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GLOBAL NONEXISTENCE OF SOLUTIONS FOR EQUATIONS WITH A NONLINEAR MEMORY TERM ON THE HEISENBERG GROUP

Abd Elhakim Lamairia *
LAMIS Laboratory
Larbi Tebessi University-Tebessa
Haouam Kamel
LAMIS Laboratory
Larbi Tebessi University-Tebessa

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Abstract

This paper investigates the nonexistence of global solutions for a nonlinear hyperbolic equation with a memory term posed on the Heisenberg group. Using the modified test function method, combined with Youngs inequality and tools from fractional calculus, we establish sufficient conditions guaranteeing finite-time blow-up of weak solutions. The analysis relies on the structure of the Heisenberg group, the properties of the sub-Laplacian, and the interplay between nonlinear damping, source terms, and fractional integral operators. A detailed study of auxiliary test functions is developed, allowing precise estimates of all nonlinear and memory contributions. By deriving sharp integral inequalities and identifying critical exponents, we show that global solutions cannot exist whenever the nonlinearities exceed certain thresholds connected to the geometry of the underlying group. Our results improve and generalize several known blow-up criteria for fractional and ultra-parabolic models. These findings highlight the significant influence of nonlinear memory kernels and the non-Euclidean geometry of the Heisenberg group on the qualitative behavior of solutions.

1.Introduction

Partial differential equations with nonlinear terms and memory effects play a central role in the modeling of complex physical and engineering phenomena.

*hakim24039@gmail.com

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In recent years, the study of fractional operators and their applications on non-Euclidean structures such as the Heisenberg group has attracted significant attention. These tools provide a powerful framework for analyzing the qualitative behavior of solutions, especially regarding their existence, uniqueness, and possible blow-up in finite time. The present work is devoted to investigating the conditions under which global solutions fail to exist, highlighting the influence of nonlinear memory terms and the geometry of the underlying group.

To begin with, let $t \in (0,+\infty), \eta \in \mathbb{H}^n, u = u(t,\eta)$. We look at the following Cauchy problem with a nonlinear memory term

$$\begin{cases} u_{tt} - \Delta_{\mathbb{H}} u + (t+1)^r |u|^{m-1} u_t = \int_0^t (t-s)^{-w} |u|^p ds \\ u(t=0,\eta) = g(\eta)\ ,\ u_t(t=0,\eta) = h(\eta), \end{cases} \tag{1}$$

where u is the unknown real-valued function,$n > 0, m > 1, p > 1$, and $r, w \in \Omega = (0,1)$, and $g(x), h(x)$ are the initial data.

2.Preliminaries

In this section, we present some basic definitions and results that we will need later to state and prove the main results.

Now, we present some results from fractional derivative calculus that we will need in the last section. Like the symbol $D^{\alpha}_{0|t}$ denotes the fractional derivative of order α, and it is defined as follows.

Definition 1: [Integrals fractionnaire de RiemannLiouville]

$$I^{\alpha}_{0|t} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{-(1-\alpha)} f(s) ds, \tag{2}$$

$$I^{\alpha}_{t|T} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{-(1-\alpha)} f(s) ds,$$

Definition 2: [Dérivée fractionnaire de RiemannLiouville]

Let $f \in C^1([0,\infty))$, and let $\alpha \in (0,1)$ a real number. The RiemannLiouville derivative of order α of the function f is defined as follows:

$$D^{\alpha}_{0|t} f(t) = \frac{d}{dt} I^{1-\alpha}_{0|t} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s)\, ds, \tag{3}$$

where Γ is the Gamma function.

$$D^{\alpha}_{t|T} f(t) = -\frac{d}{dt} I^{1-\alpha}_{t|T} f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T (s-t)^{-\alpha} f(s)\, ds, \quad t < T$$

Definition 3: [Caputo fractional derivative]
Let $f \in C^1([0,\infty))$ et $\alpha \in (0,1)$. The Caputo derivative of the function f of

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order α is given by the formula:

$$ {}^{C}D^{\alpha}_{0|t}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha} f'(s)\,ds. \tag{4}$$

remark 1: Il est bien connu que si $f(0) = 0$, alors :

$$D^{\alpha}_{0|t}f(t) = {}^{C}D^{\alpha}_{0|t}f(t).$$

Proposition 1:

For $0 < \alpha < 1$, $-\infty < t < T < +\infty$ we have the following identities

$$D^{\alpha}_{0|t} \circ I^{\alpha}_{0|t}u(t) = u(t),\ t \in (0,T) \tag{5}$$

Moreover, for all $u \in C^{l}[0,T]$, $T > 0$, the following rule holds

$$(-1)^{l} d^{l}_{t} D^{\alpha}_{t|T}u(t) = D^{\alpha+l}_{t|T}u(t), \tag{6}$$

for $l \in \mathbb{N}$

Proposition 2:

Let $g, h \in C[0,T]$, if $D^{\alpha}_{t|T}h(t)$ and $D^{\alpha}_{0|t}g(t)$ exist and belong to $C[0,T]$, then

$$\int_0^t g(s) D^{\alpha}_{s|T}h(s)ds = \int_0^t h(s) D^{\alpha}_{0|s}g(s)ds,$$

for all $0 \leq t \leq T$. **Corollary 1:**

If $\varphi(t) = \left(1 - \frac{t}{T}\right)^{\beta}_{+}, t \geq 0, T > 0, \beta \geq 1$ Then

$$D^{\alpha+i}_{t|T}\varphi(t) = CT^{-\beta}(T-t)^{\beta-\alpha-i}_{+}, \alpha \in (0,1), i = 0,1,2, \tag{7}$$

$$\left(D^{\alpha+i}_{t|T}\varphi\right)(0) = CT^{-\alpha-i}, \alpha \in (0,1), i = 0,1,2, \tag{8}$$

Definition 4:

- The Heisenberg group :

$$\mathbb{H}^{n} = \mathbb{R}^{n}_{x} \times \mathbb{R}^{n}_{y} \times \mathbb{R}_{\tau}$$

with a general element

$$(z,\tau) = (x,y,\tau), \quad z = (x,y) \in \mathbb{R}^{2n}$$

and the group law

$$(x,y,\tau) \circ (x^{'},y^{'},\tau^{'}) = \left(x + x^{'}, y + y^{'}, \tau + \tau^{'} + \frac{1}{2}(x.y^{'} - y.x^{'})\right)$$

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3.Main Results

In this section, we prove that the solution of system (1) blows up. We begin with a definition that clarifies the meaning of the solution to system (1).

Definition 5:

Let $T > 0$, $r \in (0,1)$, $0 < w < 1$, $\alpha = 1 - w$, $b(t) = (t+1)^r$, and $g \in L^1_{loc}(\mathbb{H}^n) \cap L^m_{loc}(\mathbb{H}^n)$, $h \in L^1_{loc}(\mathbb{H}^n)$, We call u a weak solution, if $u \in L^p\left((0,T), L^p_{loc}(\mathbb{H}^n)\right) \cap L^m\left((0,T), L^m_{loc}(\mathbb{H}^n)\right)$, and it satisfies

$$\Gamma(\alpha)\int_0^T\int_{\mathbb{H}^n} I^{\alpha}_{0|t}\left(|u|^p\right)\varphi d\eta dt + \int_{\mathbb{H}^n} h(\eta)\varphi(0,\eta)d\eta$$

$$-\int_{\mathbb{H}^n} g(\eta)\varphi_t(0,\eta)d\eta + \frac{1}{m}\int_{\mathbb{H}^n}|g|^{m-1}g\varphi(0,\eta)d\eta$$

$$= \int_0^T\int_{\mathbb{H}^n} u\varphi_{tt}d\eta dt - \frac{1}{m}\int_0^T\int_{\mathbb{H}^n} \acute{b}(t)|u|^{m-1}u\varphi d\eta dt \tag{9}$$

$$-\frac{1}{m}\int_0^T\int_{\mathbb{H}^n} b(t)|u|^{m-1}u\varphi_t d\eta dt - \int_0^T\int_{\mathbb{H}^n} u\Delta_{\mathbb{H}^n}\varphi d\eta dt,$$

for nonnegative functions (test)

$$\varphi \in C^2\left(([0,T] \times \mathbb{H}^n\right)$$

where

$$\varphi(T,\eta) = \varphi_t(T,\eta) = 0$$

In the sequel, we have $g > f$ and $\exists C > 0$, where $f \le Cg$, and the nonnegative constant C is supposed to be independent of $T > 0$.

Theorem :

Let $n > 0$ and $0 < w < 1$, and let $m > p > 1$. Under the following conditions on the functions (g,h)

$$\int_{\mathbb{H}^n} g(\eta)d\eta > 0$$

$$\int_{\mathbb{H}^n} h(\eta)d\eta > 0$$

$$\int_{\mathbb{H}^n} |g(\eta)|^{m-1}g(\eta)d\eta > 0, \tag{10}$$

and if

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$$\frac{p}{p-1} > \inf_{d>0} \max\left\{\frac{1+\frac{(n+1)d}{2}}{\alpha+d}, \sqrt{\left(\frac{B}{2A}\right)^2 - \frac{C}{A}} - \frac{B}{2A}\right\}, \tag{11}$$

with

$$A = (1-m)(r+m-2)(1+\rho) - \alpha(r+m-2) - (1-r)(1-m),$$

$$B = [(2m-3+r)(1+\rho) + (1-r) + \alpha,$$

$$C = -m^2(\rho+1),$$

Within the framework of definition (3.1), the weak solution of equation (1) is shown not to exist globally in the time variable t.

Proof:

To prove the result by contradiction, we assume that u is a nontrivial weak solution of problem (1), whose existence extends over all time intervals. We adopt the test function method, and for this purpose, for some positive number T, we choose the test function as follows.

$$\varphi = D^{\alpha}_{t|T}\Psi = \varphi_1^l(\eta) D^{\alpha}_{t|T}\varphi_2(t), \tag{12}$$

where $\varphi_1(\eta) = \Phi\left(\frac{|\eta|^2}{\delta}\right)$ and $\varphi_2(t) = \left(1 - \frac{t}{T}\right)_+^{\beta}$ with $1 \leq \beta$ and $0 < \delta$, $\Phi \in C^{\infty}(\mathbb{R}_+)$ is a cutoff nonincreasing function satisfying

$$\Phi(z) = \begin{cases} 1, & 0 \leq z \leq 1, \\ 0, & 2 \leq z. \end{cases}$$

We will now reformulate the weak solution problem as follows:

$$\Gamma(\alpha)\int_0^T \int_{\mathbb{H}^n} I^{\alpha}_{0|t}\left(|u|^p\right) D^{\alpha}_{t|T}\Psi d\eta dt + \int_{\mathbb{H}^n} h(\eta) D^{\alpha}_{t|T}\Psi(0,\eta) d\eta$$

$$- \int_{\mathbb{H}^n} g(\eta)\partial_t D^{\alpha}_{t|T}\Psi(0,\eta) d\eta + \frac{1}{m}\int_{\mathbb{H}^n} |g|^{m-1} g D^{\alpha}_{t|T}\Psi(0,\eta) d\eta$$

$$= \int_0^T \int_{\mathbb{H}^n} u\partial_t^2 D^{\alpha}_{t|T}\Psi d\eta dt - \frac{1}{m}\int_0^T \int_{\mathbb{H}^n} \acute{b}(t)|u|^{m-1} u D^{\alpha}_{t|T}\Psi d\eta dt \tag{13}$$

$$-\frac{1}{m}\int_0^T \int_{\mathbb{H}^n} b(t)|u|^{m-1} u\partial_t D^{\alpha}_{t|T}\Psi d\eta dt - \int_0^T \int_{\mathbb{H}^n} u\Delta_{\mathbb{H}^n} D^{\alpha}_{t|T}\Psi d\eta dt,$$

By using the identity (6) in proposition 1, we obtain

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$$\Gamma(\alpha)\int_0^T\int_{\mathbb{H}^n} I_{0|t}^{\alpha}\left(|u|^p\right) D_{t|T}^{\alpha}\Psi d\eta dt+\int_{\mathbb{H}^n} h(\eta)\varphi_1^l(\eta)D_{t|T}^{\alpha}\varphi_2(0)d\eta$$

$$-\int_{\mathbb{H}^n} g(\eta)\varphi_1^l(\eta)D_{t|T}^{\alpha}\varphi_2(0)d\eta+\frac{1}{m}\int_{\mathbb{H}^n}|g|^{m-1}g\varphi_1^l(\eta)D_{t|T}^{\alpha}\varphi_2(0)d\eta$$

$$=\int_0^T\int_{\mathbb{H}^n} u\varphi_1^l(\eta)D_{t|T}^{\alpha+2}\varphi_2(t)d\eta dt-\frac{1}{m}\int_0^T\int_{\mathbb{H}^n}\acute{b}(t)|u|^{m-1}u\varphi_1^l(\eta)D_{t|T}^{\alpha}\varphi_2(t)d\eta dt \tag{14}$$

$$-\frac{1}{m}\int_0^T\int_{\mathbb{H}^n} b(t)|u|^{m-1}u\varphi_1^l(\eta)D_{t|T}^{\alpha+1}\varphi_2(t)d\eta dt-\int_0^T\int_{\mathbb{H}^n} u\Delta_{\mathbb{H}^n}\varphi_1^l(\eta)D_{t|T}^{\alpha}\varphi_2(t)d\eta dt,$$

Applying Corollary 1 yields

$$\Gamma(\alpha)\int_0^T\int_{\mathbb{H}^n} I_{0|t}^{\alpha}\left(|u|^p\right) D_{t|T}^{\alpha}\Psi d\eta dt+CT^{-\alpha}\int_{\mathbb{H}^n} h(\eta)\varphi_1^l(\eta)d\eta$$

$$+CT^{-\alpha-1}\int_{\mathbb{H}^n} g(\eta)\varphi_1^l(\eta)d\eta+CT^{-\alpha}\int_{\mathbb{H}^n}|g|^{m-1}g(\eta)\varphi_1^l(\eta)d\eta$$

$$=\int_0^T\int_{\mathbb{H}^n} u\varphi_1^l(\eta)D_{t|T}^{\alpha+2}\varphi_2(t)d\eta dt-\frac{1}{m}\int_0^T\int_{\mathbb{H}^n}\acute{b}(t)|u|^{m-1}u\varphi_1^l(\eta)D_{t|T}^{\alpha}\varphi_2(t)d\eta dt \tag{15}$$

$$-\frac{1}{m}\int_0^T\int_{\mathbb{H}^n} b(t)|u|^{m-1}u\varphi_1^l(\eta)D_{t|T}^{\alpha+1}\varphi_2(t)d\eta dt-\int_0^T\int_{\mathbb{H}^n} u\Delta_{\mathbb{H}}\varphi_1^l(\eta)D_{t|T}^{\alpha}\varphi_2(t)d\eta dt,$$

To simplify the first term on the left-hand side of (15), we use integration by parts, relying on Proposition 2 and relation (5). We then obtain the following result:

$$\Gamma(\alpha)\int_0^T\int_{\mathbb{H}^n} I_{0|t}^{\alpha}\left(|u|^p\right) D_{t|T}^{\alpha}\Psi d\eta dt=\int_0^T\int_{\mathbb{H}^n} D_{0|t}^{\alpha}\left(I_{0|t}^{\alpha}\left(|u|^p\right)\right)\Psi d\eta dt$$

$$=\int_0^T\int_{\mathbb{H}^n}|u|^p\Psi d\eta dt, \tag{16}$$

For the remaining terms, we denote by $\Omega_{T^{\frac{d}{2}}}$ the support of φ_1 such that $\delta=T^{\frac{d}{2}}$ and $d>0$.

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$$\Omega_{T^{\frac{d}{2}}} = \sup\left\{\eta \in \mathbb{H} : |\eta|_{\mathbb{H}}^2 \leq 2T^{\frac{d}{2}}\right\}$$

Now, using the Lebesgue dominated convergence theorem, we obtain

$$\lim_{T\to+\infty}\int_{\Omega_T} g(\eta)\varphi_1^l(\eta)d\eta = \int_{\mathbb{H}^n} g(\eta)d\eta$$

$$\lim_{T\to+\infty}\int_{\Omega_T} |g(\eta)|^{m-1}g(\eta)\varphi_1^l(\eta)d\eta = \int_{\mathbb{H}^n} |g(\eta)|^{m-1}g(\eta)d\eta$$

$$\lim_{T\to+\infty}\int_{\Omega_T} h(\eta)\varphi_1^l(\eta)d\eta = \int_{\mathbb{H}^n} h(\eta)d\eta$$

We also have

$$\lim_{T\to+\infty} g(\eta)\varphi_1^l(\eta) = g(\eta)$$

$$\lim_{T\to+\infty} h(\eta)\varphi_1^l(\eta) = h(\eta)$$

$$\lim_{T\to+\infty} |g(\eta)|^{m-1}g(\eta)\varphi_1^l(\eta) = |g(\eta)|^{m-1}g(\eta)$$

We assume that we have $(g,h) \in L^m_{Loc}(\mathbb{H}^n) \times L^1_{Loc}(\mathbb{H}^n)$ satisfying

$$\int_{\mathbb{H}^n} g(\eta)d\eta > 0$$

$$\int_{\mathbb{H}^n} h(\eta)d\eta > 0$$

$$\int_{\mathbb{H}^n} |g(\eta)|^{m-1}g(\eta)d\eta > 0$$

And this leads to

$$\int_{\mathbb{H}^n} g(\eta)\varphi_1^l(\eta)d\eta > 0$$

$$\int_{\mathbb{H}^n} h(\eta)\varphi_1^l(\eta)d\eta > 0$$

$$\int_{\mathbb{H}^n} |g(\eta)|^{m-1}g(\eta)\varphi_1^l(\eta)d\eta > 0$$

When $T > 0$, we now proceed to study the right-hand side of equation (15) in the weak formulation of solutions, based on the following identity.

$$\Delta_{\mathbb{H}}\varphi_1^l(\eta) = \Delta_{\mathbb{H}}\Phi^l\left(\frac{|\eta|_{\mathbb{H}}^2}{T^{\frac{d}{2}}}\right) = \Delta_{\mathbb{H}}\Phi_l(\psi)$$

where $\psi(\eta) = \frac{|\eta|_{\mathbb{H}}^2}{T^{\frac{d}{2}}}$, with $\Phi_l \in C^\infty$
$\Phi_l \equiv 1$ on $[0,1]$ and $\Phi_l \equiv 0$ on $[2,\infty)$.

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Notation:

- $\eta = (x, y, \tau) \in \mathbb{H}^n$,
- $s = \sum_{i=1}^n x_i^2 + \sum_{i=1}^n y_i^2 = |x|^2 + |y|^2$,
- $Q = \tau^2 + s^2$,
- $\rho = |\eta|_{\mathbb{H}} = Q^{\frac{1}{4}}$ so $\rho^2 = Q^{\frac{1}{2}}$,
- $\psi(\eta) = \frac{\rho^2}{T^{\frac{d}{2}}} = \frac{|\eta|_{\mathbb{H}}^2}{T^{\frac{d}{2}}}$,
- We define the horizontal vector fields X_i, Y_i, where $X_i = \partial_{x_i} + 2y_i\partial_\tau$, $Y_i = \partial_{y_i} - 2x_i\partial_\tau, \quad \Delta_{\mathbb{H}} = \sum_{i=1}^n \left(X_i^2 + Y_i^2\right)$

Composition formula (chain rule)
For an composition $\Phi(\psi)$ we have:

$$\Delta_{\mathbb{H}}(\Phi(\psi)) = \Phi^{''}(\psi) \sum_{i=1}^n \left((X_i\psi)^2 + (Y_i\psi)^2\right) + \Phi^{'}(\psi) \sum_{i=1}^n \left((X_i\psi)^2 + (Y_i\psi)^2\right),$$

Calculation of elementary terms
1.Partial derivatives of $\psi = \frac{Q^{\frac{1}{2}}}{T^{\frac{d}{2}}}$ as
$\partial_{x_i} Q = 4sx_i$, $\partial_{y_i} Q = 4sy_i$, $\partial_\tau Q = 2\tau$,we obtain

$$\partial_{x_i}\psi = \frac{2sx_i}{T^{\frac{d}{2}}\rho^2} \text{ , } \partial_{y_i}\psi = \frac{2sy_i}{T^{\frac{d}{2}}\rho^2} \text{ , } \partial_\tau\psi = \frac{\tau}{T^{\frac{d}{2}}\rho^2}$$

2.Apply X_i , Y_i:

$$X_i\psi = \partial_{x_i}\psi + 2y_i\partial_\tau\psi = \frac{2}{T^{\frac{d}{2}}\rho^2}(sx_i + y_i\tau)$$

$$Y_i\psi = \partial_{y_i}\psi - 2x_i\partial_\tau\psi = \frac{2}{T^{\frac{d}{2}}\rho^2}(sy_i - x_i\tau)$$

3.Sum of squares key simplification:

$$\sum_{i=1}^n \left((X_i\psi)^2 + (Y_i\psi)^2\right) = \frac{4}{T^d\rho^4} \sum_{i=1}^n \left((sx_i + y_i\tau)^2 + (sy_i - x_i\tau)^2\right) = \frac{4s}{T^d}$$

The cross terms cancel after summation, and $\sum_{i=1}^n \left(x_i^2 + y_i^2\right) = s$, hence the simplification $= \frac{4s}{T^d}$.

4.Calculation of $\Delta_{\mathbb{H}}\psi$. Recall of the calculation for ρ^α.

$$\Delta_{\mathbb{H}}(\rho^\alpha) = \alpha(\alpha + 2n)s\rho^{\alpha-4},$$

By taking $\alpha = 2$ (then dividing by $T^{\frac{d}{2}}$):

$$\Delta_{\mathbb{H}}\psi = \Delta_{\mathbb{H}}(\rho^2) = \frac{4(1+n)s}{T^{\frac{d}{2}}\rho^2}$$

Final result

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By substituting into the composition formula, we obtain, for $\eta \neq 0$ (and classically where Φ_l is smooth):

$$\Delta_{\mathbb{H}}\varphi_1^l(\eta) = \Phi_l^{''}(\psi(\eta))\frac{4s(\eta)}{T^d} + \Phi_l^{'}(\psi(\eta))\frac{4(n+1)s(\eta)}{T^{\frac{d}{2}}\rho^2(\eta)}$$

By grouping (and using $\rho^2 = T^{\frac{d}{2}}\psi$), it can also be written in the often-used compact form:

$$\Delta_{\mathbb{H}}\varphi_1^l(\eta) = \frac{4s(\eta)}{T^d}\left(\Phi_l^{''}\left(\frac{|\eta|_{\mathbb{H}}^2}{T^{\frac{d}{2}}}\right) + (n+1)\frac{\Phi_l^{'}\left(\frac{|\eta|_{\mathbb{H}}^2}{T^{\frac{d}{2}}}\right)}{\frac{|\eta|_{\mathbb{H}}^2}{T^{\frac{d}{2}}}}\right) \tag{17}$$

Note 1: $\Delta_{\mathbb{H}}\varphi_1^l(\eta)$ is supported in the homogeneous annulus $\left\{T^{\frac{d}{2}} < |\eta|_{\mathbb{H}}^2 < 2T^{\frac{d}{2}}\right\}$ (since $\Phi_l^{'}$ and $\Phi_l^{''}$ vanish for $\psi \leq 1$ and for $\psi \geq 2$).

1.Prendre la valeur absolue :

$$\left|\Delta_{\mathbb{H}}\varphi_1^l(\eta)\right| \leq \frac{4|s(\eta)|}{T^d}\left(\left|\Phi_l^{''}\left(\frac{|\eta|_{\mathbb{H}}^2}{T^{\frac{d}{2}}}\right)\right| + (n+1)\left|\frac{\Phi_l^{'}\left(\frac{|\eta|_{\mathbb{H}}^2}{T^{\frac{d}{2}}}\right)}{\frac{|\eta|_{\mathbb{H}}^2}{T^{\frac{d}{2}}}}\right|\right)$$

Note 2:

If $s(\eta)$ and Φ_l are bounded, this estimate can be simplified to a constant C, where

$$\left|\Delta_{\mathbb{H}}\varphi_1^l(\eta)\right| \leq \frac{C}{T^d}$$

Perfect! We will simplify the expression into a simple upper bound.
Suppose that:
- $s(\eta)$ is bounded: $|s(\eta)| \leq \|s\|_\infty$
- $\Phi_l^{'}$ and $\Phi_l^{''}$ are bounded on their support:

$$\left|\Phi_l^{'}\left(\frac{|\eta|_{\mathbb{H}}^2}{T^{\frac{d}{2}}}\right)\right| \leq M_1$$

$$\left|\Phi_l^{''}\left(\frac{|\eta|_{\mathbb{H}}^2}{T^{\frac{d}{2}}}\right)\right| \leq M_2$$

- Moreover, since Φ_l has compact support, we can bound $\frac{|\eta|_{\mathbb{H}}^2}{T^{\frac{d}{2}}}$ from below by a constant $c > 0$ on this support, so that

$$\left|\frac{\Phi_l^{'}\left(\frac{|\eta|_{\mathbb{H}}^2}{T^{\frac{d}{2}}}\right)}{\frac{|\eta|_{\mathbb{H}}^2}{T^{\frac{d}{2}}}}\right| \leq \frac{M_1}{c}$$

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We can define:

$$C = 4\|s\|_\infty \left(M_2 + (n+1)\frac{M_1}{c} \right),$$

Thus

$$\int_0^T \int_{\mathbb{H}} \left| u \Delta_{\mathbb{H}} \varphi_1^l(\eta) D_{t|T}^{\alpha} \varphi_2(t) \right| d\eta dt \leq C \int_0^T \int_{\mathbb{H}} |u| T^{-d} \left| D_{t|T}^{\alpha} \varphi_2(t) \right| d\eta dt$$

Using the triangle inequality, we rewrite the right-hand side of equation (15) in the weak form of the solutions as follows:

$$\begin{aligned}\Gamma(\alpha)\int_0^T \int_{\mathbb{H}^n} |u|^p \Psi d\eta dt < & \int_0^T \int_{\mathbb{H}^n} \left| u \varphi_1^l(\eta) D_{t|T}^{\alpha+2} \varphi_2(t) \right| d\eta dt \\ & + \frac{1}{m} \int_0^T \int_{\mathbb{H}^n} \left| \acute{b}(t) \right| |u|^m \varphi_1^l(\eta) \left| D_{t|T}^{\alpha} \varphi_2(t) \right| d\eta dt \\ & + \frac{1}{m} \int_0^T \int_{\mathbb{H}^n} \left| b(t) \right| |u|^m \varphi_1^l(\eta) \left| D_{t|T}^{\alpha+1} \varphi_2(t) \right| d\eta dt \\ & + C \int_0^T \int_{\mathbb{H}} |u| T^{-d} \left| D_{t|T}^{\alpha} \varphi_2(t) \right| d\eta dt \end{aligned} \tag{18}$$

The -Youngs inequality states that, for all $A, B, p, q \in \mathbb{R}_+^*$

$$AB \leq \epsilon A^p + C(\epsilon) B^q, \quad pq = q + p.$$

Now, using the -Young inequality on the terms of the right-hand side of equation (18), we obtain:

$$\begin{aligned} & \int_0^T \int_{\mathbb{H}^n} |u| \varphi_1^l(\eta) \left| D_{t|T}^{\alpha+2} \varphi_2(t) \right| d\eta dt \\ & = \int_0^T \int_{\mathbb{H}^n} |u| \Psi^{\frac{1}{p}} \Psi^{\frac{-1}{p}} \varphi_1^l(\eta) \left| D_{t|T}^{\alpha+2} \varphi_2(t) \right| d\eta dt \end{aligned} \tag{19}$$

$$< \epsilon \int_0^T \int_{\mathbb{H}^n} \left[|u| \Psi^{\frac{1}{p}} \right]^p d\eta dt + C(\epsilon) \int_0^T \int_{\mathbb{H}^n} \left[\Psi^{\frac{-1}{p}} \varphi_1^l(\eta) \left| D_{t|T}^{\alpha+2} \varphi_2(t) \right| \right]^q d\eta dt$$

$$= \epsilon \int_0^T \int_{\mathbb{H}^n} |u|^p \Psi d\eta dt + C(\epsilon) \int_0^T \int_{\mathbb{H}^n} \varphi_1^l(\eta) \varphi_2^{\frac{-1}{p-1}}(t) \left| D_{t|T}^{\alpha+2} \varphi_2(t) \right|^{\frac{p}{p-1}} d\eta dt$$

and

$$\frac{1}{m} \int_0^T \int_{\mathbb{H}^n} \left| \acute{b}(t) \right| |u|^m \varphi_1^l(\eta) \left| D_{t|T}^{\alpha} \varphi_2(t) \right| d\eta dt$$

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$$= \frac{1}{m}\int_0^T\int_{\mathbb{H}^n} |u|^m \Psi^{\frac{m}{p}}\Psi^{\frac{-m}{p}}\left|\acute{b}(t)\right|\varphi_1^l(\eta)\left|D_{t|T}^{\alpha}\varphi_2(t)\right|d\eta dt \tag{20}$$

$$< \epsilon\frac{1}{m}\int_0^T\int_{\mathbb{H}^n}|u|^p\Psi d\eta dt + C(\epsilon)\frac{1}{m}\int_0^T\int_{\mathbb{H}^n}\varphi_1^l(\eta)\varphi_2^{\frac{-m}{p-m}}(t)\left|\acute{b}(t)\right|^{\frac{p}{p-m}}\left|D_{t|T}^{\alpha}\varphi_2(t)\right|^{\frac{p}{p-m}}d\eta dt$$

Then,

$$\frac{1}{m}\int_0^T\int_{\mathbb{H}^n}\left|b(t)\right||u|^m\varphi_1^l(\eta)\left|D_{t|T}^{\alpha+1}\varphi_2(t)\right|d\eta dt$$

$$= \frac{1}{m}\int_0^T\int_{\mathbb{H}^n}|u|^m\Psi^{\frac{m}{p}}\Psi^{\frac{-m}{p}}\left|b(t)\right|\varphi_1^l(\eta)\left|D_{t|T}^{\alpha+1}\varphi_2(t)\right|d\eta dt \tag{21}$$

$$< \epsilon\frac{1}{m}\int_0^T\int_{\mathbb{H}^n}|u|^p\Psi d\eta dt + C(\epsilon)\frac{1}{m}\int_0^T\int_{\mathbb{H}^n}\varphi_1^l(\eta)\varphi_2^{\frac{-m}{p-m}}(t)\left|b(t)\right|^{\frac{p}{p-m}}\left|D_{t|T}^{\alpha+1}\varphi_2(t)\right|^{\frac{p}{p-m}}d\eta dt$$

and

$$C\int_0^T\int_{\mathbb{H}^n}|u|T^{-d}\left|D_{t|T}^{\alpha}\varphi_2(t)\right|d\eta dt$$

$$< \epsilon C\int_0^T\int_{\mathbb{H}^n}|u|^p\Psi d\eta dt + C(\epsilon)C\int_0^T\int_{\mathbb{H}^n}\varphi_1^{\frac{l}{1-p}}(\eta)\varphi_2^{\frac{1}{1-p}}(t)T^{\frac{-dp}{p-1}}\left|D_{t|T}^{\alpha}\varphi_2(t)\right|^{\frac{p}{p-1}}d\eta dt \tag{22}$$

By choosing an estimation for small enough, from (19) − (22) with (18), we obtain

$$\int_0^T\int_{\mathbb{H}^n}|u|^p\Psi d\eta dt < I_1 + I_2 + I_3 + I_4 \tag{23}$$

with

$$I_1 = \int_0^T\int_{\mathbb{H}^n}\varphi_1^l(\eta)\varphi_2^{\frac{-1}{p-1}}(t)\left|D_{t|T}^{\alpha+2}\varphi_2(t)\right|^{\frac{p}{p-1}}d\eta dt$$

$$= \left(\int_{\mathbb{H}^n}\varphi_1^l(\eta)d\eta\right)\left(\int_0^T\varphi_2^{\frac{-1}{p-1}}(t)\left|D_{t|T}^{\alpha+2}\varphi_2(t)\right|^{\frac{p}{p-1}}dt\right) = I_1^1 I_1^2 \tag{24}$$

By performing the scaling $\eta = T^{\frac{d}{4}}\xi$ and denoting by Λ the homogeneous dimension of the Heisenberg group (for $\mathbb{H}^n$, $\Lambda = 2n+2$), we obtain

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$$I_1^1 = \int_{\left\{|\eta|_{\mathbb{H}} \le \sqrt{2}T^{\frac{d}{4}}\right\}} \Phi^l\left(\frac{|\eta|_{\mathbb{H}}^2}{T^{\frac{d}{2}}}\right) d\eta = T^{\frac{d\Lambda}{4}} \int_{\left\{|\xi|_{\mathbb{H}} \le \sqrt{2}\right\}} \Phi^l\left(|\xi|_{\mathbb{H}}^2\right) d\xi = CT^{\frac{(n+1)d}{2}} \tag{25}$$

Thus, for fixed T, finiteness follows from the second integral. If one requires a bound uniform in both l and T, it is necessary to control simultaneously $\sup_l \|\Phi^l\|_\infty$ and the growth in T (the family grows like $T^{\frac{d\Lambda}{4}}$ provided the second integral does not depend on T). Also,

$$I_1^2 = \int_0^T \varphi_2^{\frac{-1}{p-1}}(t)\left|D_{t|T}^{\alpha+2}\varphi_2(t)\right|^{\frac{p}{p-1}} dt$$

$$= T^{1-(\alpha+2)q}\int_0^1 (1-s)^{(\beta-(\alpha+2))q-\frac{\beta}{p-1}} ds = CT^{1-(\alpha+2)q}. \tag{26}$$

Applying Corollary 1 yields the integral I_1^2 exists.

Now, by (23)(25) , we obtain

$$I_1 = CT^{1-(\alpha+2)q+\frac{(n+1)d}{2}} \tag{27}$$

For I_2, we have

$$I_2 = \left(\int_{\mathbb{H}^n} \varphi_1^l(\eta)d\eta\right)\left(\int_0^T \varphi_2^{\frac{-m}{p-m}}(t)\left|\acute{b}(t)\right|^{\frac{p}{p-m}}\left|D_{t|T}^{\alpha}\varphi_2(t)\right|^{\frac{p}{p-m}} dt\right) = I_2^1 I_2^2, \tag{28}$$

"It is clear that the estimate of I_2^1 is the same as that of I_1^1.
Sinse $b^{'} = r(t+1)^{r-1}$,$0 < r < 1$, we obtain

$$I_2^2 = \int_0^1 \left[r(1+Ts)^{r-1}\right]^{\frac{p}{p-m}} [1-s]^{\frac{-\beta m}{p-m}} T^{-\alpha\frac{p}{p-m}} (1-s)^{(\beta-\alpha)\frac{p}{p-m}} T ds,$$

$$= r^{\frac{(r-1)p}{p-m}} T^{1-\alpha\frac{p}{p-m}} \int_0^1 [1+Ts]^{\frac{(r-1)p}{p-m}} (1-s)^{-\alpha\frac{p}{p-m}+\beta} ds \tag{29}$$

we split the integral $\int_0^1 = \int_0^{T^{-\lambda}} + \int_{T^{-\lambda}}^1$ into two integrals where $\lambda \in (0,1)$,by taking $\beta >> 1$.
we can estimate the first integral as follows,

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$$\int_0^{T^{-\lambda}} [1+Ts]^{\frac{(r-1)p}{p-m}} (1-s)^{-\alpha\frac{p}{p-m}+\beta} ds \le CT^{-\lambda}, \tag{30}$$

For the second integral, we have

$$[1+Ts]^{\frac{(r-1)p}{p-m}} \le \left(1+T^{1-\lambda}\right)^{\frac{(r-1)p}{p-m}},$$

Thus,

$$\int_{T^{-\lambda}}^1 [1+Ts]^{\frac{(r-1)p}{p-m}} (1-s)^{-\alpha\frac{p}{p-m}+\beta} ds \le \left(1+T^{-\lambda}\right)^{\frac{(r-1)p}{p-m}} \int_{T^{-\lambda}}^1 (1-s)^{-\alpha\frac{p}{p-m}+\beta} ds$$

$$\le C\left(1+T^{1-\lambda}\right)^{\frac{(r-1)p}{p-m}}, \tag{31}$$

We set the notations

$$F = \frac{p(r-1)}{p-m}, \quad G = \frac{p(r-1)}{-(p-m)+p(r-1)}, \tag{32}$$

The equation to solve is

$$T^{-\lambda} + (T^{1-\lambda}+1)^F = T^{-G} \tag{33}$$

We can give approximate solutions depending on the relative size of $T^{1-\lambda}$.
• If $T^{1-\lambda} \gg 1$ (so $T^{1-\lambda}+1 \approx T^{1-\lambda}$)then $(T^{1-\lambda}+1)^F \approx T^{F(1-\lambda)}$ and (33) becomes approximately

$$T^{-\lambda} + T^{F(1-\lambda)} = T^{-G}$$

In this case one of the two left terms dominates:

·If $F(1-\lambda) > -\lambda$ then approximately

$$F(1-\lambda) \approx -G \Rightarrow \lambda \approx 1 + \frac{G}{F} = \frac{p(r-1)}{p(r-1)-(p-m)},$$

·If $F(1-\lambda) < -\lambda$ then approximately

$$-\lambda \approx -G \Rightarrow \lambda \approx G = \frac{G}{F} = \frac{p(r-1)}{p(r-1)-(p-m)},$$

• If $T^{1-\lambda} \ll 1$ (so $T^{1-\lambda}+1 \approx 1$), than $(T^{1-\lambda}+1)^F \approx 1$ and (33) becomes

$$T^{1-\lambda} + 1 = T^{-G} \Rightarrow T^{-\lambda} \approx T^{-G} - 1$$

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Because $T^{-\lambda} > 0$, we need $T^{-G} > 1$ than $\lambda \approx -\log_T(T^{-G} - 1)$
Now if $G < 0$ with large absolute value,then $T^{-G} = T^{|G|} \gg 1$
so

$$T^{-G} - 1 \approx T^{-G}$$

Thus

$$-\log_T(T^{-G}) = -(-G) = G \Rightarrow \lambda \approx G$$

If $G = 0$, then $T^{-G} - 1 = 0$, and the logarithm is undefined.
If $G > 0$, then $T^{-G} < 1$. Hence $T^{-G} - 1 < 0$,which is not defined for the logarithm (in the reals).So no real approximation is possible in this case.

Set

$$\lambda = \frac{p(r-1)}{p(r-1) - (p-m)}$$

Then,

$$T^{-\lambda} \sim T^{(1-\lambda)\frac{p(r-1)}{p-m}},$$

and Thus

$$I_2 \leq CT^{1-\alpha\frac{p}{p-m} - \frac{p(r-1)}{-(p-m)+p(r-1)} + \frac{(n+1)d}{2}} \tag{34}$$

Now, for I_3, we put

$$I_3 = \left(\int_{\mathbb{H}^n} \varphi_1^l(\eta) d\eta\right)\left(\int_0^T \varphi_2^{\frac{-m}{p-m}}(t)\Big|b(t)\Big|^{\frac{p}{p-m}}\Big|D_{t|T}^{\alpha+1}\varphi_2(t)\Big|^{\frac{p}{p-m}} dt\right) = I_3^1 I_3^2, \tag{35}$$

It clear that the estimates for I_3^1 coincide with the above estimate for I_1^1.
For I_3^2,weknow that $b(t) = (t+1)^r$ is a strictly increasing function for $0 < t < 1$ with $0 < r < 1$, Then,For all $0 < t < 1$,we have $b(t) \leq b(T) = (1+T)^r$. Since $p - m > 0$,we obtain

$$(b(t))^{\frac{p}{p-m}} \leq (T+1)^{\frac{rp}{p-m}},$$

That is, when T is large enough, we get:

$$I_3^2 = \int_0^1 \left[(1-s)_+^{\beta}\right]^{\frac{-m}{p-m}} \Big|b(t)\Big|^{\frac{p}{p-m}} \left[CT^{-\alpha-1}(1-s)^{\beta-\alpha-1}\right]^{\frac{p}{p-m}} ds$$

$$\leq (T+1)^{\frac{rp}{p-m}} CT^{1-(\alpha+1)\frac{p}{p-m}} \int_0^1 (1-s)^{\beta-(\alpha+1)\frac{p}{p-m}} ds \tag{36}$$

$$\leq CT^{1-(\alpha+1-r)\frac{p}{p-m}}$$

. Hence,

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$$I_3 \leq CT^{1-(\alpha+1-r)\frac{p}{p-m}+\frac{(n+1)d}{2}}, \tag{37}$$

Finally, for I_4, we have

$$I_4 = C\left(\int_{\mathbb{H}^n} \varphi_1^{\frac{l}{1-p}}(\eta)d\eta\right)\left(\int_0^T \varphi_2^{\frac{1}{1-p}}(t)T^{\frac{-dp}{p-1}}\left|D_{t|T}^{\alpha}\varphi_2(t)\right|^{\frac{p}{p-1}}dt\right) = CI_4^1 I_4^2 \tag{38}$$

It clear that the estimates for I_4^1 coincide with the above estimate for I_1^1. and

$$I_4^2 = T^{-(\alpha+d)\frac{p}{p-1}+1}\int_0^1 (1-s)^{\beta-\alpha\frac{p}{p-1}}ds = CT^{-(\alpha+d)\frac{p}{p-1}+1}, \tag{39}$$

We conclude that;

$$I_4 \leq CT^{-(\alpha+d)\frac{p}{p-1}+1+\frac{(n+1)d}{2}}, \tag{40}$$

From (27), (34), (37), and (40) we obtain

$$\int_0^T \int_{\mathbb{H}^n} |u|^p \Psi d\eta dt < I_1 + I_2 + I_3 + I_4,$$

$$\begin{aligned}\int_0^T \int_{\mathbb{H}^n} |u|^p \Psi d\eta dt < C\Big(&T^{1-(\alpha+2)q+\frac{(n+1)d}{2}} \\ &+T^{1-\alpha\frac{p}{p-m}-\frac{p(r-1)}{-(p-m)+p(r-1)}+\frac{(n+1)d}{2}} \\ &+T^{1-(\alpha+1-r)\frac{p}{p-m}+\frac{(n+1)d}{2}} \\ &+T^{-(\alpha+d)\frac{p}{p-1}+1+\frac{d(n+1)}{2}}\Big).\end{aligned} \tag{41}$$

Our next goal is to make the exponents of T negative.It remains to see that

$$-\frac{(\alpha+d)p}{p-1} + \frac{(n+1)d}{2} + 1 < 0, \tag{42}$$

and

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$$1-\alpha\frac{p}{p-m}-\frac{p(r-1)}{-(p-m)+p(r-1)}+\frac{(n+1)d}{2}<0. \tag{43}$$

By taking $(p-1)q=p$, $2\rho=(n+1)d$ from (42), we obtain

$$1+\rho<q\left(\alpha+\frac{2\rho}{n+1}\right), \tag{44}$$

and from (43) with m $<$ p, after some calculations, we obtain

$$q>\sqrt{\left(\frac{B}{2A}\right)^2-\frac{C}{A}}-\frac{B}{2A}, \tag{45}$$

where:

$$A=(1-m)(r+m-2)(1+\rho)-\alpha(r+m-2)-(1-r)(1-m),$$

$$B=[(2m-3+r)(1+\rho)+(1-r)+\alpha,$$
$$C=-m^2(\rho+1),$$

Consequently,

$$q>\max\left\{\frac{1+\rho}{\alpha+\frac{2\rho}{n+1}},\sqrt{\left(\frac{B}{2A}\right)^2-\frac{C}{A}}-\frac{B}{2A}\right\}, \tag{46}$$

Since $=$ nd$\rho=\frac{(n+1)d}{2}$ and $q=\frac{p}{p-1}$, we obtain the condition

$$\frac{p}{p-1}>\inf_{d>0}\max\left\{\frac{1+\frac{(n+1)d}{2}}{\alpha+d},\sqrt{\left(\frac{B}{2A}\right)^2-\frac{C}{A}}-\frac{B}{2A}\right\}, \tag{47}$$

and from (41) and (47), we find the estimate

$$\int_0^T\int_{\mathbb{H}^n}|u|^p\Psi d\eta dt<CT^{-\chi}, \tag{48}$$

With $\chi=\chi(p,n,m,r,w)$ So, when we take the limit at $T\to+\infty$,using the dominated convergence Theorem, and the fact that

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$$\lim_{T\to+\infty} \quad \Psi = 1 \ for \ 0 < t < T,$$

We find that $$\int_0^T \int_{\mathbb{H}^n}$$

$$|u|^p \Psi d\eta dt = 0,$$

This implies that $u \equiv 0$, which contradicts (10).□ Conflict of Interest Statement: The authors declare that there is no financial or personal conflict of interest that could have influenced the content of this research.

Compliance with Ethical Standards
not applicable.

References

[1].A.Lamairia, Nonexistence of global solutions of some nonlinear ultra-parabolic equations on the Heisenberg group,ITALIAN JOURNAL OF PURE AND APPLIED MATHEMATICS N. 492023 (5470

[2].A.Lamairia, E., Haouam, K., Rebiai, B. (2018). Nonexistence of global solutions to a fractional nonlinear ultra-parabolic system. Italian Journal of Pure and Applied Mathematics, 40, 748755.

[3].B. Rebiai, K. Haouam, Nonexistence of global solutions to a nonlinear frac- tional reaction-diffusion system, IAENG International Journal of Applied Mathematics, 45(4) (2015), 259-262.

[4].H. Gimperlein, M. Ott, E. Stiegler, Fractional wave equations on Lie groups, J. Funct. Anal., 282 (2022), 109336.

[5].M. DAbbicco, G. Lucente, A modified test function method for damped wave equations, Adv. Nonlinear Stud., 13 (2013), 867892.

[6].L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations, 32 (2007), 12451260.

[7].K.Haouam, Nacer-Eddine Hamri , François Hamel. Existence et non-existence de solutions deséquations différentielles fractionnaires, Constantine; 2007.

[8]. K. Haouam, M. Sfaxi, Critical exponent for nonlinear hyperbolic system with spatio-temporal fractional derivatives, I. J. of App. Math., 6 (2011), 661-871

[9].K. Fujiwara, Blow-up of solutions to semilinear damped wave equations in the Heisenberg group, J. Differential Equations, 265 (2018), 45674605.

[10].T. Kato, Blow-up of solutions of some nonlinear hyperbolic equations, Comm. Pure Appl. Math., 33 (1980), 501505.

[11].V. Georgiev, G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source terms, J. Differential Equations, 109 (1994), 295308.

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