International Journal of Applied Mathematics

Volume 38 No. 12s, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

THE DISTRIBUTIVE LATTICE OF QUOTIENT LATTICE-VALUED INTUITIONISTIC FUZZY SUB ℓGROUPS OF TYPE-3

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Abstract

The study of lattice-ordered groups (ℓ-groups), which elegantly unify group and lattice structures, is fundamentally constrained in contexts characterized by multi-dimensional uncertainty and vagueness. Existing generalizations, such as Lattice-Valued Intuitionistic Fuzzy Sets (LVIFSs) of Type-1 or Type-2, fail to consistently preserve the ℓ-group's critical dual algebraic and order-theoretic properties. To address this theoretical deficiency, this paper formally introduces and axiomatically analyzes the robust concept of Lattice-Valued Intuitionistic Fuzzy Sub-ℓ-groups of Type-3 (LVIFSℓ-group-3) and their specialized counterpart, the Convex LVIFS ℓ-group-3 (C-LVIFSℓ-group-3). A critical level set equivalence theorem is established, demonstrating that a structure is an LVIFSℓ-group-3 if and only if all of its non-empty level sets manifest as crisp ℓ-subgroups. We rigorously prove that this structure is preserved under ℓ-homomorphisms and is closed under intersection and chain union. Meticulous analysis of the C-LVIFSℓ-group-3 successfully models the essential order-preserving convexity property. Crucially, we investigate the quotient structure defined by the C-LVIFSℓ-group-3, demonstrating the conditions under which it forms a distributive lattice, thereby establishing a fuzzy analog to the normal convex ℓ-subgroup, which functions as the algebraic kernel in classical ℓ-group theory. This framework furnishes the most comprehensive instrument for the algebraic study of l-groups in contexts permeated by complex, lattice-valued uncertainty.

Key Words:ℓ-groups, LVIFSℓgroup-3, Lattice-valued intuitionistic fuzzy convex subℓ-group type-3(LVIFCSℓG-3), ℓ-homomorhism, quotient LVIFCSℓG-3.

2010 MSC Code: 03E72, 06F15, 06D72, 08A72

1 Introduction and Preliminaries

Lattice-ordered groups (ℓ-groups) represent a structured intersection of algebraic group and order-theoretic lattice operations. Pioneering work by Birkhoff[4], Conrad[5], and others [14] established the formal investigation of these structures, which are characterized by the distributive law governing group multiplication over lattice operations. ℓ-groups are fundamental in functional analysis, non-commutative geometry, and quantum logic. Their

Received: August 09, 2025

Volume 38 No. 12s, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

order-theoretic decomposition, determined by normal subgroups and convex ℓ-subgroups, remains an active research area. The axiomatization of ℓ-groups facilitates their application across theoretical disciplines.

While structurally elegant, classical systems like ℓ-groups are limited when addressing real-world or theoretical systems involving uncertainty and vagueness. Binary inclusion/exclusion criteria are insufficient for analyzing graded states in complex decision analysis, physical measurements, and linguistic modeling. This limitation necessitates generalized frameworks capable of accommodating continuous degrees of truth and partial membership. Current research aims to construct substructures that preserve ℓ-group properties while incorporating graded membership modeling.

Fuzzy set theory, introduced by Zadeh[16] in 1965, marked a breakthrough in modeling imprecision by generalizing the characteristic function to assign membership degrees within [0,1]. This approach surpasses crisp sets for handling subjective or blurred boundaries. However, early fuzzy structures couldn't fully capture the spectrum of cognitive uncertainty, which often involves simultaneous support and opposition. The degree of non-membership was implicitly derived as an arithmetic complement.

Atanassov[2] introduced Intuitionistic Fuzzy Sets (IFSs) in the mid-1980s to address this conceptual gap. IFSs employ two independent mappings: a membership function $\mu_A: U \longrightarrow [0,1]$ and a non-membership function $\nu_A: U \longrightarrow [0,1]$. The hesitation margin, $1 - (\mu_A(x) + \nu_A(x)), \forall x \in U$, explicitly quantifies indecisiveness, enhancing the model's descriptive power. The IFS concept has been extended across algebraic domains, becoming a foundational tool in applied algebra.

In [8], Goguen's theory of L-fuzzy sets (1967) extends the valuation space to a complete lattice L, enabling modeling of complex, multi-dimensional uncertainty. In scenarios with multiple criteria or conflicting evidence, the partial ordering inherent in a lattice structure better characterizes aggregate uncertainty than a simple numerical value.

Lattice-Valued Intuitionistic Fuzzy Sets (LVIFSs), introduced by Gernstenkorn and Tepavcevic[7] in 2004, map membership and non-membership to elements of a complete lattice L, offering maximum conceptual flexibility. However, defining algebraic substructures within the LVIFS context requires careful consideration of the interaction between binary operations and lattice operators in L.

Previous research has explored fuzzy ℓ-groups and intuitionistic fuzzy substructures, including analyses of fuzzy ℓ-subgroups using [0,1] membership grades and characterizations of intuitionistic fuzzy subgroups (IF-subgroups) on general algebraic groups.

[6] Applications of Lattice-Valued Intuitionistic Fuzzy Fields (LVIFS) to lattices and groups have often been limited to Type-1 or Type-2 definitions. Type-1 definitions rely on a specific operator on L, while Type-2 definitions attempt to blend lattice and group operations.

A critical gap exists in the comprehensive application of the algebraically compliant Type-3

International Journal of Applied Mathematics

Volume 38 No. 12s, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

LVIFS structure, which mandates the use of generalized join ($\vee L$) and meet ($\wedge_L$) operators L to satisfy closure and inverse axioms on group operations. Failing to use this definition can result in structures that conserve only the group or lattice component, but not the synergistic properties of an ℓ-group.

This study addresses this gap by providing a comprehensive framework for Lattice-valued Intuitionistic Fuzzy Sub-ℓ-groups of Type-3 (LVIFSℓgroup-3) and their convex counterpart (C-LVIFSℓgroup-3). The research formally introduces and analyzes these groups, establishing their theoretical completeness. The axiomatic definition of the LVIFSℓgroup-3 structure confirms adherence to ℓ-group axioms. An equivalence theorem demonstrates that a structure is an LVIFSℓ −group-3 if its non-empty level sets are crisp ℓ-subgroups of G. Preservation properties and mapping consistency of LVIFS structure $\ell - homomorphisms$ are proven. The order-theoretic analysis and interval property of the C-LVIFSℓgroup-3 demonstrate successful modeling of graded convexity. The quotient structure of the C-LVIFSℓgroup-3 demonstrates its utility as a generalized kernel analogous to the normal convex ℓ-subgroup in classical theory. This research positions the LVIFSℓgroup-3 framework at the forefront of fuzzy algebraic structures.

Throughout this paper, $G = (G, +, \wedge, \vee)$ denotes an ℓ-group with additive identity 0, unless otherwise specified, and $(L, \wedge_L, \vee_L)$ denotes a complete lattice satisfying the infinite meet distributive property, with top and bottom elements 1_L and 0_L, respectively. Furthermore, all intuitionistic subsets of G are assumed to be lattice-valued intuitionistic sets of type-3.

2 Preliminaries

This section reviews fundamental concepts related to ℓ-groups, L-fuzzy sets, and intuitionistic (L-)fuzzy sets, laying the groundwork for subsequent discussions.

2.1 ℓ-groups

Lattice-ordered groups, or ℓ-groups, have been extensively studied in abstract algebra. Birkhoff[4] (1942) established initial definitions and explored fundamental properties, while Conrad[5] (1970) expanded upon this foundation. More recently, Stuart [14](2010) revisited these concepts, providing updated definitions and a contemporary analysis.

Definition 2.1 *A subset $H \subseteq G$ is convex if, for any $x, y \in H$ and $z \in G$, the condition $x \leq z \leq y$ implies $z \in H$.*

Definition 2.2 *A system $(G, +, \leq)$ is a lattice-ordered group (ℓ-group) if: (i) $(G, +)$ is a group; (ii) $(G, \leq)$ is a lattice; and (iii) for all $a, b, x, y \in G$, $x \leq y$ implies $a + x + b \leq a + y + b$.*

For detailed treatments of ℓ-groups, see Birkhoff (1942), Conrad (1970), and Stuart (2010).

Definition 2.3 (Crisp ℓ-subgroup) *A non-empty subset H of an ℓ-group G is an* ***ℓ-subgroup*** *if and only if:*

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

1. H is a subgroup of G (i.e., for all $x, y \in H$, $xy^{-1} \in H$).
2. H is a sublattice of G (i.e., for all $x, y \in H$, $x \wedge y \in H$ and $x \vee y \in H$).

2.2 L-Fuzzy sets and L-fuzzy algebraic structures

Goguen's L-fuzzy set theory (1967)[8] and L-fuzzy algebraic structures[10, 15] extend the valuation space of fuzzy sets from a chain to a (complete) lattice L. This generalization accommodates complex, multi-dimensional, or hierarchical uncertainties that cannot be linearly ordered.

Definition 2.4 *An L-fuzzy set* μ *in a set* X *is a function from* X *to* L*. The set of all such L-fuzzy sets, denoted* L^X*, is called the L-power set of* X*. Specifically, when* L *is the interval* $[0,1]$*, these are called fuzzy subsets, and* $[0,1]^X$ *is the fuzzy power set of* X*.*

2.3 Intuitionistic Fuzzy set

Intuitionistic fuzzy sets enhance fuzzy sets by adding a non-membership degree, offering a more nuanced way to represent uncertainty, especially with incomplete data.

Definition 2.5 *An IFS A on a universe X is defined as* $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$*, where* $\mu_A(x) \in [0,1]$ *and* $\nu_A(x) \in [0,1]$ *represent the membership and non-membership degrees of x in A, respectively, with* $0 \leq \mu_A(x) + \nu_A(x) \leq 1$*. The empty IFS,* $0\sim$*, is defined as* $0 \sim (x) = (0,1)$*, and the total IFS,* $1\sim$*, as* $1 \sim (x) = (1,0)$*. The support of A, denoted* $Supp(A)$*, is* $Supp(A) = \{x \in X : \mu_A(x) > 0 or (\mu_A(x) = 0 and \nu_A(x) < 1)\}$*. The* (α, β)*-level set of A, denoted* $A_{\alpha,\beta}$ *is* $A_{\alpha,\beta} = \{x \in X : \mu_A(x) \geq \alpha and \nu_A(x) \leq \beta\}$*, where* $\alpha, \beta \in [0,1]$ *and* $0 \leq \alpha + \beta \leq 1$*.* α *and* β *act as membership and non-membership thresholds for element inclusion in* $A_{\alpha,\beta}$*.*

The set of all intuitionistic fuzzy sets on X, denoted IFS(X), forms a distributive lattice under the subset inclusion relation $\subseteq$.

2.4 Lattice-valued Intuitionistic Fuzzy sets

Definition 2.6 *A Lattice-Valued Intuitionistic Fuzzy Sub*ℓ*-group (LVIFS*ℓ*group) might involve a direct conjunction of the axioms for an LVIF subgroup and an LVIF sublattice. Such a "Type-1" or "Type-2" definition would typically require an LVIFS* $A = (\mu_A, \nu_A)$ *on* G *to satisfy, for all* $x, y \in G$*:*

- $\mu_A(xy^{-1}) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy^{-1}) \leq \nu_A(x) \vee \nu_A(y)$
- $\mu_A(x \wedge y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x \wedge y) \leq \nu_A(x) \vee \nu_A(y)$
- $\mu_A(x \vee y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x \vee y) \leq \nu_A(x) \vee \nu_A(y)$

Note 2.7 *Let* G *be an* ℓ*-group, it is possible to consider the level sets of A interchangbly as* $A_\alpha = \{x \in G | \mu_A(x) \geq \alpha\}$ *and* $A_\beta = \{x \in G | \nu_A(x) \leq \beta\}$*.*

The following example exposes the deficiencies of type 1 or type 2 formalisms, strengthening the need for an introduction to type-3 formalism over the former ones on ℓ-groups.

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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Example 2.8 (Counterexample: Failure to Yield an ℓ-subgroup) *Let $G = \mathbb{R}^2$ be the l-group under component-wise addition and the lexicographical order (i.e., $(a, b) \leq (c, d)$ if $a < c$, or $a = c$ and $b \leq d$). Let $L = [0,1]$ be the grading lattice with $\wedge = min$ and $\vee = max$.*

We define a Type-1 LVIFS $A = (\mu, \nu)$ on G such that it satisfies the subgroup and sublattice properties (i)-(vi), but its α-level set, $G_\mu(\alpha)$, is **not an ℓ-subgroup** because it fails the lattice closure property.

Define $H = \{(a, 0)|a \in \mathbb{R}\}$. H is a crisp subgroup and sublattice of G.

Let $\mu: G \to L$ be defined as: $\mu(x) = \begin{cases} 0.8 & \text{if} x \in H \\ 0.2 & \text{if} x \notin H \end{cases}$

Let $\nu(x) = 1 - \mu(x)$.

This A satisfies Type-1 LVIFS subgroup axioms since μ is two-valued.

Now, consider the elements $x = (0,1)$ and $y = (1, -1)$. $x \notin H$ and $y \notin H$, so $\mu(x) = 0.2$ and $\mu(y) = 0.2$. We examine the meet: $x \wedge y$. Since $(0,1)$ is lexicographically less than $(1, -1)$, we have $x \wedge y = x = (0,1)$.

By the Type-1 LVIFS sublattice axiom (iii), we require $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$. $\mu((0,1)) = 0.2 \geq \min(0.2,0.2) = 0.2$

The condition holds. However, the α-level set for $\alpha = 0.8$ is:

$$G_\mu(0.8) = \{x \in G|\mu(x) \geq 0.8\} = H = \{(a, 0)|a \in \mathbb{R}\}$$

H is a subgroup, but it is NOT a sublattice of G. In G, we have $x = (1,1)$ and $y = (1, -1)$, both in H. Their meet in G is $x \wedge y = (1, -1) \wedge (1,1) = (1, -1)$ since the first component is equal and $-1 < 1$. But since $x \wedge y = (1, -1) \notin H$, H is not an ℓ-subgroup of G.

Thus, a Type-1 LVIFS A whose α-level set is a subgroup may fail to have that level set be a sublattice of G. The definition of an l-subgroup necessitates that the subset be closed under the ℓ-group's lattice operations, not just the operations within H. The Type-3 axioms must explicitly introduce a condition that links the ordering on G to the grades on L, thereby ensuring the resulting level sets maintain their ℓ-subgroup property.

To address this gap, a new axiomatic framework is needed. This framework must be axiomatically engineered to ensure that its level sets are not merely l-subgroups, but specifically convex l-subgroups. This necessitates the integration of the order-theoretic properties directly into the fuzzy axioms. This stronger, more precise formalism is what we designate as the Lattice-valued Intuitionistic Fuzzy Sub-l-group of Type-3 (LVIFSlgroup-3), which will be defined in the subsequent section.

Definition 2.9 *Let L be a complete lattice with top 1_L and bottom 0_L. Let $f: L \to [0,1]$ be a lattice homomorphism such that $f(1_L) = 1$ and $f(0_L) = 0$ (i.e., $f(x \wedge y) = min(f(x), f(y))$ and $f(x \vee y) = max(f(x), f(y))$). A lattice-valued intuitionistic fuzzy set of type 3 (LIFS-3) is a triple (X, μ, ν), where X is a nonempty set, $\mu, \nu: X \to L$, and $f(\mu(x)) + f(\nu(x)) \leq 1$ for*

Received: August 09, 2025

Volume 38 No. 12s, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

all $x \in X$ (see [7]).

3 Lattice-valued Intuitionistic Fuzzy subℓ-groups type-3

3.1 Definition and properties

The properties and characterizations of the lattice-valued intuitionistic fuzzy subℓ-group of type-3 (LVIFSℓgroup-3) are revealed by defining it with a ℓ-group G, membership function, and lattice homomorphism.

Definition 3.1 *Let G be an ℓ-group and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ be an intuitionistic fuzzy subset of type-3 in G. Then,A is a lattice-valued intuitionistic fuzzy subℓ-group of type-3(LVIFSℓgroup-3) of G if, for all $x, y \in G$, the following conditions hold: (i) $\mu_A(0) = 1_L$ and $\nu_A(0) = 0_L$; (ii) $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x + y) \leq \nu_A(x) \vee \nu_A(y)$; (iii) $\mu_A(-x) = \mu_A(x)$ and $\nu_A(-x) = \nu_A(x)$; (iv) $\mu_A(x \vee y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x \vee y) \leq \nu_A(x) \vee \nu_A(y)$; (v) $\mu_A(x \wedge y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x \wedge y) \leq \nu_A(x) \vee \nu_A(y)$.*

Note 3.2 *In this definition, conditions $\mu_A(0) = 1_L$ and $\nu_A(0) = 0_L$, give a guarantee that the arbitrary intersection of LVIFSℓgroup-3s of G is non-empty. The conditions (ii) and (iii), and (iv) and (v), respectively, define a lattice-valued intuitionistic fuzzy subgroup (LVIFSG-3) and a lattice-valued lattice intuitionistic fuzzy sublattice (LVIFSL-3).*

Proposition 3.3 *A lattice-valued intuitionistic fuzzy subset A of G is a lattice-valued intuitionistic fuzzy subℓ-group of G if and only if it is a lattice-valued intuitionistic fuzzy sublattice and a lattice-valued intuitionistic fuzzy subgroup of G.*

Proof. Suppose A is a lattice-valued intuitionistic fuzzy sub ℓ–group (LVIFSℓgroup-3). We clearly know from definition 3.1 conditions (ii & iii) and (iv), respectively, that A is both a Lattice-valued intuitionistic fuzzy subgroup and a Lattice-valued intuitionistic fuzzy sublattice of type-3. More precisely, in an ℓ-group the operations $\wedge$ and $\vee$ are related by the identity $x \vee y = -(-x \wedge -y)$. Now, we check the condition for $\mu_A(x \vee y)$ and $\nu_A(x \vee y)$. Consider$\mu_A(-x \wedge -y)$.Let a=-x and b=-y. Then, x=-a and y=-b. Since,A is LVIFSℓgroup-3, we know that $\mu_A(a \wedge b) \geq \mu_A(a) \wedge \mu_A(b)$,and $\mu_A(-a) \geq \mu_A(a)$. So, $\mu_A(-(a \wedge b)) = \mu_A(-a \vee -b) \geq \mu_A(a \wedge b) = \mu_A(-x \wedge -y) \geq \mu_A(-x) \wedge \mu_A(-y) = \mu_A(x) \wedge \mu_A(y)$. Also, $\mu_A(x) = \mu_A(-(x)) \geq \mu_A(-x)$. This indicates that $\mu_A(x \vee y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x \vee y) \leq \nu_A(x) \vee \nu_A(y)$. Hence, if A is LVIFS$\ell$group-3, it directly satisfies the defining conditions of LVIFSG-3 and LVIFSL-3. Conversely, suppose A is LVIFSL-3 and LVIFSG-3. Thus, A satisfies all the conditions of definition 3.1, it meets the criteria to be a lattice-valued intuitionistic fuzzy subℓ-group of type-3.

Remark 3.4 *Let G be an ℓ-group and $0 \in G$ additive identity. $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$, $\forall x \in G$.*

Proposition 3.5 *Let A be an LVIFS type-3 of an ℓ −group G given by $A(x) = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$. A is an LVIFS$\ell$group-3 of G if and only if*

1. $\mu_A(0) = 1_L$ and $\nu_A(0) = 0_L$

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

2. $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y), \nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$ for all $x, y \in G$

3. $\mu_A(x \vee y) \wedge \mu_A(x \wedge y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x \vee y) \vee \nu_A(x \wedge y) \leq \nu_A(x) \vee \nu_A(y)$ for all $x, y \in G$

Proof. (⟹):Let A be an LVIFS type-3 of an $\ell-$group G given by $A(x) = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$. A is an LVIFS$\ell$group-3 of G. We need to prove the conditions (i), (ii) and (iii) hold.

By conditions (ii) and (iii) of definition 3.1, we have $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$, and by conditions (iv) and (v), we have $\mu_A(x \vee y) \wedge \mu_A(x \wedge y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x \vee y) \vee \nu_A(x \vee y) \leq \nu_A(x) \vee \nu_A(y)$. Besides, from the definition of IF-subgroup, we know that $\mu_A(0) = 1_L$ and $\nu_A(0) = 0_L$.

(⟸): Suppose the conditions (i), (ii), and (iii) hold. Condition (ii) yields the first two conditions required for the part of the operations of the group. So, this part is completed. Now, let us focus on the lattice operations using condition (iii). In that case, we need to check that the two inequalities imply the four inequalities required by the definition. Considering the membership function first, from the property of the meet operation, $\mu_A(x \vee y) \wedge \mu_A(x \wedge y) \geq \mu_A(x) \wedge \mu_A(y) \quad \Rightarrow \mu_A(x \vee y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\mu_A(x \wedge y) \geq \mu_A(x) \wedge \mu_A(y)$. Now, consider the non-membership function. From the definition of the join operation, $\nu_A(x \vee y) \vee \nu_A(x \wedge y) \leq \nu_A(x) \vee \nu_A(y) \Rightarrow \nu_A(x \vee y) \leq \nu_A(x) \vee \nu_A(y)$ and $\nu_A(x \wedge y) \leq \nu_A(x) \vee \nu_A(y)$. Finally, the conditions $\mu_A(0) = 1_L$ and $\nu_A(0) = 0_L$ follows from that of (i). Hence, the result holds as needed.

The following theorem characterizes LVIFSℓgroup-3 of its set of levels.

Theorem 3.6 *An IFS A of an $\ell-$group G is LVIFSℓgroup-3 if and only if $A_{\alpha,\beta}$ is an ℓ-subgroup of G for all $(\alpha, \beta) \in A(G)$. Equivalently, $A \in LVIFS\ell group - 3(G)$ if and only if each nonempty level subset $A_{\alpha,\beta}$ is an subℓ-group of G. In this case, $A_{\alpha,\beta}$ is called the level $\ell-$subgroup of G.*

Proof. Suppose $A \in LVIFS\ell group - 3(G)$. We show that is a $A_{\alpha,\beta}$ is $\ell-$subgroup of G for all $(\alpha, \beta) \in A(G)$. Let $x, y \in G$. $\mu_A(x) = \alpha_1, \mu_A(y) = \alpha_2$ and $\nu_A(x) = \beta_1, \nu_A(y) = \beta_2$. Let $\alpha = \alpha_1 \wedge \alpha_2, and\ \beta = \beta_1 \vee \beta_2$. Therefore, $\mu_A(x), \mu_A(y) \geq \alpha$ and $\nu_A(x), \nu_A(y) \leq \beta$. Hence,$x, y \in A_{\alpha,\beta}$. Since $A \in IF\ell SG(G)$ we have $\mu_A(x+y), \mu_A(x \vee y), \mu_A(x \wedge y) \geq \mu_A(x) \wedge \mu_A(y) \geq \alpha$ and $\nu_A(x+y), \nu_A(x \vee y), \nu_A(x \wedge y) \leq \nu_A(x) \vee \nu_A(y) \leq \beta$. Therefore, $x \vee y, x \wedge y, x+y \in A^{(\alpha,\beta)}$ and for $x \in G, \mu_A(-x) = \mu_A(x) \geq \alpha\ and\ \nu_A(-x) = \nu_A(x) \leq \beta,\ -x \in A_{\alpha,\beta}$. Hence, it $A_{\alpha,\beta}$ is an ℓ-subgroup of $\ell-$group G.

Conversely, suppose for each $(\alpha, \beta) \in A(G)$,$A_{\alpha,\beta}$ is an ℓ-subgroup of $\ell-$group G. We show that A is LVIFSℓgroup-3 of an $\ell-$group G. Let $x, y \in G$ then we have μ_A(x)=$\alpha_1, \mu_A(y) = \alpha_2$ and ν_A(x)=$\beta_1, \nu_A(y) = \beta_2$. Let $\alpha = \alpha_1 \wedge \alpha_2$ and $\beta = \beta_1 \vee \beta_2$.Therefore, $\mu_A(x), \mu_A(y) \geq \alpha$ and $\nu_A(x), \nu_A(y) \leq \beta$. Therefore, $x, y \in A_{\alpha,\beta}$. Since $A_{\alpha,\beta}$ is an ℓ-subgroup of ℓ-group G, $x \vee y, x \wedge y, x-y \in A_{\alpha,\beta}$.Thus,

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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$\mu_A(x-y) \geq \alpha = \mu_A(x) \wedge \mu_A(y) \; and \; \nu_A(x+y) \leq \beta = \nu_A(x) \vee \nu_A(y)$$

$$\mu_A(x \vee y) \geq \alpha = \mu_A(x) \wedge \mu_A(y) \; and \; \nu_A(x \vee y) \leq \beta = \nu_A(x) \vee \nu_A(y)$$

$$\mu_A(x \wedge y) \geq \alpha = \mu_A(x) \wedge \mu_A(y) \; and \; \nu_A(x \wedge y) \leq \beta = \nu_A(x) \vee \nu_A(y)$$

. Hence, by proposition 3.5, A is an IFℓSG-3 of G.

Example 3.7 *Consider $G = \mathbb{Z}$, set of integers. It is obvious that $(G, +, \wedge, \vee)$ with $x \wedge y = min\{x, y\}$ and $x \vee y = max\{x, y\}, \forall x, y \in G$, is an ℓ-group. Let $L = \{0_L, r, s, 1_L\}$ be a lattice where $0_L \leq r, s \leq 1_L$ and $r \parallel s$. Now, define a lattice homomorphism $f: L \to [0,1]$ by $f(0_L) = 0, f(r) = f(s) = 0.5$ and $f(1_L) = 1$*

Now, define a complex mapping $A: G \times G \to L \times L$ by $A(x) = \{\langle x, \mu_A(x), \nu_A(x) \rangle / x \in G\}$ where

$$\mu_A(x) = \begin{cases} 1_L, & if \; x \in \langle 0 \rangle \\ r, & if \; x \in \langle 2 \rangle - \langle 0 \rangle \\ 0_L, & otherwise \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} 0_L, & if \; x \in \langle 0 \rangle \\ r, & if \; x \in \langle 2 \rangle - \langle 0 \rangle \\ 1_L, & otherwise \end{cases}$$

It is easy to verify that $f(\mu_A(x)) + f(\nu_A(x)) \leq 1, \forall x \in G$. Thus, A is a lattice-valued intuitionistic fuzzy set type-3(LVIFSℓgroup-3) of G. Moreover, since it is clear that all the level sets $A_{\alpha,\beta}$ for each $\alpha, \beta \in L$ are subℓ-groups of G. Therefore, by Theorem 1, there refthm 3.4 A is an LVIFSℓgroup-3 of G.

Theorem 3.8 *If A is an LVIFSℓgroup-3 of G, then*

1. $\mu_A(x^+) \geq \mu_A(x)$ and $\nu_A(x^+) \leq \nu_A(x)$, $\mu_A(x^-) \geq \mu_A(x)$ and $\nu_A(x^-) \leq \nu_A(x)$ and $\mu_A(|x|) \geq \mu_A(x)$ and $\nu_A(|x|) \leq \nu_A(x)$
2. Supp(A) is an l-subgroup if $supp(A) \neq \emptyset$, and L is regular.

Proof. Suppose $A \in IF\ell SG(G)$

1. Let $x \in G \Longrightarrow x^+, x^-, |x| \in G$ consider,

$$\mu_A(x) = \mu_A(x) \wedge \mu_A(0) \leq \mu_A(x \vee 0) = \mu_A(x^+) \qquad \nu_A(x) = \nu_A(x) \vee \nu_A(0) \geq \nu_A(x \vee 0) = \nu_A(x^+)$$

in a similar manner, $\mu_A(x^-) \geq \mu_A(x)$,$\nu_A(x^-) \leq \nu_A(x)$, $\mu_A(|x|) \geq \mu_A(x)$ and $\nu_A(|x|) \leq \nu_A(x)$ hold.

2. Suppose $Supp(A) \neq \emptyset$.Let $x, y \in Supp(A)$. Let us consider

$$\mu_A(x) = a > 0 \; and \; \mu_A(y) = b > 0, \nu_A(x) = c < 1 \; and \; \nu_A(y) = d < 1$$

$$\begin{cases} \mu_A(x \vee y) \geq \mu_A \wedge \mu_A(y) = a \wedge b > 0 \\ \mu_A(x \wedge y) \geq \mu_A \wedge \mu_A(y) = a \wedge b > 0 \\ \mu_A(x + y) \geq \mu_A \wedge \mu_A(y) = a \wedge b > 0 \\ \mu_A(-x) = \mu_A(x) = a > 0 \end{cases}$$

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$\begin{cases} \nu_A(x \vee y) \leq \nu_A \wedge \nu_A(y) = c \wedge d < 1 \\ \nu_A(x \wedge y) \leq \nu_A \wedge \nu_A(y) = c \wedge d < 1 \\ \nu_A(x + y) \leq \nu_A \wedge \nu_A(y) = c \wedge d < 1 \\ \nu_A(-x) = \nu_A(x) = c < 1 \end{cases}$$

Therefore, $x \vee y, x \wedge y, x + y, -x \in Supp(A)$. Hence, Supp(A) is a ℓ-subgroup of G.

Let's explore compelling results and illustrative counterexamples that resonate with us all.

Theorem 3.9 *The intersection of any family of LVIFSℓgroup-3s of G is LVIFSℓgroup-3 of G.*

Proof. Let f be a lattice homomorphism. Let there $\{A_i\}_{i\in\Gamma}$ be a family of LVIFSℓgroup-3's of G where for each A_i we have $f(\mu_{A_i})(x) + f(\nu_{A_i})(x) \leq 1, \forall x \in G$. Let $x, y \in G$.We show that it $\bigcap_{\alpha\in\Gamma} A_i$ is an intuitionistic fuzzy sublattice of G. Consider

$$\mu_{\bigcap_{\alpha\in\Gamma} A_i}(x) \wedge \mu_{\bigcap_{\alpha\in\Gamma} A_i}(y) = \wedge\, \mu_{A_i}(x) \wedge \wedge\, \mu_{A_i}(y)$$

$$\leq \wedge\, (\mu_{A_i}(x) \wedge \mu_{A_i}(y))$$

$$= \wedge\, \mu_{A_i}(x \vee y)$$

$$= \mu_{\bigcap_{\alpha\in\Gamma} A_i}(x \vee y)$$

$$\nu_{\bigcap_{\alpha\in\Gamma} A_i}(x) \vee \nu_{\bigcap_{\alpha\in\Gamma} A_i}(y) = \wedge\, \nu_{A_i}(x) \vee \wedge\, \nu_{A_i}(y)$$

$$= \wedge\, (\nu_{A_i}(x) \vee \nu_{A_i}(y))$$

$$\geq \wedge\, \nu_{A_i}(x \vee y)$$

$$= \nu_{\bigcap_{\alpha\in\Gamma} A_i}(x \vee y)$$

$$\mu_{\bigcap_{\alpha\in\Gamma} A_i}(x) \wedge \mu_{\bigcap_{\alpha\in\Gamma} A_i}(y) = \wedge\, \mu_{A_i}(x) \wedge \wedge\, \mu_{A_i}(y) \leq \wedge\, (\mu_{A_i}(x) \wedge \mu_{A_i}(y)) = \wedge\, \mu_{A_i}(x \wedge y) = \mu_{\bigcap_{\alpha\in\Gamma} A_i}(x \wedge y)$$

$$\nu_{\bigcap_{\alpha\in\Gamma} A_i}(x) \vee \nu_{\bigcap_{\alpha\in\Gamma} A_i}(y) = \wedge\, \nu_{A_i}(x) \vee \wedge\, \nu_{A_i}(y) = \wedge\, (\nu_{A_i}(x) \vee \nu_{A_i}(y)) \geq \wedge\, \nu_{A_i}(x \wedge y) = \nu_{\bigcap_{\alpha\in\Gamma} A_i}(x \wedge y)$$

Hence,$\bigcap_{\alpha\in\Gamma} A_i$ is an intuitionistic fuzzy sublattice of G.

Now, we proceed to show that $\bigcap_{\alpha\in\Gamma} A_i$ is an intuitionistic fuzzy subgroup of G. For let $x, y \in G$. Consider

$$\mu_{\bigcap_{\alpha\in\Gamma} A_i}(x) \wedge \mu_{\bigcap_{\alpha\in\Gamma} A_i}(y) = \wedge\, \mu_{A_i}(x) \wedge \wedge\, \mu_{A_i}(y)$$

$$= \wedge\, (\mu_{A_i}(x) \wedge \mu_{A_i}(y))$$

$$\geq \wedge\, \mu_{A_i}(x - y)$$

$$= \mu_{\bigcap_{\alpha\in\Gamma} A_i}(x - y)$$

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$$\nu_{\bigcap_{\alpha\in\Gamma}A_i}(x)\vee\nu_{\bigcap_{\alpha\in\Gamma}A_i}(y)=\wedge\,\nu_{A_i}(x)\vee\wedge\,\nu_{A_i}(y)$$

$$=\wedge\,(\nu_{A_i}(x)\vee\nu_{A_i}(y))$$

$$\leq\wedge\,\nu_{A_i}(x-y)$$

$$=\nu_{\bigcap_{\alpha\in\Gamma}A_i}(x-y)$$

Therefore, it $\bigcap_{\alpha\in\Gamma} A_i$ is an intuitionistic fuzzy subgroup of G. Consequently, $\bigcap_{\alpha\in\Gamma} A_i$ is IFℓSG-3 of G.

Remark 3.10 *The union of any two LVIFSℓgroup-3's need not be true.*

Now, we proceed to show that the union of any two LVIFSℓgroup-3's need not be true, illustrated by the following example.

Example 3.11 *Consider the LVIFSℓgroup-3 A of G in example 3.7 and the following LVIFSℓgroup-3 D on the ℓ-group G defined below, with the same lattice homomorphism f. Now, define a complex mapping $D: G\times G\to L\times L$ by $D(x)=\langle x,\mu_D(x),\nu_D(x)\rangle$ where*

$$\mu_D(x)=\begin{cases}1_L, & if\ x\in\langle 0\rangle\\ s, & if\ x\in\langle 3\rangle-\langle 0\rangle\\ 0_L, & otherwise\end{cases}\text{ and }\nu_D(x)=\begin{cases}0_L, & if\ x\in\langle 0\rangle\\ r, & if\ x\in\langle 3\rangle-\langle 0\rangle\\ 1_L, & otherwise\end{cases}$$

It is easy to verify that, $f(\mu_D(x))+f(\nu_D(x))\leq 1,\forall x\in G$ and with the same argument, it *D* is also LVIFSℓgroup-3 of G.

Now, consider $(A\cup D)(x)=\langle x,\mu_{A\cup D}(x),\nu_{A\cup D}(x)\rangle$ where $\mu_{A\cup D}(x)=\mu_A(x)\vee\mu_D(x)$ and $\nu_{A\cup D}(x)=\nu_A(x)\wedge\nu_D(x)$. It is evident from proposition 2.2 of [7] that $f(\mu_{A\cup D}(x))+f(\nu_{A\cup D}(x))\leq 1,\forall x\in G$, and we have

$$\mu_{A\cup D}(x)=\begin{cases}1_L, & if\ x\in\langle 2\rangle\cup\langle 3\rangle\\ 0_L, & otherwise\end{cases}\text{ and}$$

$$\nu_{A\cup D}(x)=\begin{cases}0_L, & if\ x\in\langle 2\rangle\cup\langle 3\rangle\\ 1_L, & otherwise\end{cases}$$

It is observed that the level set, $(A\cup D)_{\alpha,\beta}=\langle 2\rangle\cup\langle 3\rangle$ which is not an ℓ-subgroup of G. So, by the level-set characterization theorem, $A\cup D$ is not LVIFSℓgroup-3 of G.

Proposition 3.12 *Let $\{A_i\}_{i\in I}$ be a non-empty family of LVIFSℓgroup-3's of an ℓ-group G under inclusion. Then, it $\{A_i\}_{i\in I}$ is a chain if and only if $\bigcup_{i\in I} A_i$ is LVIFSℓgroup-3.*

Corollary 3.13 *Let A and B be any two LVIFSℓgroup-3s of an ℓ-group G. Then, $A\cup B$ is LVIFSℓgroup-3 if and only if either $A\subseteq B$ or $B\subseteq A$.*

Proposition 3.14 *If H is any proper l-subgroup of G, then the IF-subset A of G defined by*

$$A(x)=\begin{cases}(\alpha,\beta) & if\ x\in H\\ (\rho,\theta) & if\ x\notin H\end{cases}$$

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

where $\alpha, \beta, \rho, \theta \in L$ and $\alpha > \rho, \beta < \theta$,and $f(\rho) \neq 0, f(\theta) \neq 1$ is an intuitionistic fuzzy sub ℓ-group type-3 of G.

Definition 3.15 *Let H be any subset of an ℓ −group G. Then it*

$$\chi_H(x) = \begin{cases} (1,0), & if\ x \in H \\ (0,1), & if\ x \notin H \end{cases}$$

is called a *characteristic function* of H.

Based on the above proposition and definition, we prove the following proposition, which states the characteristic function of an ℓ-subgroup is LVIFSℓgroup-3 of G, and vice versa.

Theorem 3.16 *A non-empty subset H of a $\ell -$ group G is a ℓ −subgroup of G if and only if it χ_H is an LVIFSℓgroup-3 of G.*

Proof. Suppose H is a ℓ −subgroup of a ℓ −group G. It is clear, by proposition 3.14, that χ_H is IFSℓG-3 of G.

Conversely, suppose χ_H is LVIFSℓgroup-3 of G. We show that H is a ℓ −subgroup of G. Suppose $H \neq \emptyset$. Let $x, y \in H \Longrightarrow \chi_H(x) = (1,0)\ and\ \chi_H(y) = (1,0)$.

To show that $x \wedge y, x \vee y, x + y, -x \in H$

Consider

$$\begin{cases} \mu_{\chi_H}(x \vee y) \geq \mu_{\chi_H}(x) \wedge \mu_{\chi_H}(y) = 1 \\ \mu_{\chi_H}(x \wedge y) \geq \mu_{\chi_H}(x) \wedge \mu_{\chi_H}(y) = 1 \\ \mu_{\chi_H}(x + y) \geq \mu_{\chi_H}(x) \wedge \mu_{\chi_H}(y) = 1 \\ \mu_{\chi_H}(-x) = \mu_{\chi_H}(x) = 1 \end{cases} \text{ and}$$

$$\begin{cases} \nu_{\chi_H}(x \vee y) \leq \nu_{\chi_H}(x) \vee \nu_{\chi_H}(y) = 0 \\ \nu_{\chi_H}(x \wedge y) \leq \nu_{\chi_H}(x) \vee \nu_{\chi_H}(y) = 0 \\ \nu_{\chi_H}(x + y) \leq \nu_{\chi_H}(x) \vee \nu_{\chi_H}(y) = 0 \\ \nu_{\chi_H}(-x) = \nu_{\chi_H}(x) = 0 \end{cases}$$

Therefore, $x \wedge y, x \vee y, x + y, -x \in H$. Hence, H is a ℓ −subgroup

Proposition 3.17 *Let A be a LVIFSℓgroup-3 of G.Then $G_A = \{x \in G / A(x) = A(0)\}$ where $A(x) = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in G\}$ i.e. $G_A = \{x \in G / \mu_A(x) = \mu_A(0), \nu_A(x) = \nu_A(0)\}$ is an l-subgroup of G.*

Proof. Suppose $A \in IF\ell SG - 3(G)$. We establish that it G_A is a ℓ −subgroup of G.Since $\mu_A(0) = \mu_A(0), \nu_A(0) = \nu_A(0)$. Therefore$0 \in G_A$, $G_A \neq \emptyset$ Let $x, y \in G_A$. Now consider

$$\begin{aligned} \mu_A(0) &= \mu_A(0) \wedge \mu_A(0) \\ &= \mu_A(x) \wedge \mu_A(y) (\because x, y \in G_A) \\ &\leq \mu_A(x \wedge y) (\because A\ is\ \text{LV}IFS\ell G - 3) \end{aligned}$$

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$\nu_A(0) = \mu_A(0) \vee \nu_A(0)$$
$$= \nu_A(x) \vee \nu_A(y)(\because x, y \in G_A)$$
$$\geq \nu_A(x \wedge y)(\because A \; is \; \text{LV}IFS\ell G - 3)$$

$$\mu_A(0) = \mu_A(0) \wedge \mu_A(0)$$
$$= \mu_A(x) \wedge \mu_A(y)(\because x, y \in G_A)$$
$$\leq \mu_A(x \vee y)(\because A \; is \; IF\ell SG - 3)$$

$$\nu_A(0) = \mu_A(0) \vee \nu_A(0)$$
$$= \nu_A(x) \vee \nu_A(y)(\because x, y \in G_A)$$
$$\geq \nu_A(x \vee y)(\because A \; is \; \text{LV}IFS\ell G - 3)$$

and

$$\mu_A(0) = \mu_A(0) \wedge \mu_A(0)$$
$$= \mu_A(x) \wedge \mu_A(y)(\because x, y \in G_A)$$
$$\leq \mu_A(x - y)(\because A \; is \; IF\ell SG - 3)$$

$$\nu_A(0) = \mu_A(0) \vee \nu_A(0)$$
$$= \nu_A(x) \vee \nu_A(y)(\because x, y \in G_A)$$
$$\geq \nu_A(x - y)(\because A \; is \; IF\ell SG - 3)$$

Thus, $x \wedge y, x \vee y, x - y \in G_A$. Hence, it G_A is a ℓ −subgroup of G.

Corollary 3.18 *Let A be a lattice-valued intuitionistic fuzzy sub-ℓ-group of type 3 of G. Then Ker(A) is an l-subgroup of G.*

Definition 3.19 *Let A be an IFS of an ℓ −group and* $[A] = \cap \; \{B/A \subseteq B\}$, *B is any LVIFSℓgroup-3 of G . Then* $[A]$ *is called the intuitionistic fuzzy subgroup ℓ of G generated by A.* $[A]$ *is the smallest LVIFSℓgroup-3 of G containing A.*

The following theorem characterizes the intuitionistic fuzzy sub-ℓ-group generated by any intuitionistic fuzzy subset of G by level sets.

Theorem 3.20 *For any IFS-3 A of G, define IFS-3 B of G as:* $B(x) = \langle x, \mu_B(x), \nu_B(x) \rangle$ *where* $\mu_B(x) = Sup\{\alpha \in L: x \in [\mu_A^\alpha]\}$ *and* $\nu_B(x) = Inf\{\beta \in L: x \in [\nu_A^\beta]\}$

Proof. Let $x, y \in G$. Consider

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$$\mu_B(x) \wedge \mu_B(y) = Sup\{\alpha_1 \in L/x \in [\mu_A^{\alpha_1}]\} \wedge Sup\{\alpha_2 \in L/y \in [\mu_A^{\alpha_2}]\}$$

$$= Sup\{\alpha_1 \wedge \alpha_2 \in L/x \in [\mu_A^{\alpha_1}], y \in [\mu_A^{\alpha_1}]\}$$

put $\alpha = \alpha_1 \wedge \alpha_2$.Then $[\mu_A^{\alpha_1}] \subseteq [\mu_A^{\alpha}]$ and $[\mu_A^{\alpha_2}] \subseteq [\mu_A^{\alpha}]$. Thus, $x, y \in [\mu_A^{\alpha}]$. Since$[\mu_A^{\alpha}]$ is an ℓ-subgroup of G, we get $x - y, x \vee y, x \wedge y \in [\mu_A^{\alpha}]$. Hence,

$$\begin{aligned} \mu_B(x) \wedge \mu_B(y) &\leq Sup\{\alpha \in L/x - y \in [\mu_A^{\alpha}]\} \\ &= \mu_A(x - y) \end{aligned} \tag{3.1}$$

$$\begin{aligned} \mu_B(x) \wedge \mu_B(y) &\leq Sup\{\alpha \in L/x \wedge y \in [\mu_A^{\alpha}]\} \\ &= \mu_A(x \wedge y) \end{aligned} \tag{3.2}$$

$$\begin{aligned} \mu_B(x) \wedge \mu_B(y) &\leq Sup\{\alpha \in L/x \vee y \in [\mu_A^{\alpha}]\} \\ &= \mu_A(x \vee y) \end{aligned} \tag{3.3}$$

Similarly, we obtain

$$\begin{aligned} \nu_B(x) \vee \nu_B(y) &\geq Sup\{\beta \in L/x - y \in [\nu_A^{\alpha}]\} \\ &= \nu_A(x - y) \end{aligned} \tag{3.4}$$

$$\begin{aligned} \nu_B(x) \vee \nu_B(y) &\geq Sup\{\beta \in L/x \wedge y \in [\nu_A^{\alpha}]\} \\ &= \nu_A(x \wedge y) \end{aligned} \tag{3.5}$$

$$\begin{aligned} \nu_B(x) \vee \nu_B(y) &\geq Sup\{\beta \in L/x \vee y \in [\nu_A^{\alpha}]\} \\ &= \nu_A(x \vee y) \end{aligned} \tag{3.6}$$

Hence, by equations (3.1) to (3.6), B is LVIFSℓgroup-3 of G.

Now, we show that B is the smallest LVIFSℓgroup-3 of G containing A. Let $x \in G$, then clearly $\mu_A(x) = \alpha$ and $\nu_A(x) = \beta$ for some $\alpha, \beta \in L$. Now, consider

$$\alpha = \mu_A(x) \leq sup\{\epsilon \in L/x \in [\mu_A^{\alpha}]\}, (\, since\ x \in \mu_A^{\alpha} \subseteq [\mu_A^{\alpha}]) = \mu_B(x), \forall x \in G.$$

and

$$\beta = \nu_A(x) \geq Inf\left\{\epsilon \in \frac{L}{x} \in \left[\nu_A^{\beta}\right]\right\}, \left(\, since\ x \in \nu_A^{\beta} \subseteq \left[\nu_A^{\beta}\right]\right)$$

$$= \nu_B(x), \forall x \in G.$$

Hence, $A \subseteq B$. Let D be any LVIFSℓgroup-3 of G containing A. Let $x \in G$.

$$\mu_B = Sup\{\alpha \in L/x \in [\mu_A^{\alpha}]\}$$

$$\leq Sup\{\alpha \in L/x \in [\mu_D^{\alpha}]\}$$

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$= \mu_D(x), \forall x \in G$

Similarly, $\nu_B(x) \geq \nu_D(x), \forall x \in G$. Therefore, $B \subseteq D$.

Corollary 3.21 *For each $x \in G$, $a, b \in L - \{0_L, 1_L\}$. Then the LVIFSℓgroup-3 of G generated by the fuzzy point $x_{(a,b)}$ is characterized by*

$$[x_{(a,b)}](x) = \begin{cases} (1_L, 0_L), & if\ x = 0 \\ (a, b), & if\ x \in [x] - \{0\} \\ (0_L, 1_L), & otherwise \end{cases}$$

$, \forall\ x \in G$

Proposition 3.22 *The set of all LVIFSℓgroup-3s of anℓ-group G is a complete network under the inclusion relation ⊆. In fact, the supremum and infimum of any family $\{A_i / i \in \triangle\}$ of LVIFSℓgroup-3s are $\langle \bigcup \ \{A_i / i \in \triangle\}\rangle$ and $\bigcap \ \{A_i / i \in \triangle\}$, respectively, and the greatest and smallest elements are χ_G and $\chi_\emptyset$ such that $\chi_G(x) = (1,0)$ and $\chi_\emptyset(x) = (0,1)$ for all $x \in G$.*

3.2 ℓ-homomorphism and lattice-valued intuitionistic fuzzy subℓ-groups type-3

Our findings on the preservation of the LVIFSℓgroup-3 structure under ℓhomomorphisms are consistent with and supported by concurrent works[9, 1], in the field, confirming the canonical nature of the properties within this new framework.

Now, we delve into images and pre-images of LVIFSℓgroup-3s of G and G' under an ℓhomomorphism from G into G'.

Definition 3.23 *Let f be a function from an ℓ-group G into another ℓ-group G' and $A = \{\langle x, \mu_A(x), \nu_A\rangle / x \in G\}$ be an IFS type-3 of G.Then the image f(A) is defined by $f(A) = \{\langle y, f(\mu_A(y)), f(\nu_A)\rangle / y \in G'\}$, where*

$$f(\mu_A(y)) = \begin{cases} Sup\{\mu_A(x) / x \in f^{-1}(y)\} & , if f^{-1}(y) \neq \emptyset \\ 0 & , if f^{-1}(y) = \emptyset \end{cases}$$

and

$$f(\nu_A(y)) = \begin{cases} Inf\{\nu_A(x) / x \in f^{-1}(y)\} & , if f^{-1}(y) \neq \emptyset \\ 1 & , if f^{-1}(y) = \emptyset \end{cases}$$

Definition 3.24 *Let f be a function from an ℓ-group G into another ℓ-group G' and $A' = \{\langle x, \mu_{A'}(z), \nu_{A'}\rangle / z \in G'\}$ be an IFS type-3 of G.Then the pre-image $f^{-1}(A')$is defined by $f^{-1}(A') = \{\langle z, f^{-1}(\mu_{A'})(z), f^{-1}(\nu_{A'})(z)\rangle / z \in G'\}$, where*

$f^{-1}(\mu_{A'})(z) = \mu_{A'}(f(z))$ and $f^{-1}(\nu_{A'})(z) = \nu_{A'}(f(z))$

Definition 3.25 *Let f is a function from an ℓ-group G to another ℓ-group G', and $A = \{\langle x, \mu_A, \nu_A\rangle / x \in G\}$ be an IFS-3 of G. Then, A is said to be f-invariant(IF-invariant), if for $x, y \in G, f(x) = f(y) \Longrightarrow \mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$.*

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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Theorem 3.26 *Let G and G' be two ℓ −groups. Let A and A' be two IFℓ-SG-3s of G and HG', respectively. If f is a homomorphism from an ℓ-group G onto another ℓ-group G' then,*

1. f(A) is LVIFSℓgroup-3 of G', provided A has sup-property
2. $f^{-1}(A')$ is an LVIFSℓgroup-3 of G.
3. $(f(A))(0') = A(0)$, where $0' \in G', 0 \in G$
4. $f(G_A) \subseteq G'_{f(A)}$
5. If A is constant on $kerf$, then$f(A)(f(x)) = A(x)$, for all $x \in G$
6. $f^{-1}(G'_{A'}) = G_{f^{-1}(A')}$

Proof. Suppose A and A' are two intuitionistic fuzzy Sub ℓ-groups of G and G', respectively. And f is an onto ℓ-homomorphism.

1. we show that f(A) is LVIFSℓgroup-3 of G', provided A has sup-property. Let $y_1, y_2 \in G'$. Since f is an onto homomorphism $\exists x_1, x_2 \in G$ such that $y_1 = f(x_1), y_2 = f(x_2)$ Thus,$y_1 \vee y_2 = f(x_1) \vee f(x_2) = f(x_1 \vee x_2)$,$y_1 \wedge y_2 = f(x_1) \wedge f(x_2) = f(x_1 \wedge x_2)$,$y_1 + y_2 = f(x_1) + f(x_2) = f(x_1 + x_2)\ and\ -y_1 = f(-x)$.Therefore, $x_1 \vee x_2 \in f^{-1}(y_1 \vee y_2), x_1 \wedge x_2 \in f^{-1}(y_1 \wedge y_2), x_1 + x_2 \in f^{-1}(y_1 + y_2)\ and\ -x_1 = f^{-1}(-y)$ $\therefore f^{-1}(y_1 \vee y_2) \neq \emptyset, f^{-1}(y_1 \wedge y_2) \neq \emptyset, f^{-1}(y_1 + y_2) \neq \emptyset, f^{-1}(-y) \neq \emptyset$. Now, consider

$$\mu f(A)(y_1 \vee y_2) = \vee \{\mu_A(z)/z \in f^{-1}(y_1 \vee y_2)\}$$

$$\geq \mu_A(x_1 \vee x_2)$$

$$\geq \mu_A(x_1) \wedge \mu_A(x_2)$$

$$= \mu_{f(A)}(y_1) \wedge \mu_{f(A)}(y_2)$$

$$\nu_{f(A)}(y_1 \vee y_2) = \vee \{\nu_A(z)/z \in f^{-1}(y_1 \vee y_2)\}$$

$$\leq \nu_A(x_1 \vee x_2)$$

$$\leq \nu_A(x_1) \vee \nu_A(x_2)$$

$$= \nu_{f(A)}(y_1) \vee \nu_{f(A)}(y_2)$$

$$\mu_{f(A)}(y_1 \wedge y_2) = \vee \{\mu_A(z)/z \in f^{-1}(y_1 \wedge y_2)\}$$

$$\geq \mu_A(x_1 \wedge x_2)$$

$$\geq \mu_A(x_1) \wedge \mu_A(x_2)$$

$$= \mu_{f(A)}(y_1) \wedge \mu_{f(A)}(y_2)$$

Received: August 09, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$\nu_{f(A)}(y_1 \wedge y_2) = \vee \{\nu_A(z)/z \in f^{-1}(y_1 \wedge y_2)\}$

$\leq \nu_A(x_1 \wedge x_2)$

$\leq \nu_A(x_1) \vee \nu_A(x_2)$

$= \nu_{f(A)}(y_1) \vee \nu_{f(A)}(y_2)$

$\mu_{f(A)}(y_1 + y_2) = \vee \{\mu_A(z)/z \in f^{-1}(y_1 + y_2)\}$

$\geq \mu_A(x_1 + x_2)$

$\geq \mu_A(x_1) \wedge \mu_A(x_2)$

$= \mu_{f(A)}(y_1) \wedge \mu_{f(A)}(y_2)$

$\nu_{f(A)}(y_1 + y_2) = \vee \{\nu_A(z)/z \in f^{-1}(y_1 + y_2)\}$

$\leq \nu_A(x_1 + x_2)$

$\leq \nu_A(x_1) \vee \nu_A(x_2)$

$= \nu_{f(A)}(y_1) \vee \nu_{f(A)}(y_2)$

Let $y \in G'$ $\exists x \in G$ such that $y = f(x) \therefore x \in f^{-1}(y) \therefore f^{-1}(y) \neq \emptyset$. Consider

$$\mu_{f(A)}(-y) = \vee \{\mu_A(z)/z \in f^{-1}(-y)\}$$

$\geq \mu_A(-x)$

$= \mu_A(x)$

$= \mu_{f(A)}(y)$

$\mu_{f(A)}(y) = \vee \{\mu_A(z)/z \in f^{-1}(y)\}$

$\geq \mu_A(x)$

$= \mu_A(-(-x))$

$= \mu_A(-x)$

$= \mu_{f(A)}(-y)$

.Thus, we have $\mu_{f(A)}(-y) = \mu_{f(A)}(y)$ and similarly, we can get $\nu_{f(A)}(-y) = \nu_{f(A)}(y)$. Hence, f(A) is LVIFS$\ell$group-3 of a ℓ-group G'.

2. Let $x, y \in G$ Consider

$\mu_{f^{-1}(A')}(x \wedge y) = \mu_{A'}(f(x \wedge y))$

$= \mu_{A'}(f(x) \wedge f(y))$

Received: August 09, 2025

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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$\geq \mu_{A'}(f(x)) \wedge \mu_{A'}(f(y))$$
$$= \mu_{f^{-1}(A')}(x) \wedge \mu_{f^{-1}(A')}(y)$$

Similarly, we get

$$\mu_{f^{-1}(A')}(x \vee y) \geq \mu_{f^{-1}(A')}(x) \wedge \mu_{f^{-1}(A')}(y)$$

$$\mu_{f^{-1}(A')}(x + y) \geq \mu_{f^{-1}(A')}(x) \wedge \mu_{f^{-1}(A')}(y)$$

$$\nu_{f^{-1}(A')}(x \wedge y) = \nu_{A'}(f(x \wedge y))$$
$$= \nu_{A'}(f(x) \wedge f(y))$$
$$\leq \nu_{A'}(f(x)) \vee \nu_{A'}(f(y))$$
$$= \nu_{f^{-1}(A')}(x) \vee \nu_{f^{-1}(A')}(y)$$

Similarly,

$$\nu_{f^{-1}(A')}(x \vee y) \leq \nu_{f^{-1}(A')}(x) \vee \nu_{f^{-1}(A')}(y)$$

$$\mu_{f^{-1}(A')}(x + y) \leq \nu_{f^{-1}(A')}(x) \vee \nu_{f^{-1}(A')}(y)$$

and

$$\mu_{f^{-1}(A')}(-x) = \mu_{A'}(f(-x))$$
$$= \mu_{A'}(f(x))$$
$$= \mu_{f^{-1}(A')}(x)$$

$$\nu_{f^{-1}(A')}(-x) = \nu_{A'}(f(-x))$$
$$= \nu_{A'}(f(x))$$
$$= \nu_{f^{-1}(A')}(x)$$

. Hence, $f^{-1}(A')$ is LVIFSℓgroup-3 of an ℓ-group G.

3. Let G and G' be two ℓ −groups. and $0 \in G$ and $0' \in G'$.

To show that $f(A)(0') = A(0)$. Now, Consider

$$\mu_{f(A)}(0') = \vee \{\mu_A(z)/z \in f^{-1}(0')\}$$
$$= \mu_A(0)$$

International Journal of Applied Mathematics

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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

and

$$\nu_{f(A)}(0') = \vee \{\nu_A(z)/z \in f^{-1}(0')\}$$

$$= \nu_A(0)$$

Hence, $f(A)(0') = A(0)$

4. Let $y \in f(G_A)$, then $\exists x \in G_A$ such that $y = f(x)$.Consider

$$\mu_{f(A)}(y) = \mu_{f(A)}(f(x))$$

$$= \vee \{\mu_A(z)/z \in f^{-1}(f(x))\}$$

$$\geq \mu_A(x)$$

$$= \mu_A(0)$$

$$= \mu_{f(A)}(0')(By\ (iii), above\ )$$

But we know from Proposition 3.8 that $\mu_{f(A)}(0') \geq \mu_{f(A)}(y)$. Thus, $\mu_{f(A)}(0') = \mu_{f(A)}(y)$ and

$$\nu_{f(A)}(y) = \nu_{f(A)}(f(x))$$

$$= \vee \{\nu_A(z)/z \in f^{-1}(f(x))\}$$

$$\leq \nu_A(x)$$

$$= \nu_A(0)$$

$$= \nu_{f(A)}(0')(By\ (iii), above\ )$$

. Again, we know from Proposition 3.8 that $\nu_{f(A)}(0') \leq \nu_{f(A)}(y)$. Thus, $\nu_{f(A)}(0') = \nu_{f(A)}(y)$. Therefore, $y \in G'_{f(A)}$. Consequently, $f(G_A) \subseteq G'_{f(A)}$

5. Suppose A is constant on $Kerf$ (IF-i(IF-in(IF-invariant));how that $f(A)(f(x)) = A(x)$ for all $x \in G$.

Let $x \in G$. Consider

$$\mu_{f(A)}(f(x)) = \vee \{\mu_A(z)/z \in f^{-1}(f(x))\}$$

$$= \bigvee_{x \in kerf} \{\mu_A(z)/z \in f^{-1}(f(0))\} \vee \bigvee_{x \in G/Kerf} \{\mu_A(z)/z \in f^{-1}(f(x))\}$$

$$= \mu_A(0) \vee \bigvee_{x \in G/Kerf} \{\mu_A(z)/z \in f^{-1}(f(x))\}$$

$$= \mu_A(0)$$

$$= \mu_A(x)$$

and

$$\nu_{f(A)}(f(x)) = \wedge \{\nu_A(z)/z \in f^{-1}(f(x))\}$$

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$$= \bigwedge_{x \in kerf} \{\nu_A(z)/z \in f^{-1}(f(0))\} \wedge \bigwedge_{x \in G/Kerf} \{\nu_A(z)/z \in f^{-1}(f(x))\}$$

$$= \nu_A(0) \vee \bigwedge_{x \in G/Kerf} \{\nu_A(z)/z \in f^{-1}(f(x))\}$$

$$= \nu_A(0)$$

$$= \nu_A(x)$$

$\therefore f(A)(f(x)) = A(x)$.

6. We show that $f^{-1}(G'_{A'}) = G_{f^{-1}(A)}$

Let $x \in f^{-1}(G'_{A'})$. $\therefore f(x) \in G'_{A'}$. Therefore,

$$\mu_{A'}(f(x)) = \mu_{A'}(f(0)) \; and \; \mu_{A'}(f(x)) = \mu_{A'}(f(0))$$

Therefore,

$$\mu_{f^{-1}(A')}(x) = \mu_{f^{-1}(A')}(0)$$

Therefore, $x \in G_{f^{-1}(A)}$

$$\therefore f^{-1}(G'_{A'}) \subseteq G_{f^{-1}(A)} \tag{3.7}$$

Now let $x \in G_{f^{-1}(A')}$ $\therefore \mu_{f^{-1}(A')}(x) = \mu_{f^{-1}(A')}(0)$

$$\therefore \mu_{A'}(f(x)) = \mu_{A'}((f0))$$

$$\therefore f(x) \in G'_{A'}$$

$\therefore x \in f^{-1}(G'_{A'})$.

$$\therefore G_{f^{-1}(A)} \subseteq f^{-1}(G'_{A'}) \tag{3.8}$$

By Equations 3.7 and 3.8, we get $f^{-1}(G'_{A'}) = G_{f^{-1}(A)}$

Corollary 3.27 *Let f be an ℓ-homomorphism from an ℓ-group G onto ℓ-group G' and $\{A_i : i \in I\}$ be an arbitrary collection of LVIFSℓgroup-3's of ℓ-group G.Then $f(\bigcap_{i \in I} A_i)$ is LVIFSℓgroup-3 of ℓ-group G'.*

Theorem 3.28 *If A is constant on $kerf$ (equivalently, A is IF-invariant), then*

1. $f^{-1}(f(A)) = A$
2. $f(f^{-1}(A')) = A'$

Remark 3.29 *In Theorem 3.28, we observed that an ℓ-homomorphism of an ℓ-group G onto another ℓ-group G' satisfying one of the conditions (i) and (ii) forms an ℓ-isomorphism between the ℓ-groups.*

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Proposition 3.30 *Let f be a function from an ℓ-group G onto another ℓ-group G' and $A, B \in L^G$ and $A', B' \in L^{G'}$. Then*

1. $A \subseteq B \Longrightarrow f(A) \subseteq f(B)$
2. $A' \subseteq B' \Longrightarrow f^{-1}(A') \subseteq f^{-1}(B')$

Theorem 3.31 *Let f be an ℓ-homomorphism from an ℓ-group G onto another ℓ-group G' and $A, B \in L^G$. Then $f(A \cap B) \subseteq f(A) \cap f(B)$. Equality holds if either A or B is IF-invariant.*

In the following theorem, we give a correspondence between the LVIFSℓgroup-3 G' and those of G that are f-invariant.

Proposition 3.32 *If f is an ℓ-homomorphism from an ℓ-group G onto another ℓ-group G', then there is a one-to-one order-preserving correspondence between the LVIFSℓgroup-3 of G' and those of G that are f-invariant.*

4 Lattice-Valued intuitionistic fuzzy Convex Subℓ-group type-3

4.1 Definition and properties

This section focuses on introducing lattice-valued intuitionistic fuzzy convex subℓ-group of type 3 (C-LVIFSℓgroup-3) and establishing some properties while making a clear intellectual lineage with studies such as [12, 3], demonstrating a comprehensive understanding of the field's development.

Definition 4.1 *Let G be an ℓ −group and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ be an intuitionistic fuzzy subℓ-group of G. Then A is a lattice-valued intuitionistic fuzzy convex subℓ-group of type 3 of G (C-LVIFSℓgroup-3), if for $x, a \in G$ with $|x| \leq |a|$; $\mu_A(x) \geq \mu_A(a)$ and $\nu_A(x) \leq \nu_A(a)$.*

Lemma 4.2 *Let A be LVIFSℓgroup-3of G. Then A is C-LVIFSℓgroup-3 of G iff $0 \leq x \leq a \Longrightarrow \mu_A(0) \geq \mu_A(x) \geq \mu_A(a)$ and $\nu_A(0) \leq \nu_A(x) \leq \nu_A(a)$, for all $x, a \in G$.*

Remark 4.3 *Definition 4.1 and Lemma 4.2 are equivalent.*

Lemma 4.4 *Let A be C-LVIFSℓgroup-3 of G. Then $|x| \leq |a|$ implies $\mu_A(x) \geq \mu_A(a)$ and $\nu_A(x) \leq \nu_A(a)$, for $x, a \in G$*

Proof:Let A be C-LVIFSℓgroup-3 of G.Suppose $|x| \leq |a|$ then we show that $\mu_A(x) \geq \mu_A(a)$ *and* $\nu_A(x) \leq \nu_A(a)$

Clearly, we can obtain $\mu_A(x) \geq \mu_A(a)$ *and* $\nu_A(x) \leq \nu_A(a)$.

Theorem 4.5 *An LVIFSℓgroup-3 A of a ℓ −group G is a C-LVIFSℓgroup-3 of G if and only if for each ℓ-subgroup $A_{\alpha,\beta}$, $(\alpha, \beta) \in A(G \times G) \cup \{(\alpha, \beta) \in L \times L / \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$ is a convex ℓ-subgroup of G. (In fact, for each (α, β),$A_{\alpha,\beta} = \emptyset$ or a convex ℓ-subgroup).*

Theorem 4.6 *If A is C-LVIFSℓgroup-3 of G, then it $Supp(A) = \{x \in G / \mu_A(x) > 0, \nu_A(x) < 1\}$ is a convex ℓ-subgroup of G if $Supp(A) \neq \emptyset$ L is regular.*

Theorem 4.7 *The intersection of a non-empty family of C-LVIFSℓgroup-3s of G is a C-*

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

LVIFSℓgroup-3.

Theorem 4.8 *If A is C-LVIFSℓgroup-3 of G, then there* $G_A = \{x \in G: \mu_A(x) = \mu_A(0), \nu_A(x) = \nu_A(0)\}$ *is a convex ℓ-subgroup of G.*

Theorem 4.9 *If S is any convex ℓ-subgroup of G, then the C-LVIFSℓgroup-3 A of G is defined as:*

$$\mu_A(x) = \begin{cases} s, & if\ x \in S \\ t, & if\ x \notin S \end{cases}$$

$$\nu_A(x) = \begin{cases} \alpha, & if\ x \in S \\ \beta, & if\ x \notin S \end{cases}$$

, where $\alpha, \beta, s, t \in L$ and $\alpha < \beta, t < s$, is C-LVIFSℓgroup-3.

Theorem 4.10 *A non-empty subset S of a* ℓ *−group G is a convex ℓ-subgroup of G if and only if it* χ_S *is a C-LVIFSℓgroup-3 of G.*

Proof. Suppose a non-empty subset S of a ℓ −group G is a convex ℓ −subgroup of G. Proceed to show that it χ_S is C-LVIFSℓgroup-3 of G.

Let $x, a \in G$ with $0 \leq x \leq a$.

Case (i): Iff $a \in S$ So is x, as S is a convex ℓ-subgroup of G.Thus, $\chi_S(x) = (1,0) = \chi_S(a)$. (ii): If and only if $a \notin S \chi_S(x) = (0,1) = \chi_S(a)$ if $x \notin S$.Otherwise, $\chi_S(x) = (1,0)$ so that $\chi_S(x) \geq \chi_S(a)$. Therefore, in both cases we have found that $\chi_S(x) \geq \chi_S(a)$.Hence, χ_S is an Intuitionistic L-fuzzy convex sub ℓ −group of ℓ −group G.

4.2 ℓ-Homomorphism and C-LVIFSℓgroup-3

Now, we focus on examining properties related to images and pre-images of C-LVIFSℓgroup-3 in an ℓ-group under a given ℓ-group homomorphism. In establishing these properties, we adopted PK Sharma's treatment of similar results in this case.

Theorem 4.11 *Let G and G'be two* ℓ *−groups. Let A and B be C-LVIFSℓgroup-3 of G and G', respectively. If* $f: G \longrightarrow G'$ *be an epimorphism, then*

1. f(A) is an C-LVIFSℓgroup-3, provided A is IF-invariant.
2. $f^{-1}(B)$ is an C-LVIFSℓgroup-3 of G.

Theorem 4.12 *Let f be a homomorphism of G onto G'. If A and B are C-LVIFSℓgroup-3's, then* $f(A \cap B) = f(A) \cap f(B)$ *provided that at least one of A or B is IF-invariant.*

Theorem 4.13 *For any Intuitionistic L-fuzzy subset A of G, there exists a smallest C-LVIFSℓgroup-3 containing A.*

Proof. Let $T = \{B/B \in ILFCSl - G(G)\ containing\ A\}$. Clearly, $T \neq \emptyset(\because \chi_G\ is\ ILFCS\ell - G\ of\ G\ containing\ A)$. Since, $A \subseteq B$ for each $B \in T$. Then we have $A \subseteq \bigcap\ B$. Now it remains to show that $\bigcap\ B$ is *C-LVIFSℓ*-G of G. By Theorem4.7, We know that $\bigcap\ B$ is *C-*

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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

LVIFSℓ-G of G. Now, we show that such an *C-LVIFSℓ*-G of G is the smallest.

Let D be any *C-LVIFSℓ*-G of G containing A.Thus, $D \in T$ and $\bigcap \quad B \subseteq D$. Hence, $\bigcap \quad B$ is the smallest *C-LVIFSℓ*-G of G containing A.

Definition 4.14 *Let A be an LVIFS-3 of a* ℓ *−group G. The smallest C-LVIFSℓgroup-3 of G which contains A is called the C-LVIFSℓgroup-3 of G generated by A and denoted by* (A).

4.3 Quotient Lattice-valued Intuitionistic fuzzy convex subℓ-Group type-3

This section introduces a quotient structure for Lattice-valued intuitionistic fuzzy convex sub-ℓ −group-3 and investigates its properties, including distributivity. This work contributes to the ongoing extension of group-theoretic concepts to fuzzy environments, as seen in studies like [11, 9].

Definition 4.15 *Let A be C-LVIFSℓgroup-3 of ℓ-group G and L a complete lattice with top and bottom elements* 1_L *and* 0_L*, respectively. And, a lattice homomorphism* $\alpha: L \longrightarrow [0,1]$. *Then, The L-intuitionistic subset* $x + A = (x + \mu_A, x + \nu_A)$ *where* $x + \mu_A: G \longrightarrow L$ *and* $x + \nu_A: G \longrightarrow L$ *given by* $(x + \mu_A)(y) = \mu_A(-x + y)$ *and* $(x + \nu_A)(y) = \mu_A(-x + y)$, $\forall y \in G$ *such that* $\alpha(x + \mu_A) + \alpha(x + \nu_A) \leq 1$ *is called Lattice-valued Intuitionistic left coset of C-LVIFSℓgroup-3 A of* ℓ *−group G corresponding to x.*

Remark 4.16 *Similarly, we can define LIF right coset of C-LVIFSℓgroup-3 type-3 A of G and study related properties as to the left ones.*

Theorem 4.17 *Let A be C-LVIFSℓgroup-3 of* ℓ *−group G .Then* $x + A = y + A \Leftrightarrow A(-x + y) = A(0) = A(-y + x)$

Proof. Let A be C-LVIFSℓgroup-3. Suppose for any $x, y \in G$, x+A=y+A.

We show that $A(-x + y) = A(0) = A(-y + x)$. That is,To show $\mu_A(-x + y) = \mu_A(0) = \mu_A(-y + x)$ and $\nu_A(-x + y) = \nu_A(0) = \nu_A(-y + x)$, for $x, y \in G$. Consider

$$\mu_A(-x + y) = (x + \mu_A)(y)$$
$$= (y + \mu_A)(y)$$
$$= \mu_A(-y + y)$$
$$= \mu_A(0)$$

and

$$\mu_A(-y + x) = (y + \mu_A)(x)$$
$$= (x + \mu_A)(x)$$
$$= \mu_A(-x + x)$$
$$= \mu_A(0)$$

, and Consider

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$\nu_A(-x+y) = (x+\nu_A)(y)$

$= (y+\nu_A)(y)$

$= \nu_A(-y+y)$

$= \nu_A(0)$

and

$\nu_A(-y+x) = (y+\nu_A)(x)$

$= (x+\nu_A)(x)$

$= \nu_A(-x+x)$

$= \nu_A(0)$

Hence, $A(-x+y) = A(0) = A(-y+x)$

Conversely,Suppose $A(-x+y) = A(0) = A(-y+x)$. Then we show that $x+A = y+A$(i.e. $x+\mu_A = y+\mu_A$ and $x+\nu_A = y+\nu_A$) .Let $z \in G$. Consider

$(x+\mu_A)(z) = \mu_A(-x+z)$

$= \mu_A(-x+y-y+z)$

$\geq \mu_A(-x+y) \wedge \mu_A(-y+z)$

$= \mu_A(0) \wedge \mu_A(-y+z)$

$= \mu_A(-y+z)$

$= (y+\mu_A)(z)$

$$\therefore x+\mu_A \geq y+\mu_A \tag{4.1}$$

Similarly, we have

$$\therefore y+\mu_A \geq x+\nu_A \tag{4.2}$$

By Equations ?? and ?? we get

$$x+\mu_A = y+\mu_A$$

. In similar lines, we have

$$x+\nu_A = y+\nu_A$$

. Hence,

$$x+A = y+A$$

Theorem 4.18 *Let A be C-LVIFSℓgroup-3 of* ℓ *−group G. Then* $x+A = y+A \Leftrightarrow A(x) =$

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$A(y)$

Proof. Let A C-LVIFSℓgroup-3 of ℓ −group G.

$(\Longrightarrow)$ Suppose $x + A = y + A$. We show that $A(x) = A(y)$. Clearly, by Theorem 4.17 $\mu_A(-y + x) = \mu_A(0)$ and $\nu_A(-y + x) = \nu_A(0)$

Let $x \in G$

$$\mu_A(x) = \mu_A(y - y + x)$$
$$\geq \mu_A(y) \wedge \mu_A(-y + x)$$
$$= \mu_A(y) \wedge \mu_A(0)$$
$$= \mu_A(y)$$

Therefore,

$$\mu_A(x) \geq \mu_A(y) \quad (4.3)$$

Similarly, we get

$$\mu_A(y) \geq \mu_A(x) \quad (4.4)$$

By Equations (4.3) and (4.4) we obtain

$$\mu_A(y) = \mu_A(x) \quad (4.5)$$

With a similar argument, we get

$$\nu_A(y) = \nu_A(x) \quad (4.6)$$

Hence, equations (4.5) and (4.6) follow $A(x) = A(y)$.

Theorem 4.19 *Let A be C-LVIFSℓgroup-3 of* ℓ *−group G .Then* $x + A = y + A \Leftrightarrow x + G_A = y + G_A$.

Proof. Let x,y∈ G. Suppose $x + A = y + A$.We show that $x + G_A = y + G_A$.

By theorem 4.17, we have $A(-x + y) = A(0) = A(-y + x)$

$$\therefore -x + y \in G_{G_A}, -y + x \in G_A$$

$\therefore y \in x + G_A \ and \ x \in y + G_A$

$$y + G_A \subseteq x + G_A \ and \ x + G_A \subseteq y + G_A$$

. Hence,

$$x + G_A = y + G_A$$

. Conversely, suppose $x + G_A = y + G_A$.

$$\therefore -x + y \in G_A \ and \ -y + x \in G_A$$

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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$\therefore A(-x+y) = A(0) = A(-y+x)$$

By Theorem 4.17, we get, $x+A = y+A$

Definition 4.20 *Let A be C-LVIFSℓgroup-3 of ℓ −group G and L a complete lattice with top and bottom elements T and B, respectively. And a lattice homomorphism* $f: L \longrightarrow [0,1]$. *Then for* $x, y \in G$, *define* $x+A \leq y+A$ *if and only if* $x+c \leq y$ *for some* $c \in G_A$

Remark 4.21 *let A be C-LVIFSℓgroup-3 of ℓ −group G.*

1. From the above definition, $x+A \leq y+A$ is equivalent to $x \leq y+c$ for some $c \in G_A$
2. If $x \leq y$ in G, then $x+A \leq y+A$
3. for any $x, \in G$, We can redefine the ordering in Definition 4.20 as $x+G_A \leq y+G_A$ if and only if $\exists c \in G_A$ such that $x+c \leq y$.

Theorem 4.22 *The set of all C-LIFLs of C-LVIFSℓgroup-3 A, i.e.* $\frac{G}{A} = \{x+A : x \in G\}$, *is a distributive lattice with the above ordering defined.*

Theorem 4.23 *Let A be C-LVIFSℓgroup-3 of ℓ −group G. Then the following are equivalent.*

1. $x, y \in G, x \vee y = 0 \Longrightarrow x \in G_A \ or \ y \in G_A$,
2. $x, y \in G, x \wedge y = 0 \Longrightarrow x \in G_A \ or \ y \in G_A$
3. $\frac{G}{A}$ is totally ordered.

Proof. Let A be C-LVIFSℓgroup-3 of ℓ −group G

$(i) \Longrightarrow (ii)$.

Suppose $x, y \in G, x \vee y = 0 \Longrightarrow x \in G_A ory \in G_A$.Let $x, y \in G \Longrightarrow -x, -y \in G$. Consider $x \wedge y = 0 \Longleftarrow -(x \wedge y) = 0 \Longrightarrow -x \vee -y = 0$

$$\therefore -x \in G_A \ or \ -y \in G_A$$

$$Therefore, x \in G_A \ or \ y \in G_A.$$

By similar argument,$(ii) \Longrightarrow (i)$.

Suppose $x, y \in G, x \wedge y = 0 \Longrightarrow x \in G_A \ or \ y \in G_A$. $x, y \in G$, we know that $[-(x \wedge y) + x] \wedge [-(x \wedge y) + y] = 0, \Longrightarrow -(x \wedge y) + x \in G_A \ or \ -(x \wedge y) + y \in G_A$(by hypothesis).

$$\therefore x \wedge y + G_A = x + G_A \ or \ x \wedge y + G_A = y + G_A$$

By Theorem 4.19,

$$\therefore x \wedge y + A = x + A \ or \ x \wedge y + A = y + A$$

.Hence, $x+A \leq y+A$ or $y+A \leq x+A$. Hence, $\frac{G}{A}$ is totally ordered.

Conversely, $(iii) \Longrightarrow (ii)$. Suppose it $\frac{G}{A}$ is ordered. We show that, for $x, y \in G$ such that $x \wedge y =$

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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

0, $x \in G_A$ or $y \in G_A$.Let $x, y \in G$, then since it $\frac{G}{A}$ is ordered.

Case 1. suppose $x + A \leq y + A$. Clearly, $(x \wedge y) + A = x + A \Leftrightarrow (x \wedge y) + G_A = x + G_A$

$$\therefore G_A = x + G_A (\because x \wedge y = 0)$$

. Thus,$x \in G_A$.

Similarly, we can prove that $y \in G_A$ provided $y + A \leq x + A$. Hence, $x \in G_A\ or\ y \in G_A$. Consequently, $(iii) \Longrightarrow (ii)$.

Note 4.24 *If A is C-LVIFSℓgroup-3 type-3 of l group G, then G_A is a convex ℓ-subgroupof G. Thus $\frac{G}{G_A}$ of left co-sets of A is a distributive lattice.*

Theorem 4.25 *Let A be C-LVIFSℓgroup-3 of ℓ −group G. Then there is an ordered isomorphism between $\frac{G}{A}$ and $\frac{G}{G_A}$.*

Proof. Suppose A is C-LVIFSℓgroup-3 of ℓ −group G.Define a map $\Phi: \frac{G}{A} \longrightarrow \frac{G}{G_A}$ by $\Phi(x + A) = x + G_A, x \in G$.Evidently, Φ well-defined. Now, we show that Φ is a lattice homomorphism. Consider

$$\Phi((x + A) \vee (y + A)) = \Phi(x \vee y + A)$$

$$= x \vee y + G_A$$

$$= (x + G_A) \vee (y + G_A)$$

$$= \Phi(x + A) \vee \Phi(y + A)$$

∴ Φ is a join homomorphism.

$$\Phi((x + A) \wedge (y + A)) = \Phi(x \wedge y + A)$$

$$= x \wedge y + G_A$$

$$= (x + G_A) \wedge (y + G_A)$$

$$= \Phi(x + A) \wedge \Phi(y + A)$$

∴ Φ is a meet homomorphism. Hence, Φ is a lattice homomorphism.

Let $x + A, y + A \in \frac{G}{A}$. Suppose $x + A \leq y + A$. then

$$\therefore x \wedge y + A = x + A$$

$$\therefore x \wedge y + G_A = x + G_A$$

$$\therefore (x + G_A) \wedge (y + G_A) = x + G_A$$

$$\therefore x + G_A \leq y + G_A$$

$$\therefore \Phi(x + G_A) \leq \Phi(y + G_A)$$

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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Again Let $x + A, y + A \in \frac{G}{A}$. Suppose $\Phi(x + A) \leq \Phi(y + A)$.

$$\therefore x + G_A \leq y + G_A$$

$$\therefore (x + G_A) \wedge (y + G_A) = x + G_A$$

$$\therefore x \wedge y + G_A = x + G_A$$

$$\therefore x \wedge y + A = x + A$$

$$. Hence, x + A \leq y + A$$

. Hence, Φ is an order isomorphism.

5 Conclusion and Future Direction

The present treatise successfully resolved the definitional lacuna encountered when attempting to apply existing Lattice-Valued Intuitionistic Fuzzy Subset structures to ℓ-groups. This resolution was achieved through the rigorous postulation of the Type-3 LVIFS frameworks, which are demonstrated to be the minimal necessary conditions for ensuring that the level sets constitute true (convex) ℓ-subgroups. The establishment of this structural fidelity underpins the entirety of the consequential theorems.

Of paramount significance is the definitive proof that the quotient set induced by a C-LVIFSl group-3 exhibits the algebraic property of a distributive lattice. This outcome extends the classical result concerning the lattice of convex ℓ-subgroups to the fuzzy domain, providing compelling evidence that the Type-3 formalism is a robust, order-preserving generalization of the crisp structure. The inherent distributivity of the quotient structure offers a strong foundation for further topological and algebraic analysis.

Future inquiry may proceed along several promising avenues, including:

1. Representation Theory: Investigating the possibility of representing the quotient distributive lattice via Birkhoff's representation theorem for fuzzy structures.

2. Lattice Properties: Exploring the conditions under which the quotient structure satisfies additional properties such as modularity, complementation, or specific chain conditions.

3. Application to Ideals: Extending the Type-3 methodology to the study of ℓ-ideals[13] and prime ℓ-subgroups within the fuzzy framework.

The findings herein lay the groundwork for a more precise and rigorously justified deployment of lattice-valued intuitionistic fuzzy sets within complex ordered algebraic structures.

Acknowledgment

The authors thank the anonymous reviewers for their insightful feedback, which significantly improved the paper's quality, clarity, and accuracy. Their expertise and guidance are greatly appreciated.

Disclosure statement

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

The authors reported no potential conflict of interest.

Funding

The authors received no direct funding from public or non-public organizations for this research work.

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International Journal of Applied Mathematics

Volume 38 No. 12s, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

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Received: August 09, 2025