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FOUR-DIMENSIONAL REAL PRE-HILBERT ALGEBRAS SATISFYING $\|x^2\| = \|x\|^2$ AND $(x,x,x) = 0$

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Abstract

This paper deals with some results concerning the four-dimensional Pre-Hilbert third-power associative algebras satisfying $\|x^2\| = \|x\|^2$. We show that if A contains a central idempotent e such that $||ex|| = ||x||$, then A is flexible and is isomorphic to $\mathbb{H}^{(\alpha)}$ or $\overset{*}{\mathbb{H}}^{(\alpha)}$ where $\alpha \neq \frac{1}{2}$.

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Key Words: Pre-Hilbert Algebra, Flexible Algebra, Central idempotent, Third-power associative algebra.

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1 Introduction

We denote by $(.|.)$ the inner product (see [8], [9], [13], [15] and [10]). This notation will be used throughout this work to represent the scalar product in the considered Hilbert space.

A is said to be a division algebra if the right multiplication operator R_x and the left multiplication operator L_x by a non-zero element x of A are bijectives.

Frobenius (1878, see [4]) gave a boost to the problem of determining finite-dimensional real division algebras (RDA). He proved that $\{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ classifies the associative RDA. This illustrates the uniqueness of the non-commutative nature of $\mathbb{H}$ (the real algebra of quaternions, Hamilton 1843) among these algebras. Zorn (1931, see [16]) provided an alternative analogue by showing that $\mathbb{O}$ (the real algebra of octonions, 1845) is the unique alternative and non-associative RDA. By using arguments from algebraic geometry and functional analysis, L. Ingelstam (1962, see [6]), M. F. Smiley (1965, see [13]), and Froelich (1993, see [5]) arrived at the same result as Frobenius and proved that normed pre-Hilbert associative unital algebras with unit e such that $\|e\| = 1$ are division algebras. Subsequently, these are isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. This result was extended to the non-associative case by B. Zalar (1995, see [15]), who showed that normed pre-Hilbert alternative unital algebras with unit e such that $\|e\| = 1$ and $\|x^2\| \leq \|x\|^2$ for all $x \in A$ are isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.

A. Cuenca Mira (2007, see [1]) generalized Ingelstam's result by replacing the conditions of associativity and the existence of a unit e with weaker conditions, proving that alternative pre-Hilbert algebras containing a central idempotent e such that $\|ex\| = \|e\|\|x\|$ for all $x \in A$ are isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. Motivated by these facts, we became interested in the study of pre-Hilbert algebras satisfying $\|x^2\| = \|x\|^2$ and containing a non-zero central idempotent e such that $\|ex\| = \|x\|$ for all $x \in A$. In Section 2, we introduce the basic tools for our study. In Section 3, we classify algebraically the Four-dimensional pre-Hilbert algebras satisfying $\|x^2\| = \|x\|^2$

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and containing a central idempotent e such that $\|ex\| = \|x\|$ for all $x \in A$. We show the following theorem:

Theorem 1. *Let A be a Four-dimensional real pre-Hilbert third-power associative algebra satisfying $\|x^2\| = \|x\|^2$ and containing a non-zero central idempotent e such that $\|ex\| = \|x\|$ for all $x \in A$. Then A is isomorphic to $\mathbb{H}^{(\alpha)}$ or $\overset{*}{\mathbb{H}}{}^{(\alpha)}$ where $\alpha \neq \frac{1}{2}$.*

In Section 2, we introduce the basic tools for the study of Four-dimensional third-power associative pre-Hilbert algebras.

In Section 3, we classify algebraically all four-dimensional real third-power associative pre-Hilbert algebras satisfying $\|x^2\| = \|x\|^2$ and containing a non-zero central idempotent e such that $\|ex\| = \|x\|$.

2 Notations and Preliminaries

In this paper, all algebras are assumed to be over the field $\mathbb{R}$ of real numbers and not necessarily associative. Let A be an algebra. We denote by (x, y, z) the associator $(xy)z - x(yz)$ of $x, y, z \in A$ and by $[x, y]$ the commutator $xy - yx$ of $x, y \in A$. The subalgebra of A generated by an element $x \in A$ is denoted by $A(x)$. We denote by $A^{(\lambda)}$, the λ-mutation of A(the algebra with vector space A and product defined by $x * y = \lambda xy + (1 - \lambda)yx$ for all $x, y \in A$).

We pose $V := e^{\perp}$, where e is a central idempotent of A.

Definition 2. Let A be an arbitrary algebra. An algebra A is called a:

1. Pre-Hilbert algebra if it is endowed with a space norm that comes from an inner product $(\cdot, \cdot)$ such that

$$(\cdot, \cdot) : A \times A \longrightarrow \mathbb{R}, \quad (x|y) \longmapsto \langle x, y \rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

2. Flexible algebra if it satisfies the identity $(x, y, x) = 0$ for all $x, y \in A$.

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3. Third Power Associative if it satisfies the identity $(x, x, x) = 0$ for all $x \in A$

4. Power-commutative if any subalgebra generated by a single element is commutative

5. Division algebra if $A \neq \{0\}$ and the following linear transformations
$$L_a : A \to A, \quad L_a(x) = ax$$
$$R_a : A \to A, \quad R_a(x) = xa$$
are invertible for each nonzero element $a \in A$

The following results are useful for our study.

Theorem 3 (**Segre (1954)**, see [12]). *Every finite-dimensional real or complex algebra with the property that $x^2 = 0$ implies $x = 0$ has at least one non-zero idempotent.*

Theorem 4 (**Moutassim (2008)**, see [8]). *Let A be a four-dimensional Pre-Hilbert flexible algebra, without a divisor of zero, satisfying $\|x^2\| = \|x\|^2$. Then A is isomorphic to $\overset{*}{\mathbb{H}}{}^{(\alpha)}$ or $\mathbb{H}^{(\alpha)}$, where $\alpha \neq \frac{1}{2}$.*

Theorem 5 (**Diankha & al (2016)**, see [2]). *Let A be a four-dimensional third power associative real division algebra containing a central element. Then A is power commutative.*

Theorem 6 (**Cuenca (2012)**, see [1]). *Let A be a Pre-Hilbert algebra with inner product $(\cdot \mid \cdot)$ and a norm-one central idempotent e. Then there exists an anti-commutative product $\wedge$ in $V := e^{\perp}$ which makes V a Pre-Hilbert algebra relative to the restriction of $(\cdot \mid \cdot)$, satisfying $(ev \mid v \wedge w) = 0$ for every $v, w \in V$, and such that the product in A of two arbitrary elements $\alpha e + v$ and $\beta e + w$ ($\alpha, \beta \in \mathbb{R}$, $v, w \in V$) is given by:*

$$(\alpha e + u)(\beta e + v) = (\alpha\beta - (u \mid v))e + \beta ev + \alpha eu + u \wedge v$$

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Remark 7. *We set $\alpha = \beta = 0$; we can define the product $\wedge$ of V as follows:*

$$u \wedge v = uv + (u \mid v)e. \qquad \forall u, v \in V$$

So

$$(uv \mid e) = -(u \mid v)e$$

Theorem 8 (**Moutassim & al (2022)**, see [10]). *Let A be a Two-dimensional real Pre-Hilbert algebra, without divisors of zero, and satisfying $\|a^2\| = \|a\|^2$ for all $a \in A$. Then A is isomorphic to one of $A_1(\gamma,\delta)$, $A_2(\gamma,\delta)$, or $A_3(\gamma,\delta)$ for all $(\gamma,\delta) \in \mathbb{R} \times \mathbb{R}^*$, such that $\gamma^2 + \delta^2 < \pm 2\delta$.*

$A_1(\gamma,\delta)$	e	u
e	e	$\gamma e + \delta u$
u	$-\gamma e - \delta u$	e

$A_2(\gamma,\delta)$	e	u
e	e	$\gamma e + \delta u$
u	$-\gamma e + (2-\delta)u$	-e

$A_3(\gamma,\delta)$	e	u
e	e	$\gamma e + \delta u$
u	$-\gamma e - (2+\delta)u$	-e

Corollary 9. *Let A be a Two-dimensional real Pre-Hilbert algebra, containing a nonzero central idempotent, without divisors of zero, and satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$. Then A is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$.*

Corollary 10. *Let A be a Two-dimensional real Pre-Hilbert third power associative algebra, without divisors of zero, and satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$. Then A is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$.*

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3 Main Results

In this section, we suppose that the algebra A contains a nonzero norm-one central idempotent e such that $\|e\| = 1$ and $\|ex\| = \|x\|$ for all $x \in A$.

Lemma 11. *Let $x, y \in A$. We have:*

1) $(x^2|xy + yx) = 0$

2) $\|xy + yx\|^2 + 2(x^2|y^2) = 2\|x\|^2\|y\|^2$

Proof. We obtain this lemma by a straightforward linearization of the identity $\|x^2\| = \|x\|^2$. □

Lemma 12. *Let $x, y \in A$. We have:*

1) $x^2 = -\|x\|^2 e$, *for all* $x \in V$.

2) $xy + yx = -2(x \mid y)e$

Proof. 1. Let $x \in V$, by lemma 11, we have

$$(x^2/e) = -\|x\|^2$$

Hence

$$||(x^2|e) + \|x\|^2||^2 = 0$$

This implies that

$$x^2 = -\|x\|^2 e$$

2. Immediate consequence of (1).

□

Lemma 13. *Let A be a Four-dimensional real Pre-Hilbert third-power associative algebra, without divisors of zero, satisfying $\|x^2\| = \|x\|^2$ and containing a non-zero central idempotent e. If A contains a two-dimensional subalgebra B, then $e \in B$.*

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Proof. Let $x \in B$. By Lemma 12, we have $x^2 = -\|x\|^2 e$. Thus, $e = -\|x\|^{-2}x^2 \in B$. This means that e is an omnipresent idempotent. $\square$

Lemma 14. *Let A be a Four-dimensional real Pre-Hilbert third-power associative algebra, without divisors of zero, satisfying $\|x^2\| = \|x\|^2$ and containing a non-zero central idempotent e. Then for all $x \in V$, $A(x,e) \simeq \overset{*}{\mathbb{C}}$ or $\mathbb{C}$.*

Proof. As A is power commutative (Theorem 5), $A(x)$ is a commutative subalgebra of A, so $\dim A(x) \leq 2$. If $\dim A(x) = 1$, there exists an idempotent f such that $x = \lambda f$, or $[f,e] = 0$, which implies that $A(f,e) \simeq \overset{*}{\mathbb{C}}$, then $(f \mid e) = -\frac{1}{2}$. In the oder hand we have $0 = (x \mid e) = (e \mid f)$, which is a contradiction. So $\dim A(x) = 2$. Then lemma 13 and corollary 9 ends the proof. $\square$

Lemma 15. *for all $x, y, z \in V$, we have*

$$(yx|z) = (y|xz)$$

Proof. Let $x, y, z \in V$, by Theorem 6. We have $(ex|x \wedge y) = 0$. and by lemma 14, we have $A(e,x) \simeq \overset{*}{\mathbb{C}}$ *or* $\mathbb{C}$, so $ex = \pm x$, which imply that

$$(x|xy) = 0$$

Moreover

$$0 = (x(y+z)|y+z) = (xy|z) + (xz|y)$$

So

$$(yx|z) = (y|xz)$$

$\square$

we have the main Theorem in this section.

Theorem 16. *Let A be a Four-dimensional real Pre-Hilbert Third-Power associative algebra, satisfying $\|x^2\| = \|x\|^2$ and containing a non zero central idempotent e such that $||ex|| = ||x||$ for all $x \in A$. Then A is isomorphic to $\mathbb{H}^{(\alpha)}$ or $\overset{*}{\mathbb{H}}^{(\alpha)}$ where $\alpha \neq \frac{1}{2}$.*

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Proof. Let $B = \{e, u, v, w\}$ be an orthonormal basis of A, we have by lemma 14. $A(e,u) \simeq \{\overset{*}{\mathbb{C}}, \mathbb{C}\}$, $A(e,v) \simeq \{\overset{*}{\mathbb{C}}, \mathbb{C}\}$, $A(e,w) \simeq \{\overset{*}{\mathbb{C}}, \mathbb{C}\}$. So we have

$$eu = \pm u, \quad and \quad u^2 = -e$$
$$ev = \pm v, \quad and \quad v^2 = -e$$
$$ew = \pm w, \quad and \quad w^2 = -e$$

And by remark 7 we have

$$(uv \mid e) = -(u \mid v) = 0$$

By lemma 15 and lemma 12, we have

$$(uv \mid u) = -(u \mid u^2) = (u \mid e) = 0$$

And

$$(uv \mid v) = (u \mid v^2) = (u \mid e) = 0$$

Hence $uv = -vu = \alpha w$, with $\alpha \in \mathbb{R}^*$, and by the same way we get $wu = -uw = \beta v$ and $vw = -wv = \gamma u$ with $\beta, \gamma \in \mathbb{R}^*$

In the other hand we have

$$(uv \mid w) = (u \mid vw) \quad and \quad (uw \mid v) = (u \mid wv)$$

So $\gamma = \beta = \alpha$

We distinguish the following cases.

Cas -1- If $A(e,u), A(e,v)$, and $A(e,w)$ are isomorphic to $\mathbb{C}$, then A is isomorphic to $\mathbb{H}^{(\alpha)}$.

Cas -2- If $A(e,u), A(e,v)$, and $A(e,w)$ are isomorphic to $\overset{*}{\mathbb{C}}$, then A is isomorphic to $\overset{*}{\mathbb{H}}^{(\alpha)}$.

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Cas -3- If $A(e,u), A(e,v)$ are isomorphic to $\mathbb{C}$ and $A(e,w)$ is isomorphic to $\overset{*}{\mathbb{C}}$, then the multiplication table of A is given by:

	e	u	v	w
e	e	u	v	- w
u	u	-e	α w	$-\alpha$ v
v	v	$-\alpha$ w	-e	α u
w	$-w$	α v	$-\alpha$ u	-e

In this case we have

$$(e+u+v+k, e+u+v+k, e+u+v+k) = 8\alpha(i-j) \neq 0$$

Contradiction

Cas -3- If $A(e,u), A(e,v)$ are isomorphic to $\overset{*}{\mathbb{C}}$ and $A(e,w)$ is isomorphic to $\mathbb{C}$, then the multiplication table of A is given by:

	e	u	v	w
e	e	$-u$	$-v$	w
u	$-u$	-e	α w	$-\alpha$ v
v	$-v$	$-\alpha$ w	-e	α u
w	w	α v	$-\alpha$ u	-e

in this case case we have

$$(e+u+v+k, e+u+v+k, e+u+v+k) = 4\alpha(j-i) \neq 0$$

Contradiction. □

Corollary 17. *Let A be a Four-dimensional real Pre-Hilbert algebra, satisfying $\|x^2\| = \|x\|^2$ and containing a non zero central idempotent e such that $||ex|| = ||x||$, for all $x \in A$. Then the following statements are equivalent:*

1. *A is flexible.*

2. *A is third-power assiciative.*

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3. A is isomorphic to $\mathbb{H}^{(\alpha)}$ *or* $\overset{*}{\mathbb{H}}^{(\alpha)}$, *where* $\alpha \neq \frac{1}{2}$

Proof. 1. 1) $\Longrightarrow$ 2) Trivial.

2. 2) $\Longrightarrow$ 3) By theorem 16.

3. 3) $\Longrightarrow$ 1) By a simple calculation.

□

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