International Journal of Applied Mathematics

Volume 38 No. 12s, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

SOME NEWFANGLED RESULTS IN CONNECTED INTEGRITY IN GRAPHS

K.Sorna Devi[1], M.Bhuvaneshwari[2],Selvam Avadayappan[3]

[1].Department of Mathematics,

Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi-630611,India

[2,3].Research Department of Mathematics, VHN Senthikumara Nadar College,

(Affiliated to Madurai Kamaraj University)

Virudhunagar - 626001(India)

Abstract

Facility Location Problem is a traditional optimization problem to decide the best location of a service spot in order to fulfill the needs of the entire vicinity. The optimal solution to this problem is reached by means of graph models. During the period of contagious diseases like COVID'19, the primary aim is to isolate the nodes of the network to a greater extend. Also while breaking the chain, one has the objective that the victim's region must be reduced as small as possible which in turn results in the less probability of spreading the disease. Both motives merge together in arranging medical camps in severely affected places. The foremost intention will be minimizing the cost of arranging medical camps and maximizing the easiness of medicine supply among them. The study on a graphical parameter named as Integrity yields an optimal solution to this problem. In order to share the availability of medicines in the camps, they must be inter connected. Motivated by this situation, the parameter namely, Connected Integrity, defined by $CI(G) = \min_{S \subset V(G)} \{ |S| + m(G-S) \}$ where the graph induced by S is connected and $m(G - S)$ denotes the order of the largest component of $G - S$ is introduced. In this paper, we pay attention to study the properties of CI sets, realize graphs on CI values, generate CI values for trees and characterize trees with CI value 4. Also we introduce the derived graph namely ζ-Splitting graph.

Keywords : Integrity, Vulnerability, Connected Integrity, Splitting graph

AMS Subject Classification Number (2020) : 05C(Primary)

Received: August 11, 2025

International Journal of Applied Mathematics
Volume 38 No. 12s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

1 Introduction

In this paper, G denotes only finite, simple and connected graph. Basic notations and terminology can be referred from [3]. δ and Δ represent the minimum and maximum degree of vertices in a graph respectively. α , κ denote the vertex covering number and vertex connectivity of G respectively. $\beta(G)$ represents the independence number of G. A vertex which is adjacent to all the remaining vertices in a graph is termed as a *full vertex*. A vertex with degree 1 is named as a *pendant vertex* and the edge incident to it is termed as a *leaf* and its unique neighbor is a *support*. $N(v)$, the neighbourhood set of v, denotes the set of all vertices which are adjacent to v. $< S >$ denotes the subgraph induced by S whose vertex set is S and edge set is $E(G) \cap (S \times S)$. A graph which contains unique cycle is said to be *unicyclic*. A *walk* is an alternating sequence of vertices and edges. A walk is said to be a *closed walk* if its initial and terminal vertices coincide. A closed walk with distinct edges is called a *closed trail*. A closed trail is said to be an *Eulerian trail* if it traverses through every edge of G exactly once. A graph G having an *Eulerian trail* is called an *Eulerian graph*. An edge is said to be *subdivided* if it is replaced by P_3. A graph is said to be *r-regular* if degree of v is r for all the vertices v in G. A graph is said to be *biregular* if $\{d(v)/v \in V(G)\} = \{m,k\}$ where m and k are non negative integers. The distance between two vertices u and v, denoted by $d(u,v)$ is the length of a shortest path between them. The *Eccentricity* of a vertex v is defined as $e(v) = \max\{d(u,v)/u \in V(G)\}$. Minimum(maximum) of eccentricities of all the vertices is termed as *radius(diameter)* of G denoted by $rad(G)(diam(G))$. A vertex v is said to be a *central vertex* of G if $e(v) = rad(G)$. The set of all central vertices of G is called the *center of G* denoted as $cen(G)$. A graph G is said to be *self centered* if every vertex of G is a central vertex of G. Let G and H be two graphs of order m and k respectively. A subset S of $V(G)$ is a *vertex cut* if G-S is disconnected. Bistar $B(r,t)$ is the graph obtained by joining r leaves at one end and t leaves at the other end of K_2. A double broom graph $B(d,r,t)$ is obtained from a path of length d by joining r leaves at one of its end and t leaves at the other end. A full binary tree T_t of height t is a rooted tree such that each vertex except the pendant vertices has 2 children and all the pendant vertices are at a distance t from the root. The *join* of two graphs G and H, denoted by $G + H$ is formed from disjoint copies of G and H such that $N_{G+H}(v) = N_G(v) \cup V(H)$ for all v in $V(G)$. Let G be a graph of order n and H be a graph of order m. Then the Corona graph $G \circ H$ is obtained from G by taking n copies of H and joining all the vertices in the i^{th} copy of H to the i^{th} vertex of G. Let

International Journal of Applied Mathematics
Volume 38 No. 12s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
G_1 and G_2 be two connected graphs with vertex sets $V_1 = \{u_1,u_2,...u_n\}$ and $V_2 = \{v_1,v_2,...v_m\}$ respectively. Then the graph $G_1 \times G_2$ has the vertex set $V_1 \times V_2$ and any two vertices (u_i,v_j) and (u_k,v_s) are adjacent in $G_1 \times G_2$ if ($u_i = u_k$ and v_j is adjacent to v_s) or (u_i is adjacent to u_k and $v_j = v_s$) Let G_1 and G_2 be any two connected graphs with $cen(G_1) \cong cen(G_2)$. Then center fusion of G_1 and G_2, denoted by $G_1 \Theta G_2$ is defined to be a graph with vertex set $V = V_1 \cup V_2 \setminus V(cen(G_1))$. If f is an isomorphism from $cen(G_1)$ to $cen(G_2)$ and $v_i \in cen(G_1)$, take $w_i = f(v_i)$, then edge set of $G_1 \Theta G_2$ is $E = E_1 \cup E_2 \cup \{ww_i / wv_i \in E_1 \ \& \ w \in V_1 \setminus V(cen(G_1))\}$.

Among the various applications of Graph Theory, the one that is perhaps most often discussed is Network Problem. Communication network problem is one of the renowned problems which was discussed in many literature and solved by countless graphical parameters. One such parameter is Integrity, introduced by Barefoot[4] and its formal definition is $I(G) = \min_{S \subset V(G)} \{|S| + m(G-S)\}$ where $m(G\text{-}S)$ denotes the order of the largest component of $G - S$. One can refer [1,2,5,6,7] to enhance the study in Integrity in graphs.

On this line, we introduce a new parameter namely Connected Integrity which is defined as $CI(G) = \min_{S \subset V(G)} \{|S| + m(G-S)\}$ where $< S >$ is connected and $m(G - S)$ denotes the order of the largest component of $G - S$. CI set is the set $S \subset V(G)$ for which $|S| + m(G-S) = CI(G)$.

For more results on the parameter one can refer [14]. Also the value of the parameter on special graphs has been studied in [13]. In addition Connected Integrity Polynomial [15] has been developed and analyzed.

In this paper, we intend to study some properties of CI set and realizations of graphs with given CI values. Also we present some results in trees in Section *2* and we initialize the study on ζ-Splitting graphs in Section *3*.

Let us recall the following facts and theorems from [13] and [14].

Fact 1[13] $2 \le CI(G) \le n$, for any nontrivial connected graph G.

Fact 2[13] For any cycle C_n, $n \ge 3$, $CI(C_n) = n$.

Fact 3[13] Let T be a tree. Then $CI(T) = n$ *iff* $T \cong P_2$.

Fact 4[13] $CI(K_{m,n}) = \min\{m, n\}+2$, *for* $m,n \ge 2$.

Fact 5[13] For $n \ge 2$, $CI(K_n) = n$.

Received: August 11, 2025

International Journal of Applied Mathematics
Volume 38 No. 12s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Fact 6[13] For $n \geq 2$, $CI(K_{1,n}) = 2$.

Fact 7[14] The Connected Integrity of path P_n is $CI(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

Fact 8[14] $CI(W_{1,n}) = \left\lceil 2\sqrt{n} \right\rceil$, for any wheel graph $W_{1,n}$.

Theorem A[14] If G is a cycle with a chord, then $CI(G) = n - g(G) + 2$ where $g(G)$ denotes the girth of G.

Theorem B[14] For any graph G, $CI(G) \geq \delta(G) + 1 \geq \kappa(G) + 1$. The bound is sharp for $K_{1,n}$.

Theorem C[14] Let G be any connected graph. Then $CI(G) = 3$ if and only if G is isomorphic to any one of the following: *(i)* $S_1(K_{1,n})$ *(ii)* F_m *(iii)* $F_m \Theta S_1(K_{1,n})$ *(iv)* $F_m \Theta K_{1,n}$ *(v)* $F_m \Theta S_1(K_{1,n}) \Theta K_{1,r}$ *(vi)* $S_1(K_{1,n}) \Theta K_{1,r}$ *(vii)* Cr_n *(viii) Bistar (ix)* Cr_n with some pendant vertices at both or one of its central vertices.

Theorem D[14] For any tree T of order n, $CI(T) \leq n - rad(T) + 1$.

Theorem E[14] For any tree, $CI(T) \leq \left\lceil \frac{n}{2} \right\rceil + 1$.The inequality is sharp and it is attained by even paths.

2 Main Results

Analyzing the properties of *CI* sets enables us to ease the operations in them. As an initial step towards it, we discuss the following characteristics of *CI* sets.

Proposition 2.1 Let G be a connected non complete graph with $\Delta(G) = n-1$. Then any *CI* set of G contains all the full vertices of G.

Proof If any *CI* set of G does not contain a full vertex of G, then $CI(G) = n$, which is a contradiction, since G is not isomorphic to K_n. ■

Theorem 2.2 The complement of a *CI* set of a graph G is a *CI* set iff G is isomorphic to K_n or C_n.

Proof Let S be a *CI* set of G. Then $CI(G) = |S| + m(G - S)$. Assume that S^c is also a *CI* set of G so that $CI(G) = |S^c| + m(G - S^c)$. $G - S = \langle S^c \rangle$ which is connected or trivial. Hence $m(G - S) = n - |S|$ yielding $|S| + m(G - S) = n$, that is $CI(G) = n$ which gives that $G \cong K_n$ or $G \cong C_n$. Converse is obvious. ■

Received: August 11, 2025

International Journal of Applied Mathematics
Volume 38 No. 12s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Theorem 2.3 Any graph *G* of order *n* is an induced subgraph of a graph *H* such that *V(G)* is a *CI* set of *H*.

Proof Given any graph *G* of order *n* with vertices $v_1, v_2, \ldots v_n$. Consider the graph *H* with $V(H) = \{v_1, v_2, \ldots v_n\} \cup \{u\ , u_2, \ldots u_n\} \cup \{w_1, w_2, \ldots w_n\}$ and

$E(H) = E(G) \cup \{u_i u_j, w_i w_j, v_i u_j, v_i w_j / 1 \le i, j \le n\}$. Then *H* is nothing but *(2K_n) + G* and *V(G)* serves as a *CI* set for *H* so that *CI(H) = 2n*. ■

There do exist graphs with any given positive integer, as its required parameter. Here we can discuss some realization theorems.

Theorem 2.4 Given any positive integer *m(2 ≤ m ≤ n)*, there exists a connected graph *G* on *n* vertices with *CI(G) = m*.

Proof Suppose $2 \le m \le \left\lceil \frac{n}{2} \right\rceil$. Consider the graph *G*, with $V(G) = \{v_1, v_2, \ldots v_n\}$ and $E(G) = \{v_i v_{i+1}, 1 \le i \le m-1\} \cup \{v_m v_{j,} m+1 \le j \le n\}$. *G* is nothing but a path P_m in which a pendant vertex is fused with central vertex of $K_{1,n-m}$. Then $S = \{v_m\}$ will be the *CI* set yielding *CI(G) = 1 + m - 1 = m*. Any set other than *S* will yield a greater value than *m*. A graph with *m = 5* and *n = 10* is shown below.

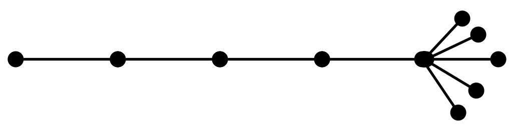


Figure *1*

Suppose $\left\lceil \frac{n}{2} \right\rceil + 1 \le m \le n-1$ .Consider the graph *G* with $V(G) = \{v_1, v_2, \ldots v_n\}$ and $E(G) = \{v_i v_{i+1}, 1 \le i \le n-1\} \cup \{v_1 v_m\}$. Then $S = \{v_1, v_m\}$ will be a *CI* set such that *CI(G) = 2 + m - 2 = m*. A graph with *m = 8* and *n = 10* is shown below.

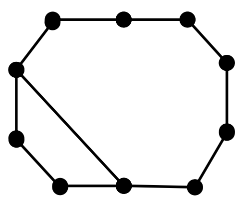


Figure *2*

Received: August 11, 2025

International Journal of Applied Mathematics
Volume 38 No. 12s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
If $m = n$, then C_n is the desired graph with $CI(C_n) = n$. ■

Theorem 2.5 Given any integer $m(m \geq 5)$, there exists a graph G with $CI(G) = \alpha = m$.

Proof Let the vertex set of the required graph G be $V(G) = \{v_1, v_2, \ldots v_{2m-1}\}$ and the edge set be $E(G) = \{v_i v_{i+1}, 1 \leq i \leq 2m-2\} \cup \{v_1 v_{m-1}\} \cup \{v_1 v_{m+2}\}$. Then $K = \{v_1, v_3, v_5, \ldots v_{2m-1}\}$ is a minimum vertex cover for G giving $\alpha(G) = m$ and $S = \{v_1, v_{m-1}, v_{m+2}\}$ is a CI set yielding $CI(G) = |S| + m(G - S) = 3 + m - 3 = m = \alpha$. Any set other than S will yield a value greater than α. Hence $CI(G) = \alpha = m$. ■

Theorem 2.6 Given any integer $k(k \geq 1)$, there exists a graph G with $CI(G) = \delta(G) + k$.

Proof Consider the graph G with the vertex set

$V(G) = \{v_1, v_2, \ldots v_m\} \cup \{w\} \cup \{u_1, u_2, \ldots u_k\}$ and the edge set

$E(G) = \{v_i v_j / i \neq j, 1 \leq i, j \leq m\} \cup \{u_i u_j / i \neq j, 1 \leq j \leq k\} \cup \{v_i w / 1 \leq i \leq m\} \cup \{v_i u_j / 1 \leq i \leq m, 1 \leq j \leq k\}$.

.That is G is $K_m + (K_1 \cup K_k)$ so that $\delta(G) = m$. Let $S = \{v_1, v_2, \ldots v_m\}$. $|S| + m(G - S) = m + k = \delta(G) + k$. Any set other than S will not yield a lesser value than $\delta(G) + k$. ■

Theorem 2.7 For any given positive integer $m \geq 2$ and any graph G of order $n \leq m - 1$, there exists a graph H containing G as an induced subgraph with $CI(H) = m$.

Proof Consider the given graph G with n vertices. Construct the graph $H \cong K_{m-n} \circ G$. Then G is an induced subgraph of H and $V(K_{m-n})$ is a CI set of H yielding $CI(H) = m - n + n = m$. ■

While developing the concepts in research, whenever a parameter is introduced, it is customary to estimate the same for family of trees. Now we enumerate the CI values for some trees.

Theorem 2.8 Let T be a tree with n vertices and p leaves. Then $CI(G) \leq n - p + 1$.

Proof It is obvious that for any tree T, the set S of all internal vertices yields $m(G - S) = 1$. We know that $|S| = n - p$ and hence $CI(T) \leq n - p + 1$. ■

Corollary 2.9 For any tree T, $CI(T) = n - 1$ iff $T \cong P_3$ *or* P_4 .

International Journal of Applied Mathematics
Volume 38 No. 12s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Proof Since for any tree, the number of leaves is at least two, $CI(T) \leq n - 1$. When we restrict $p = 2$, we get $T \cong P_n$. For a path, $CI(P_n) = n - 1$ iff $n = 3$ or 4. ■

Theorem 2.10 For a caterpillar T of order n, if $T \cong P_m \circ K_t^{\,c}$, $t \geq 2$, then $CI(T) = m+1$.

Proof T has two CI sets S_1 and S_2 where S_1 is the set of all internal vertices of T and S_2 is the set of all internal vertices of P_m. Both yield $CI(T) = m + 1$. ■

Theorem 2.11 For any tree T of order n, $CI(T \circ K_m^{\,c}) = n+1$ when $m \geq n$.

Proof It is obvious from Theorem 2*8* that $CI(T \circ K_m^{\,c}) \leq (n+1)m - nm + 1$. Since $m \geq n$, any CI set must contain all the internal vertices of $T \circ K_m^{\,c}$, resulting in $CI(T \circ K_m^{\,c}) = n+1$. ■

Remark 2.12 It should be noted that the result need not hold if $m < n$. In support of that one can think of $K_{1,n} \circ K_1$ $(n \geq 3)$ which has CI value to be 3 as a counter example.

Corollary 2.13 For a Bistar $B(r,t)$, $(r,t \geq 1)$, $CI(B(r,t) = 3$.

Proof $B(t,t)$ is nothing but $P_2 \circ K_t^{\,c}$ and the result follows from Theorem 2.10, as $m = 2$. Even the result holds when $r \neq t$. ■

Theorem 2.14 If G is a comb graph $(G \cong P_n \circ K_1)$, then $CI(G) = n$.

Proof Let $G \cong P_n \circ K_1$ with $v_i, 1 \leq i \leq n$ denoting the internal vertices such that v_i is adjacent with v_{i+1} for $1 \leq i \leq n-1$. Then any $S = \begin{cases} \left\{ v_{\left\lceil \frac{n}{2} \right\rceil \pm i} \right\}, & 0 \leq i \leq \left\lfloor \dfrac{n-2}{2} \right\rfloor, if\ n\ is\ odd \\ \left\{ v_{\frac{n}{2}-i}, v_{\frac{n}{2}+1+i} \right\}, & 0 \leq i \leq \dfrac{n-2}{2}, if\ n\ is\ even \end{cases}$ forms a CI set with $m(G - S) = n - |S|$, yielding the desired result. ■

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Theorem 2.15 Consider a double broom graph $G \cong B(d, m_1, m_2)$, $d \geq 3,\ m_1 \leq m_2, m_1, m_2 \geq 2$.

$$\text{Then } CI(G) = \begin{cases} d+1, & if\ \ m_1 + m_2 \geq d \\ \left\lceil \frac{d}{2} \right\rceil + m_1, & if\ \ m_1 + m_2 < d\,, m_1 = m_2\ \ forodd\ d \\ \left\lceil \frac{d+1}{2} \right\rceil + m_1, & if\ m_1 + m_2 < d, m_1 = m_2\ \ foreven\ d \\ \left\lceil \frac{d}{2} \right\rceil + \left\lceil \frac{m_2 - m_1}{2} \right\rceil + m_1, & otherwise \end{cases}$$

Proof Let G be a double broom graph with

$V(G) = \{v_i / 1 \leq i \leq d\} \cup \{u_i / 1 \leq i \leq m_1\} \cup \{w_i / 1 \leq i \leq m_2\}$ where u_i's and w_i's are leaves, v_1 and v_d are supports and the rest of v_i being internal vertices of degree *2*.

Case 1 $m_1 + m_2 \geq d$ .Obviously, $S = \{v_1, v_2, \ldots v_d\}$ is a set with *m(G - S) = 1*. This implies that *CI(G) = d+1*.

Case 2 $m_1 + m_2 < d$.

Subcase 1 $m_1 = m_2, d$ is odd. Then $S = \left\{ v_{\left\lceil \frac{d}{2} \right\rceil} \right\}$ forms a *CI* set which yields

$$CI(G) = \left\lfloor \frac{d}{2} \right\rfloor + 1 + m_1 = m_1 + \left\lceil \frac{d}{2} \right\rceil.$$

Subcase 2 $m_1 = m_2, d$ is even. Then $S = \left\{ v_{\left\lceil \frac{d}{2} \right\rceil} \right\}$ forms a *CI* set yielding

$$CI(G) = \left\lfloor \frac{d}{2} \right\rfloor + 1 + m_1 = m_1 + \left\lceil \frac{d+1}{2} \right\rceil.$$

Subcase 3 $m_1 \neq m_2$. Let $k = m_2 - m_1$. Then $S = \left\{ v_{\left\lceil \frac{d}{2} \right\rceil + \left\lceil \frac{k}{2} \right\rceil} \right\}$ will be a *CI* set which yields

$$CI(G) = \left\lceil \frac{d}{2} \right\rceil + \left\lceil \frac{k}{2} \right\rceil - 1 + m_1 + 1 = m_1 + \left\lceil \frac{d}{2} \right\rceil + \left\lceil \frac{k}{2} \right\rceil = m_1 + \left\lceil \frac{d}{2} \right\rceil + \left\lceil \frac{m_2 - m_1}{2} \right\rceil. \qquad \blacksquare$$

It is customary in literature to estimate the parameter in operated graphs. We too calculate the CI value for the following graphs which are derived by graph operations.

International Journal of Applied Mathematics
Volume 38 No. 12s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Theorem 2.16 Let G and H be connected graphs with n and m vertices respectively. Then $CI(G + H) = min\{m + I(G), n + I(H)\}$.

Proof Any cut set S of $G + H$ must contain $V(G)$ or $V(H)$. If S_1 and S_2 are integrity sets of G and H respectively, then $S_1 \cup V(H)$ and $S_2 \cup V(G)$ form connected integrity sets of $G + H$. Hence $CI(G + H) = min\{m + I(G), n + I(H)\}$. ■

Theorem 2.17 Let G and H be two connected graphs of order n and m respectively. Then $CI(G \, o \, H) \leq n + m$. Bound is attained if G is a cycle graph or complete graph.

Proof Let S be a CI set of $G \, o \, H$. Then the cardinality of S is at most n. In this case $m((G \, o \, H) - S) = m$. Hence $CI(G \, o \, H) = min\{|S| + m((G \, o \, H) - S) \leq n + m$. ■

Theorem 2.18 $CI(K_2 \times K_{1.k}) = 4$ where $k \geq 1$.

Proof Let $V(K_2) = \{u,v\}$ and $V(K_{1.k}) = \{w_1, w_2, ... w_k, x\}$ where x is the central vertex of $K_{1.k}$. Then $V(K_2 \times K_{1.k}) = \{(u,w_i) / 1 \leq i \leq k\} \cup \{(v,w_i) / 1 \leq i \leq k\} \cup \{(u,x),(v,x)\}$. It is noted that $S = \{(u,x),(v,x)\}$ forms a CI set of $K_2 \times K_{1.k}$ with $CI(K_2 \times K_{1.k}) = 4$. ■

Theorem 2.19 $CI(K_2 \times K_{m.k}) = 2m + 3$ where $m \leq k$, $m,k \geq 2$.

Proof Let $V(K_2) = \{u,v\}, V(K_{m.k}) = \{x_1, x_2, ... x_m, y_1, y_2, ... y_k\}$. In $K_2 \times K_{m.k}$, any vertex (u,x_i) is adjacent to the vertices $(u,y_1),(u,y_2),...(u,y_k)$ and (v,x_i). In order to disconnect the graph, the vertices $(u,x_1),(u,x_2),...(u,x_m),(v,x_1),(v,x_2),...(v,x_m)$ have to be removed. So, $S = \{(u,x_1),(u,x_2),...(u,x_m),(v,x_1),(v,x_2),...(v,x_m),(u,y_1)\}$ forms the CI set for $K_2 \times K_{m.k}$ with $m(K_2 \times K_{m.k} - S) = 2$. Hence $CI(K_2 \times K_{m.k}) = 2m + 3$. ■

Theorem 2.20 $CI(K_2 \times K_m) = \begin{cases} \dfrac{3m+1}{2} & if\ m\ is\ odd \\ \dfrac{3m}{2} + 1 & if\ m\ is\ even \end{cases}$ where $m \geq 3$.

Proof Let $V(K_2) = \{u,v\}, V(K_m) = \{w_1, w_2, ... w_m\}$.

Case *1* m is odd

In this case $S = \left\{(u,w_1),(u,w_2),...(u,w_{\lceil \frac{m}{2} \rceil}),(v,w_{\lceil \frac{m}{2} \rceil}),(v,w_{\lceil \frac{m}{2} \rceil + 1}),...(v,w_m)\right\}$ forms a CI set of cardinality $m + 1$ and $m(K_2 \times K_m - S) = \left\lfloor \dfrac{m}{2} \right\rfloor$. Hence $CI(K_2 \times K_m) = \dfrac{3m+1}{2}$.

Received: August 11, 2025

International Journal of Applied Mathematics
Volume 38 No. 12s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Case *2 m* is even

In this case $S = \left\{(u,w_1),(u,w_2),\ldots(u,w_{\frac{m}{2}}),(v,w_{\frac{m}{2}}),(v,w_{\frac{m}{2}+1}),\ldots(v,w_m)\right\}$ forms a *CI* set of cardinality $m + 1$ and $m(K_2 \times K_m - S) = \frac{m}{2}$. Hence $CI(K_2 \times K_m) = \frac{3m}{2}+1$. ■

Next let us characterize the trees with *CI* value *4*. One can refer characterization for graphs with *CI* value *3* in *[14]*. First we shall classify some families of graphs as follows.

Let $\mathcal{F}_1$ denote the family of graphs $P_3(m_1,m_2,m_3)$ obtained from P_3 by attaching m_1,m_2,m_3 pendant vertices respectively at each of its vertex subject to the following conditions. (i) $m_1,m_3 \geq 1$ and $m_2 \geq 0$ (ii) m_1 & m_3 are not both simultaneously equal to *1*. $P_3(5,2,8)$ is shown in Figure *3*.

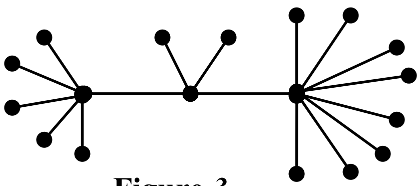


Figure *3*

It is noted that the connected integrity value of *G* is *4* for any graph $G \in \mathcal{F}_1$

Let $\mathcal{F}_2$ denote the family of graphs $B_{i,j}(m,t)$ obtained from the bistar $B(m,t)$ by subdividing respectively i $(1 \leq i \leq m)$ leaves and j $(0 \leq j \leq t)$ leaves at the support vertices subject to the following conditions. (i) $m, t \geq 1$ (ii) $j \neq 0$ if $t = 1$. $B_{2,3}(5,8)$ is shown in Figure *4*.

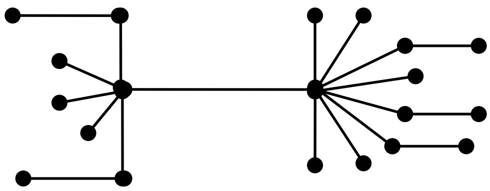


Figure *4*

Any graph in the family $\mathcal{F}_2$ has *4* as *CI* value.

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
For $m \geq 1, n \geq 2$, let $S^*_{m,n}$ denote the graph obtained by subdividing m edges of star $K_{1,n}$. Let v_i $(1 \leq i \leq n)$ and u_j $(1 \leq j \leq m)$ denote the pendant vertices and vertices of degree 2 respectively in $S^*_{m,n}$ Let $\mathcal{F}_3$ denote the family of graphs $S^*_{m,n}(t_1,t_2)$ obtained from $S^*_{m,n}$ by attaching t_1 pendant vertices one each at $v_1, v_2, .. v_{t_1}$ and t_2 pendant vertices one each at $u_{t_1+1}, u_{t_1+2}, .. u_{t_2}$ satisfying the conditions, $t_1 + t_2 \leq n$, $t_1, t_2 \geq 0$, t_1, t_2 are not simultaneously zero. $S^*_{10,12}(4,2)$ is shown in Figure 5.

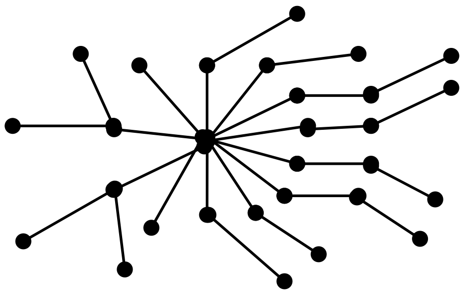


Figure 5

CI value of any graph in the family $\mathcal{F}_3$ is 4.

Theorem 2.21 Let T be tree. Then $CI(T) = 4$ iff $T \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.

Proof Let T be a tree with *CI* value 4. As $m(T - S) \geq 1$, the *CI* set S is of cardinality $1,2$ or 3.

Case 1 $|S| = 3$. This forces $m(T - S) = 1$. Then $T \cong P_3(m_1, m_2, m_3)$ .So T must be a member of $\mathcal{F}_1$.**Case 2** $|S| = 2$. Then $\langle S \rangle = K_2$ and $m(T - S) = 2$. So $T \cong B_{i,j}(m\ ,t)$ and $T \in \mathcal{F}_2$.

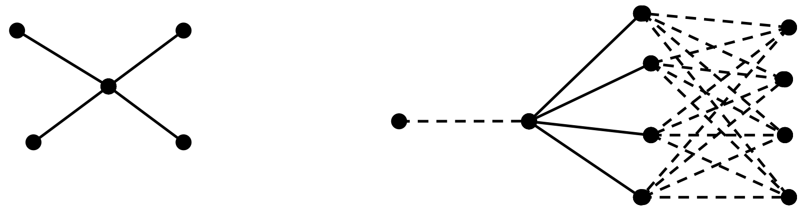
Case 3 $|S| = 1$. Then the components of $T - S$ are isomorphic to K_1, K_2 or P_3 with atleast one component isomorphic to P_3. Hence T belongs to $\mathcal{F}_3$. And the converse is obvious. ∎

3 ζ-Splitting Graphs

Sampath Kumar and Walikar [8] introduced the splitting graph of G denoted by $S(G)$ which is formed by introducing a new vertex v' corresponding to each vertex v of G and joining v' to all the vertices in $N(v)$ in G. Selvam Avadayappan and M.Bhuvaneshwari [9] initiated the study of cosplitting graph $CS(G)$ which is obtained from G, by adding a new

Received: August 11, 2025

Volume 38 No. 12s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
vertex w corresponding to each vertex v of G and joining w to all vertices which are not adjacent to v in G. A graph G and its cosplitting graph is given in Figure *6*.



G ***CS(G)***

Figure *6*

Let $S_1, S_2, \ldots S_k$ be the minimum vertex cuts in G. Then the *vertex cut splitting graph VS (G)* of a graph G which was introduced by Selvam Avadayappan, et.al., in [10] is the graph obtained from G by adding new vertices $w_1, w_2, \ldots w_k$ and joining w_i to each vertex in S_i, $1 \leq i \leq k$.

A graph G and its vertex cut splitting graph is given in Figure *7*.

G **VS(G)**

Figure *7*

[11] and [12] can be referred to know about γ-Splitting graphs and β-Splitting graphs respectively.

With the motivation of these splitting graphs, we add one more to this family called ζ-Splitting graph. First let us define ζ-set.

Definition A *CI* set of a graph G is a subset of its vertex set such that $|S| + m(G - S) = CI(G)$ where $CI(G)$ is the Connected Integrity of the graph G and $m(G - S)$ is the number of vertices in the largest component of $G - S$. The *connected integrity index* $\zeta(G)$ is the number of vertices in the smallest *CI* set, that is, $\zeta(G) = \min\{|S| \,/\, S \text{ is a CI set}\}$. Note that a *ζ-set* is a *CI* set of minimum cardinality.

As an illustration, consider the following graph shown in Figure *8*.

Received: August 11, 2025

International Journal of Applied Mathematics
Volume 38 No. 12s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

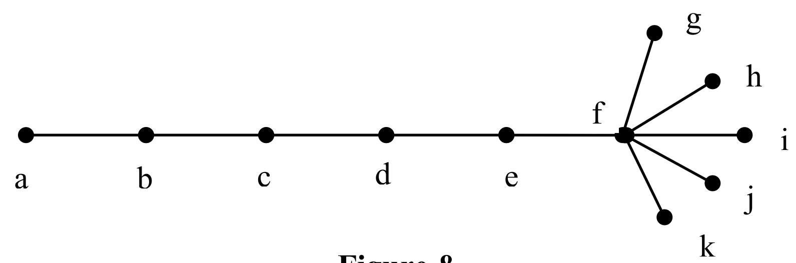


Figure *8*

Here, $CI(G) = 6$ with the following CI sets

$S_1 = \{f\}, S_2 = \{e, f\}, S_3 = \{d, e, f\}, S_4 = \{c, d, e, f\}, S_5 = \{b, c, d, e, f\}, S_6 = \{a, b, c, d, e, f\}$.

But S_1 is the smallest CI set, that is ζ-set. Therefore $\zeta(G) = 1$.

Definition Let G be a graph. Let k be the number of ζ- sets in G. Let $S_1, S_2, \ldots S_k$ be the ζ-sets in G. Then the *ζ-Splitting graph $S_\zeta(G)$* of a graph G is the graph obtained from G by adding new vertices $w_1, w_2, \ldots w_k$ and joining w_i to each vertex in S_i, $1 \le i \le k$.

The ζ-Splitting graph for the graph given in Figure *8* is shown in Figure *9*

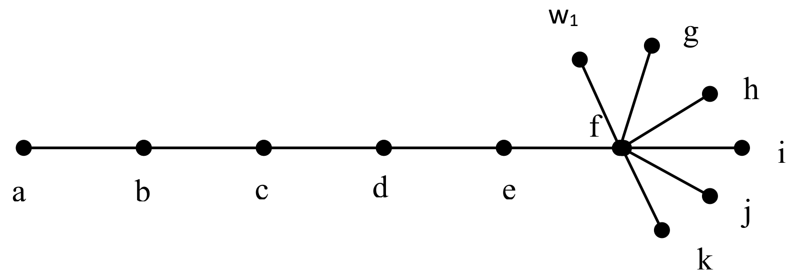


Figure *9*

Observation We observe the following facts from the definition of the ζ-Splitting graph.

Let $d^*(v)$ be the degree of the vertex v in $S_\zeta(G)$.

1. The newly added vertices $w_1, w_2, \ldots w_k$ are independent in $S_\zeta(G)$. That is, $d(w_i, w_j) \ge 2$ for any i,j, $1 \le i,j \le k$.

2.$d^*(w_i) = \zeta(G)$ for i, $1 \le i \le k$.

3. For any vertex $v \in V(G), d(v) \le d^*(v)$.

4. Every graph G is an induced subgraph of $S_\zeta(G)$. Even more G is a proper subgraph of $S_\zeta(G)$, since every graph contains atleast one ζ-set.

Received: August 11, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

5. The graphs having only one full vertex such as $K_{1,n}$ are examples for graphs whose ζ-Splitting graph contains exactly one newly added vertex.

6. $d^*(w_i) \leq n - 1$, where n is the order of G, since maximum possible cardinality of a CI set is $n - 1$ which occurs for K_n and C_n.

7. $d^*(w_i) \geq$ number of full vertices in G, as any CI set contains all full vertices of G.

8. $S\varsigma(K_{1,n}) \cong K_{1,n+1}$ for any $n \geq 2$.

9. $S\varsigma(K_n) \cong K_n \circ K_1$ for any $n \geq 2$.

10. $S\varsigma(C_n) \cong C_n \circ K_1$ for any $n \geq 3$.

It is probable to construct graphs with any given connected integrity index. Let us have the construction as in following theorem.

Theorem 3.1 Given any two integers $m \geq 1,\ n \geq 2,\ m \leq n - 2$, there exists a graph G of order n such that $\zeta(G) = m$.

Proof Given any integers $m \geq 1$, $n \geq 2$, construct a graph G with vertex set $V(G) = \{v_1, v_2, ... v_n\}$ and edge set $E(G) = \{v_i v_j / 1 \leq i \neq j \leq m\} \cup \{v_i v_k / 1 \leq i \leq m, m+1 \leq k \leq n\}$. Then the vertices $v_1 v_2, ... v_m$ are adjacent to each other and form the CI set of G with CI(G) = m + 1. Any set having fewer than m vertices gives a connected integrity value greater than $m + 1$. Hence $\zeta(G) = m$. ∎

There are graphs whose connected integrity index is same as its independence number. Not only that its complement also has same connected integrity index and independence number.

Theorem 3.2 Given any integer $m \geq 4$, there exists a graph G of order $2m$ such that $\varsigma(G) = \varsigma(\overline{G}) = \beta(G) = \beta(\overline{G}) = m.$

Proof For any integer $m \geq 4$, let us consider the graph G with vertex set $V(G) = V_1 \cup V_2$ where $V_1 = \{v_1, v_2, ... v_m\}$ and $V_2 = \{u_1, u_2, ... u_m\}$ and edge set

$E(G) = \{v_i v_j / 1 \leq i \neq j \leq m\} \cup \{v_i u_i, v_i u_{i+1} / 1 \leq i \leq m-1\}. \cup \{v_m u_m, v_m u_1\}$. Then in G, any vertex v_i of V_1 is adjacent to all the remaining vertices in V_1 and two vertices in V_2 whereas in $\overline{G}$, any vertex u_i in V_2 is adjacent to all the remaining vertices in V_2 and $(m - 2)$ vertices in V_1.

Received: August 11, 2025 ∎

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
Hence $S = V_1$ and $S' = V_2$ form the CI sets of G and $\overline{G}$ respectively with $CI(G) = CI(\overline{G}) = m + 1$.Also $\zeta(G) = \zeta(\overline{G}) = m$. It is also noted that $\beta(G) = \beta(\overline{G}) = m$.

Theorem 3.3 There exists a family of non Eulerian graphs G for which ζ-Splitting graph is Eulerian.

Proof Consider the graph $G \cong K_{2n} - e$ with $V(G) = \{v_1,v_2,...v_{2n}\}, n \geq 2$ and $e = v_1v_2$ .Then $V(G) - \{v_1,v_2\}$ is the unique ζ-set of G and is of order $2n - 2$. Note that $d(v_1) = 2n -2 = d(v_2)$ whereas $d(v_j) = 2n - 1$ for all $j, 3 \leq j \leq 2n$. Hence in $S_\zeta(G)$, $d^*(v_i) = 2n -2 = d^*(v_j)$ and $d^*(v) = 2n$ for all $v \in V(G) - \{v_1, v_2\}$ and the newly added vertex is of degree $2n$-2 so that $S_\zeta(G)$ is Eulerian. ■

Theorem 3.4 The graph $S_\zeta(K_{m,n})$ is biregular, $m,n \geq 2$.

Proof Let $V = \{v_1, v_2,...v_m\} \cup \{u_1, u_2,...u_n\}$ be the vertex set of $K_{m,n}$.

Case *1* Suppose $m = n$. Any set of the form $\{v_i,u_j / 1 \leq i \leq m\}$, $\{v_j,u_i / 1 \leq i \leq m\}$ with $1 \leq j \leq m$ is a ζ-set with $\zeta(K_{m,n}) = m + 1$. There are $2m$ number of ζ-sets of order $m + 1$. Each vertex of $K_{m,n}$ belongs to exactly $m + 1$ of its ζ-set. So $d^*(w_i) = m + 1$, $1 \leq i \leq 2m$ where w_i's are the newly added vertices and $d^*(v_i) = 2m + 1 = d^*(u_i)$, $1 \leq i \leq m$. Hence $S_\zeta(K_{m,n})$ is a $(m + 1,2m + 1)$ biregular graph.

Case *2* Suppose $m \neq n$ and $m < n$. Then there are n number of ζ-sets namely $\{v_1,v_2,...v_m,u_1\}$, $\{v_1,v_2,...v_m,u_2\}$,...$\{v_1,v_2,...v_m,u_n\}$. Each is of order $m + 1$. Therefore, $d^*(w_i) = m + 1$, $1 \leq i \leq n$, $d^*(v_i) = 2n$, $1 \leq i \leq m$ and $d^*(u_i) = m + 1$, $1 \leq i \leq n$. Hence $S_\zeta(K_{m,n})$ is a $(m + 1,2n)$ biregular graph. ■

Proposition 3.5 A vertex v is a full vertex in $S_\zeta(G)$ iff it is a full vertex in G.

Proof There is no ζ-set for any graph G with $\zeta(G) = n$ and the result follows. ■

Let us discuss the distance relation between G and S_ζ(G).

Proposition 3.6 For any graph G, $2 \leq diam(S_\zeta(G)) \leq diam(G) + 2$.

Proof Since any two newly added vertices in $S_\zeta(G)$ are independent, $diam(S_\zeta(G)) \geq 2$. Let x and y be any two vertices in $S_\zeta(G)$.

Case 1 If x and y are vertices of G, then $d(x,y) \leq diam(G)$.

Received: August 11, 2025

International Journal of Applied Mathematics
Volume 38 No. 12s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Case 2 If x is a vertex of G and y is a newly added vertex, then $d(x,y) = d(x,u) + 1$ where $u \in N(y) \subseteq V(G)$. Hence $d(x,y) \leq diam(G) + 1$.

Case 3 If both x and y are newly added vertices, then $d(x,y) = 1 + d(u,v) + 1$ where $u \in V(G) \cap N(x)$ and $v \in V(G) \cap N(y)$. Hence $d(x,y) \leq diam(G) + 2$. ■

Theorem 3.7 Any graph G of order $n \geq 3$ is an induced subgraph of a graph H such that $diam(H) = diam(S_\zeta(H)) = 2$.

Proof Given the graph G of order $n \geq 3$, construct the graph H by adding a vertex u and joining u to all the vertices of G Then $diam(H) = 2$. Also u occurs in all the ζ-sets of G and hence adjacent to all the newly added vertices. Therefore, distance between any two newly added vertices is 2 resulting that $diam(S_\zeta(H)) = 2$. ■

Next, we intend to find the CI value of $S_\zeta(G)$.

Proposition 3.8 $CI(S_\zeta(G)) = 2$ iff $G \cong K_{1,n-1}$.

Proof The result follows from the fact that $CI(G) = 2$ iff $G \cong K_{1,n}$ and $S_\zeta(G) \cong K_{1,n}$ iff $G \cong K_{1,n-1}$. ■

Theorem 3.9 Given any integer $n \geq 3$, there exists a family of graphs G of order n with $CI(S_\zeta(G)) = n - 1$.

Proof Consider the graph G with $V(G) = \{v_1,v_2,...v_n\}$ and $E(G) = \{v_iv_j / 1 \leq i \neq j \leq n\} - \{v_1v_n\}$. G is nothing but K_n-$\{v_1v_n\}$. Then $S = \{v_2,v_3,...v_{n-1}\}$ is the unique ζ-set of order $n - 2$ for G as well as for $S_\zeta(G)$ yielding $CI(S_\zeta(G)) = n - 1$. ■

Remark Normally, one can think that $CI(S_\zeta(G)) \geq CI(G)$. But this is not true.

$CI(G) = CI(S_\zeta(G))$ for $G \cong K_{1,n}$.

$CI(G) < CI(S_\zeta(G))$ for $G \cong C_n$.

$CI(G) > CI(S_\zeta(G))$ for G is a full binary tree with height ≥ 4.

The above discussion results in the following definition.

Definition A graph G is called a CI^+ *graph* if $CI(G) < CI(S_\zeta(G))$. Also G is called a CI^- *graph* if $CI(G) > CI(S_\zeta(G))$ and is called a CI^* *graph* if $CI(G) = CI(S_\zeta(G))$.

Following table shows the nature of some standard graphs.

International Journal of Applied Mathematics
Volume 38 No. 12s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Graph *G*	***CI(G)***	***CI(S_ζ(G))***	**Type**
K_n	*n*	*n + 1*	CI^+ *graph*
Cn	*n*	*n + 1*	CI^+ *graph*
$K_{1,n}$	*2*	*2*	CI^**graph*
$K_{m,n}$	*min{m,n} + 2*	*Min{m,n} + 3*	CI^+ *graph*
$B_{m,n}$	*3*	*3*	CI^**graph*

We can embed any graph G in CI^* graph, CI^- graph and CI^+graph as in following theorems.

Theorem 3.10 Any graph *G* of order $n \geq 3$ is an induced subgraph of CI^* graph.

Proof Let G be the given graph of order n ≥ 3. Consider the complete graph K_n. Take *n* copies of *G* and join the i^{th} vertex of K_n to all the vertices in the i^{th} copy of *G*, $1 \leq i \leq n$. Then all the vertices of K_n form the unique ζ-set of *H* as well as $S_\zeta(H)$. Hence $CI(H) = CI(S_\zeta(H)) = 2n$ and therefore, *H* is the required graph. ■

Theorem 3.11 Any graph *G* of order $n \geq 5$ is an induced subgraph of CI^- graph.

Proof Suppose G be any given graph of order n ≥ 5. Now consider the graph $H \cong P_n \circ G$ with $v_1, v_2, \ldots v_n$ as vertices of P_n. Then all the $n - 2$ internal vertices of P_n form the unique ζ-set of *H* giving $CI(H) = n - 2 + n + 1 = 2n - 1$. As the newly added vertex is adjacent to all the *n-2* internal vertices of P_n, we note that $S = \{v_2, v_4, \ldots\ldots v_n\}$ is the ζ-set of $S_\zeta(H)$, if *n* is even *and* $S' = \{v_2, v_4, \ldots\ldots v_{n-1}\}$ is the ζ-set of $S_\zeta(H)$, if *n* is odd. In both the cases, $CI(S_\zeta(H)) = \left\lfloor \frac{n-4}{2} \right\rfloor + 2 + 1 + n + 1 = \left\lfloor \frac{3n+4}{2} \right\rfloor$. Therefore, $H \cong P_n \circ G$ is the desired CI^-graph. ■

Theorem 3.12 Any graph *G* of order $n \geq 2$ is an induced subgraph of CI^+graph.

Proof Given any graph *G* of order $n \geq 2$, consider the graph $H \cong K_{2,n} \circ G$ with $V(K_{2,n}) = \{u_1, u_2, v_1, v_2, \ldots v_n\}$. Then $S_i = \{u_1, u_2, v_i\}, 1 \leq i \leq n$ are the ζ-sets of *H*. Hence $CI(H) = n + 4$. Any ζ-set of $S_\zeta(H)$ is of the form $S_i' = \{u_1, u_2, v_i\}$ or $S_i' = \{u_1, u_2, w_i\}, 1 \leq i \leq n$ so that $CI(S_\zeta(H)) = n + 5$. Therefore, $H \cong K_{2,n} \circ G$ is the desired graph. ■

Let us have some realization theorems.

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Theorem 3.13 For any $n \geq 5$, there exists a self centered CI^+ graph G of order n such that $S_\zeta(G)$ is also self centered of radius 2 and $CI(S_\zeta(G)) = CI(G) + 1$.

Proof Construct the graph G with vertex set $V(G) = \{v_1, v_2, \ldots v_{n-1}, u\}$ and edge set $E(G) = \{v_i v_j / 2 \leq i \neq j \leq n - 2\} \cup \{v_1 v_j / 2 \leq j \leq n - 2\} \cup \{v_n v_j / 2 \leq j \leq n - 2\} \cup \{uv_1, uv_{n-1}\}$. G is nothing but K_n with a subdivided edge. Then every vertex in G has eccentricity 2 and G is self centered. $S_i = \{v_1, v_i, v_{n-1}\}$, $2 \leq i \leq n - 2$ are ζ-sets of G so that $CI(G) = 3 + n - 4 = n - 1$. It is observed that $\{v_1, v_2, \ldots v_{n-1}\}$ is the unique ζ-set of $S_\zeta(G)$ so that $CI(S_\zeta(G)) = n$.. Hence G is the desired self centered CI^+ graph. ■

Theorem 3.14 Given any $d \geq 3$, and $n \geq d + 1$, there exists a CI^* graph of diameter d and order n.

Proof Consider the path P_{d-1} with vertices $v_1, v_{2,}, \ldots v_{d-1}$. Construct the graph G by joining v_1 of the path P_{d-1} to all the vertices of $K_{\left\lceil \frac{n-d+1}{2} \right\rceil}$ and v_{d-1} to all the vertices of $K_{\left\lfloor \frac{n-d+1}{2} \right\rfloor}$. Then diameter of G is d and $CI(G) = CI(S_\zeta(G))$. Thus G is the desired CI^* graph of order n. ■

Theorem 3.15 Given any $d \geq 1$ and $n \geq d + 2$, there exists a CI^* graph G such that $\delta(G) = d$ and $S_\zeta(G) \cong VS(G)$.

Proof Construct G as follows: $V(G) = \{u, v_1, v_2, \ldots v_{n-2}, w\}$ and $E(G) = \{v_i v_j / 1 \leq i \neq j \leq n - 2\} \cup \{uv_i / 1 \leq i \leq d\} \cup \{wv_i / 1 \leq i \leq d\}$. Then $\delta(G) = d$. $\{v_1, v_2, \ldots v_d\}$ forms the unique minimum vertex cut of G as well as ζ-set of G. Hence $S_\zeta(G) \cong VS(G)$. ■

References

1. Vecdi Aytac and Sema Bodur, *Vulnerability of some Splitting Graphs,* Bulletin of the International Mathematical Virtual Institute, Vol 5(2015),95-102.
2. K.S.Bagga, L.W.Beineke,Wayne Goddard, M.J.Lipman and R.E.Pippert, *A Survey of Integrity,* Discrete Applied Mathematics,37/38,13-28,1992.
3. R.Balakrishnan and K.Ranganathan, A Text Book of Graph Theory, Springer-Verlag,Newyork,Inc ,1999.
4. C.A.Barefoot, R.Entringer and H.Swart, *Vulnerability in Graphs*-A Comparative Survey, J.Combin.Math.Combin.Comput., 1, 12-22,1987.
5. C.A.Barefoot, R.Entringer and H.Swart, *Integrity of Trees and Powers of Cycles*, Congress Numer., 58,103-114,1987.

Received: August 11, 2025

Volume 38 No. 12s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
6. L.H.Clark, R.C. Entringer and M.R.Fellows, *Computational Complexity of Integrity*, J.Combin. Math. Combin. Comput.,2,179-191,1987.
7. W.Goddard and H.C.Swart, *Integrity in Graphs: Bounds and Basics*, J. Combin.Math.Com & Comput.,7,139-151,1990.
8. Sampath Kumar.E, Walikar.H.B. *On the Splitting graph of a graph*(1980), J.Karnatak Uni. Sci. 25:13.
9. Selvam Avadayappan and M.Bhuvaneshwari, *Cosplitting and coregular graphs,* International Journal of Mathematics and Soft Computing, Vol.5,(2015), 57-64.
10. Selvam Avadayappan and M.Bhuvaneshwari, M.Indira Devi, *Vertex Cut Splitting graphs,* Journal of Applied Science and Computations, Vol.VI,Issue III,March 2019,36-47.
11. Selvam Avadayappan and M.Bhuvaneshwari, R.Iswarya, *γ-Splitting graphs,* International Journal for Research in Applied Science & Engineering,Vol.4,(2016), 670-680.
12. Selvam Avadayappan and M.Bhuvaneshwari and B.Vijaya Lakshmi, *β-Splitting graphs,* International Journal of Scientific Review and Research in Engineering and Technology, Vol.1(3),Mar.-Apr. 2016,84-101.
13. K.Sorna Devi, M.Bhuvaneshwari, Selvam Avadayappan, *Connected Integrity in Some Special Graphs*, Proceedings of National Symposium on "Newfangled Avenues of Applied Mathematics, August 2021,153-159.
14. K.Sornadevi, M.Bhuvaneshwari, Selvam Avadayappan, *A Note on Connected Integrity in Graphs*, Advances and Applications in Mathematical Sciences, Volume 22, Issue 1,105-115,2022.
15. K.Sornadevi, M.Bhuvaneshwari, Selvam Avadayappan, *Connected Integrity Polynomial of Zero Divisor graphs*, Indian Journal of Natural Sciences, Volume 16, Issue 91, August 2025, 99835-99845 .

Received: August 11, 2025