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Convex-Fair Domination in Graphs

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Abstract

In this paper, we propose and investigate a novel domination parameter in graph theory, termed convex-fair domination, which integrates the principles of convex and fair domination. The newly introduced concept aims to balance the structural coherence of convex dominating sets with the equitable vertex coverage characteristic of fair domination. We develop a formal definition of convex-fair domination and present a comprehensive theoretical framework to support its analysis. Several theorems, lemmas, and corollaries are established to explore the fundamental properties, bounds, and existence conditions of convex-fair dominating sets. Through illustrative examples and proofs, we demonstrate the applicability of this parameter across various graph classes. Additionally, the paper encourages the

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examination of convex-fair domination in new and complex graph families, laying the groundwork for future research in this direction. The introduction of this parameter not only enriches the current landscape of domination theory but also opens pathways for its application in network optimization, resource allocation, and equitable system design.

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Key Words and Phrases: Convex domination, Fair domination, domination in graphs

1 Introduction

Let $G = (V, E)$ be a connected, finite and undirected graph with $|V(G)| = n(G)$.

The open neighbourhood of a vertex $v \in V(G)$ in G is the set of all vertices adjacent to v in G denoted by $N_G(v)$ and the closed neighbourhood is $N_G[v] = N_G(v) \cup \{v\}$.[3]

The degree of a vertex v in G is the number of edges incident to v, that is $deg_G(v) = |N_G(v)|$. The minimum and maximum degrees among all vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The length of a shortest cycle in G is the girth of G and is denoted $g(G)$.[4]

A dominating set S is a subset of $V(G)$ such that every vertex in $V(G)$ that is not in S is adjacent to at least one vertex in S. The minimum number of vertices in a dominating set of S is called the domination number, denoted by $\gamma(G)$.[7]

The distance between two vertices x and y in a connected graph G is the length of a shortest $(x - y)$ path in G which is denoted by $d_G(x, y)$. An $(x - y)$ path of length $d_G(x, y)$ is called an $(x - y)$-geodesic.[1]

A set $S \subseteq V(G)$ is convex in G if vertices from all $(x - y)$-geodesics contained in S for every two vertices $x, y \in S$. A set S is a convex dominating set if S is convex and dominating. The convex domination number $\gamma_{con}(G)$ of a graph G denotes the minimum cardinality of a convex dominating set.[2, 6]

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A fair dominating set in a graph G is a dominating set S such that all vertices not in S are dominated by the same number of vertices from S; that is, every two vertices not in S have the same number of neighbors in S. The fair domination number, $fd(G)$, of G is the minimum cardinality of a fair dominating set.[8, 5]

A set $S \subseteq V(G)$ is said to be convex-fair dominating set if it follows three following conditions.

1. S is a dominating set.
2. For every $u, v \in S$, the $(u-v)$-geodesic is contained in S.
3. For every $u, v \in V(G) \setminus S, |N(u) \cap S| = |N(v) \cap S|$.

The minimum cardinality of convex-fair dominating set is called convex-fair domination number which is denoted by $\gamma_{cf}(G)$.

2 Main Results

This section of the analysis examines a novel parameter called the convex-fair domination number across various types of graphs. Additionally, we have derived some results on the convex-fair domination number.

Observation 1. Let G be a connected graph with order n, then the following holds.

1. $\gamma(G) \leq \gamma_{cf}(G)$
2. $\gamma_{cf}(G) \leq n(G)$
3. $\gamma_{con}(G) \leq \gamma_{cf}(G)$
4. $fd(G) \leq \gamma_{cf}(G)$

Illustration 1. Graph G is shown in Figure:1 where the order of G is 8. $\gamma(G)$-set= $\{v_1, v_7\}$.
$\gamma(G) = 2$.
$\gamma_{con}(G)$-set= $\{v_1, v_2, v_3, v_7\}$.

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$\gamma_{con}(G) = 4$.
fd-set$= \{v_1, v_7\}$.
$fd(G) = 2$.
$\gamma_{cf}(G)$-set$= \{v_1, v_2, v_3, v_7, v_5, v_8\}$.
$\gamma_{cf}(G) = 6$.

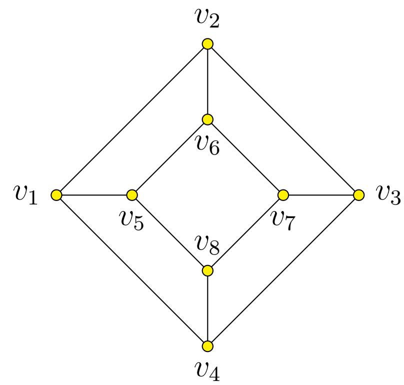

Figure:1 Graph G

Theorem 2.1. *Let G be a connected graph with order greater than 4. If $\gamma_{cf}(G) = n(G)$, then the maximum degree of graph G, $\Delta(G) \leq n(G) - 4$.*

Proof. Let G be a connected graph with order greater than 4. We will demonstrate that if $\Delta(G) > n(G) - 4$, then $\gamma_{cf}(G) < n(G)$.
Case:1 $\Delta(G) = n(G) - 1$.
If $\Delta(G) = n(G) - 1$, then it is evident that $\gamma_{cf}(G) = 1 < n(G)$.
Case:2 $\Delta(G) = n(G) - 2$
Let u be a vertex with $d_G(u) = \Delta(G) = n(G) - 2$, and let v is not in $N_G[u]$. Since G is connected, there exists a vertex $x \in N_G(v) \cap N_G(u)$, and $\{u, x\}$ forms a minimum convex-fair dominating set in G. Thus, $\gamma_{cf}(G) = 2 < n(G)$.
Case:3 $\Delta(G) = n(G) - 3$
Let x, y not in $N_G[u]$.
If $d_G(x) = 1$ or $d_G(y) = 1$, Assume that, $d_G(x) = 1$. Then $V(G) \setminus \{x\}$ becomes a convex-fair dominating set of G, which implies $\gamma_{cf}(G) < n(G)$.

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Hence, assuming $d_G(x) > 1$ and $d_G(y) > 1$, if there exists a vertex $z \in N_G(u) \cap N_G(x) \cap N_G(y)$, then $\{u, z\}$ is a minimum convex-fair dominating set in G, and $\gamma_{cf}(G) = 2 < n(G)$.
But, if x and y have no common neighbor in $N_G(u)$, then there are $v \in N_G(u) \cap N_G(x)$ and $w \in N_G(u) \cap N_G(y)$. Then $\{u, v, w\} \cup (N_G(v) \cap N_G(w))$ forms a convex-fair dominating set in G.
Hence, $\gamma_{cf}(G) < n(G)$. □

Corollary 1. If $\gamma_{cf}(G) = n(G)$, then $2 \leq \delta(G) \leq \Delta(G) \leq n(G)-4$.
(Proof mentioned as above.)
Corollary 2. If $\gamma_{cf}(G) = n(G)$, then $n(G)$ is greater than 5.
(Proof mentioned as above.)

Theorem 2.2. *For any complete bipartite graph $G = K_{m,n}$, $\gamma_{cf}(K_{m,n}) = 2$.*

Proof. Let $G = K_{m,n}$ be complete bipartite graph having $m+n$ number of vertices and two disjoint partitions say X and Y.
Assume that partition X and Y has m and n number of vertices, respectively.
So, $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$.
Dominating set S of $K_{m,n}$ contains two vertieces, say x_i, y_j, where $i \in 1, 2, ..., m$ and $j \in 1, 2, ...n$. So, $\gamma(K_{m,n}) = 2$.
Also, x_i and y_j are adjacent vertices, so we can say that the $(x_i - y_j)$-geodesic is contained in dominating set.
For every vertex u, where $u \in \{x_1, x_2, ...x_m, y_1, y_2, ...y_n\}$ but $u \neq \{x_i, y_j\}$, $|N(u) \cap S| = 1$.
Therefore, S is a convex-fair dominating set.
Hence, $\gamma_{cf}(G = K_{m,n}) = 2$. □

Corollary 3. For any complete graph $G = K_n$, $\gamma_{cf}(K_n) = 1$.
(Proof mentioned as above.)

Theorem 2.3. *For any wheel graph W_n with order $n(n \geq 4)$, $\gamma_{cf}(W_n) = 1$.*

Proof. Let G be a wheel graph $W_n = C_{n-1} + \{v_n\}$ with n vertices, say $v_1, v_2, v_3, ..., v_n$.

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(where $C_{n-1} = \{v_1, v_2, v_3, ..., v_{n-1}\}$)
Dominating set S of W_n contains only one vertex v_n. So, $\gamma(W_n) = 1$.
Here, S contains only one vertex so we can say that S is convex dominaing set.
Now, For every vertex u (where $u \in V(G) \setminus S$), $|N(u) \cap S| = 1$.
Therefore, S is a convex-fair dominating set.
Hence, $\gamma_{cf}(G = W_n) = 1$. □

Theorem 2.4. *For any helm graph H_n with order $2n + 1(n \geq 4)$, $\gamma_{cf}(H_n) = n + 1$.*

Proof. The helm graph H_n has $2n + 1$ vertices and it contains the wheel W_n and $n - 1$ pendant vertices. Let $\{v_1, v_2, v_3, ..., v_{2n+1}\}$ be the vertices of a graph.
where $v_{n+1}, v_{n+2}, v_{n+3}, ..., v_{2n}$ are pendant vertices of $v_1, v_2, v_3, ..., v_n$, respectively. and v_{2n+1} is center vertex adjacent to n vertices having degree 3.
Let us suppose, $S = v_1, v_2, v_3, ..., v_n$ is the domination set of graph $G = H_n$.
For every $v_i, v_j \in S$, the $(v_i - v_j)$-geodesic is contained in S.
But $|N(v_{2n+1}) \cap S| = n$ and $|N(v_k) \cap S| = 1$ $(k \in n+1, n+2, ...2n)$.
Therefore, we have to add one more vertex v_{2n+1} to S.
So, S has $n + 1$ vertices, $S = v_1, v_2, v_3, ..., v_n, v_{2n+1}$.
Now, for every $v \in V(G) \setminus S$, $|N(v) \cap S| = 1$.
Therefore, S is a convex-fair dominating set.
Hence, $\gamma_{cf}(G = H_n) = n + 1$. □

Corollary 4. For any closed helm graph CH_n with order $2n+1(n \geq 4)$, $\gamma_{cf}(CH_n) = n + 1$.
(Proof mentioned as above.)

Theorem 2.5. *For any gear graph G_n with $2n + 1(n \geq 4)$ vertices and $3n$ edges, $\gamma_{cf}(G_n) = n + 1$.*

Proof. Let $G = G_n$ be a gear graph with $2n + 1$ vertices.
$\{v_1, v_2, v_3, ..., v_{2n+1}\}$ is a vertex set of G_n. Where $v_{n+i}(; 1 \leq i \leq n)$

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is to degree two vertex adjancent to vertices v_i and v_{i+1}. and v_{2n+1} is center vertex adacent to n vertices of degree three.
Let us suppose, $S = v_1, v_2, v_3, ..., v_n$ is the domination set of graph $G = G_n$.
For every $v_i, v_j \in S$, the $(v_i - v_j)$-geodesic contained in S.
But $|N(v_{2n+1}) \cap S| = n$ and $|N(v_{n+i}) \cap S| = 2$ (where $1 \leq i \leq n$).
Therefore, we have to add one more vertex v_{2n+1} to S.
So, S has $n + 1$ vertices, $S = v_1, v_2, v_3, ..., v_n, v_{2n+1}$.
Now, for every $v_{n+i} \in V(G) \setminus S$, $|N(v_{n+i}) \cap S| = 1$.
Therefore, S is a convex-fair dominating set.
Hence, $\gamma_{cf}(G = G_n) = n + 1$. □

3 Conclusion

Here we introduce and rigorously define the novel concept of convex-fair domination in graphs, a hybrid extension that combines properties from both convex domination and fair domination theories. The study begins by laying a strong foundational framework that allows for the precise mathematical characterization of convex-fair dominating sets and their associated parameters. We explore various structural properties and provide a series of theorems and corollaries that support the theoretical development of the concept. Each result is accompanied by detailed proofs and illustrative examples to ensure clarity and deepen understanding. In doing so, we demonstrate how convex-fair domination behaves under different graph operations and within specific classes of graphs. Moreover, the paper encourages further investigation by proposing the study of convex-fair domination in newly defined or less commonly explored families of graphs. This opens the door to identifying unique patterns, establishing tighter bounds, and potentially uncovering broader applications within algorithm design, optimization, and network theory. The work thus not only enriches the existing body of graph domination literature but also sets the stage for future inquiries into more complex and nuanced domination pa-

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