^{(1-n)/2}\Phi(t,x,x_n)\right](\xi) = \int_0^\infty w^{\nu}_{|\xi|}(t,x_n,x'_n)x'^{(1-n)/2}_n x^{(1-n)/2}\mathcal{F}\left[\Phi_0\right](\xi,x'_n)\frac{dx'_n}{x'_n},$$

and

$$\Phi(t,x,x_n) = \int_0^\infty \mathcal{F}^{-1}\left[w^{\nu}_{|\xi|}(t,x_n,x'_n)x'^{(1-n)/2}_n\mathcal{F}\left[\Phi_0\right](\xi,x'_n)\right](x)\frac{dx'_n}{x'_n}, \tag{45}$$

$$\Phi(t,x,x_n) = (2\pi)^{-(n-1)/2}\times \int_0^\infty \mathcal{F}^{-1}\left[w^{\nu}_{|\xi|}(t,x_n,x'_n)\right](x) * \Phi_0(x,x'_n)x'^{(1-n)/2}_n\frac{dx'_n}{x'_n}, \tag{46}$$

thus

$$\begin{aligned}&\Phi(t,x,x_n)\\ &= (2\pi)^{-(n-1)/2}\int_0^\infty\int_{\mathbb{R}^{n-1}}\mathcal{F}^{-1}\left[w^{\nu}_{|\xi|}(t,x_n,x'_n)\right]\\ &\times(x-x')x_n^{(n-1)/2}x'^{(n-1)/2}_n\times\Phi_0(x',x'_n)\frac{dx'dx'_n}{x'^n_n},\end{aligned}$$

$$\Phi(t,w) = \int_{\mathbb{H}^n} w^{\nu}_n(t,w,w')\Phi_0(w')d\mu(w').$$

□

**Corollary 4.** *The classical wave kernel in hyperbolic space is given by*

$$w^0_n(t,w,w') = c_n(\cosh t - \cosh\rho)_+^{(1-n)/2}. \tag{47}$$

*with* $c_n = \frac{1}{2\pi^{(n-1)/2}\Gamma(\nu-(n-3)/2)}$.

*Proof.* For $n$ even

$$w_n(t,w,w') = \frac{1}{(2\pi)^{n/2}}(\cosh t - \cosh\rho)_+^{(1-n)/2}. \tag{48}$$

International Journal of Applied Mathematics
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For $n$ odd using the formulas see Shillov [14], p. 120

$$\frac{x_+^\lambda}{\Gamma(\lambda+1)}|_{\lambda=-n} = \delta^{(n-1)}(x). \tag{49}$$

to obtain:

$$w_n(t, w, w') = \frac{1}{(2\pi)^{(n-1)/2}} \delta^{(n-3)/2}(t-\rho). \tag{50}$$

with $\delta$ is the Dirac measure. Note that this same solution given in Lax and Phillips [9] sec.7. □

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