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EQUITABLE COLORING IN PATH, CYCLE, AND RELATED GRAPH CLASSES UNDER G^{xyz} OPERATIONS

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Abstract

Our research focuses on determining the equitable chromatic number associated with a defined operation G^{xyz}, where $\{x, y, z \in \{-\}\}$, applied to a variety of graph classes. Specifically, we analyze the behavior of this parameter across path, cycle, wheel, umbrella, tadpole, sundat, and octopus graphs. Our results provide insights into the structural implications of the G^{xyz} transformation on equitable coloring properties within these graph families.

Math. Subject Classification: 05C15.

Key Words and Phrases: Equitable Coloring; G^{---} Transformation; Path; Cycle; Wheel; Umbrella; Tadpole; Sundat; Octopus Graph

1 Introduction

An equitable h-coloring of a graph H is a proper vertex coloring in which the cardinality of any two color classes differs by at most one. The concept, introduced by Meyer [8], includes the result that every tree admits an equitable Δ-coloring.

One of the cornerstone results in equitable coloring is the Hajnal et al. [5], which guarantees that every graph with Δ admits an equitable $(\Delta + 1)$-coloring. This concept can be viewed as

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a refinement of Brooks' Theorem [2], which establishes an upper bound on the chromatic number of a connected graph. Unlike Brooks' Theorem, which guarantees the feasibility of proper coloring, equitable coloring requires that the vertex set be divided into color classes of nearly equal size.

Several well-established results support the theory of equitable coloring. Trees are equitably Δ-colorable, consistent with Meyer's original result [8]. In the case of complete bipartite graphs $K_{m,n}$, an equitable 2-coloring exists if and only if the sizes of the partite sets differ by at most one [8]. Every cubic graph is equitably 3-colorable, with the sole exception of the complete graph K_4 [3]. Furthermore, it has been shown that planar graphs with maximum degree $\Delta \geq 9$ admit equitable Δ-colorings [6].

2 Preliminaries

Definition 1. The transformation graph G^{xyz} [7, 12], where $\{x, y, z \in \{-\}\}$ of a graph G is defined on the vertex set $V(G) \cup E(G)$. Two vertices u' and v' in G^{---} are connected if and only if one of the following conditions holds:

1. $u', v' \in V[G]$ and u' and v' are not adjacent in G,
2. $u', v' \in E[G]$ and the edges u' and v' are not adjacent in G,
3. one of u' and v' lies in $V[G]$ and the other in $E[G]$, and u' and v' are not incident in G.

Definition 2. A path graph P_q [11] consists of a sequence of q vertices connected consecutively by $q-1$ edges, with no repeated vertices or edges.

Definition 3. A cycle graph C_m [11] is a closed chain such that each vertex has degree 2, and the first and last vertices are the same.

Definition 4. An octopus graph $O_{c,d}$ [1] is constructed by joining a star graph $K_{1,c}$ and a fan graph F_d $(d \geq 2)$ at a common vertex: $O_{c,d} = K_{1,c} + F_d$.

Definition 5. A wheel graph W_s [11] is generated by placing a hub vertex at the center of the cycle C_{s-1} and connecting it to each vertex on the rim.

Definition 6. An umbrella graph $U_{a,b}$ [10] is obtained by joining a path P_a to the central vertex of a fan graph F_b.

Definition 7. A tadpole graph $T_{s,t}$ [9] consists of a cycle C_s with a path P_t appended to one of its vertices, forming a tadpole-like configuration.

Definition 8. A sundat graph St_h [4] has the vertex set $V(St_h) = \{i\} \cup \{k_n, l_n, m_n \mid 1 \leq n \leq h\}$.

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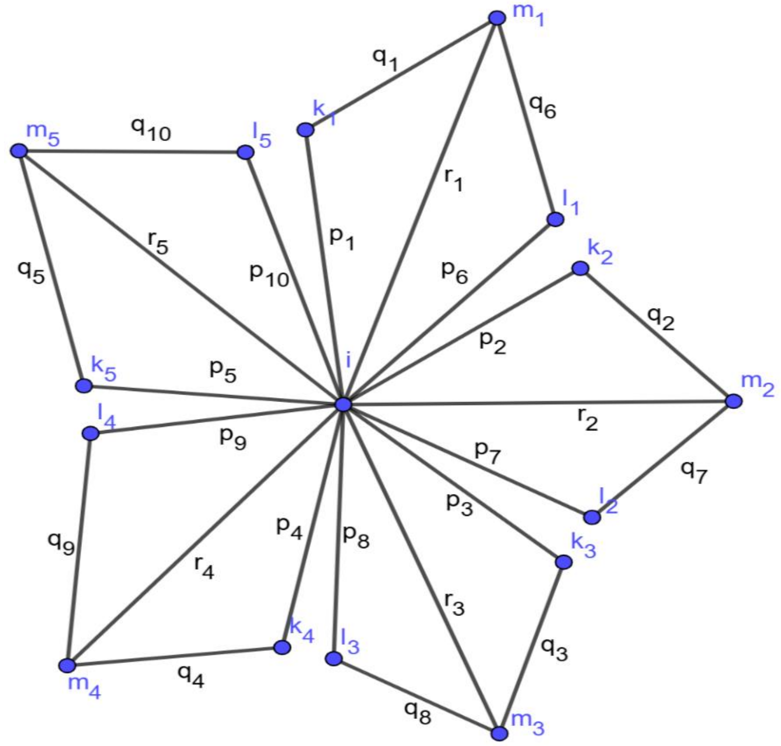

Figure 1: An example of St_5

Proposition 2.1: For any graph H, we have

$$\chi(H) \leq \chi_{=}(H) \leq \Delta(H) + 1$$

3 Results

Theorem 3.1. Let $q \geq 4$ be an integer. Then the equitable coloring of the G^{---} transformation of the path P_q is given by

$$\chi_{=}\left(G^{---}(P_q)\right) = \left\lfloor \frac{2q+1}{3} \right\rfloor$$

Proof. Let the vertices of P_q be defined as $V(P_q) = \{u'_n \mid 1 \leq n \leq q\}$, and its edge set as $E(P_q) = \{u''_{n'} \mid 1 \leq n' \leq q-1\}$, where each $u''_{n'}$ corresponds to the connection between the consecutive vertices $u'_{n'}$ and $u'_{n'+1}$.

Under the G^{---} transformation, the vertex set of the resulting graph is formed by combining the original vertex and edge set, which is formally expressed as

$$V\left(G^{---}(P_q)\right) = V(P_q) \cup \{u''_{n'} \mid 1 \leq n' \leq q-1\}$$

Consider a color function $\vartheta: V[G^{---}(P_q)] \to C$, where the color set is defined as $C = \{1,2, \dots, \left\lfloor \frac{2q+1}{3} \right\rfloor\}$

The function ϑ maps each vertex of $G^{---}(P_q)$ to a color set C according to the following rules.

- If $q \geq 4$

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$$\vartheta(u'_n) = \begin{cases} \left\lfloor \frac{2n}{3} \right\rfloor, & \text{if } n \equiv 0 \pmod 3, \\ \left\lfloor \frac{2n+1}{3} \right\rfloor, & \text{if } n \equiv 1 \pmod 3, \; 1 \leq n \leq q \\ n - \left\lfloor \frac{n+2}{3} \right\rfloor, & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

$$\vartheta(u''_{n'}) = n' - \left\lfloor \frac{n'}{3} \right\rfloor, 1 \leq n' \leq q-2$$

$\vartheta(u''_{q-1}) = \vartheta(u'_q), n' = q-1$

Thus, the vertices of $G^{---}(P_q)$ are equitably colored by the coloring function defined above. This immediately yields the upper bound

$$\chi_{=}\big(G^{---}(P_q)\big) \leq \left\lfloor \frac{2q+1}{3} \right\rfloor, q \geq 4$$

Furthermore, by Proposition 2.1, we obtain the lower bound

$$\chi_{=}\big(G^{---}(P_q)\big) \geq \chi\big(G^{---}(P_q)\big) = \left\lfloor \frac{2q+1}{3} \right\rfloor, q \geq 4$$

Combining the upper and lower bounds, it follows that

$$\chi_{=}\big(G^{---}(P_q)\big) = \left\lfloor \frac{2q+1}{3} \right\rfloor, q \geq 4$$

Theorem 3.2. Let $m \geq 4$ be an integer. Then the equitable coloring of the G^{---} transformation of the cycle C_m is given by

$$\chi_{=}(G^{---}(C_m)) = m - \left\lfloor \frac{m}{3} \right\rfloor$$

Proof. Let the vertices of C_m be defined as $V(C_m) = \{d'_n \mid 1 \leq n \leq m\}$, and its edge set as $E(C_m) = \{d''_n \mid 1 \leq n \leq m\}$, where each edge d''_n corresponds to the connection between consecutive vertices d'_n and d'_{n+1} for $1 \leq n \leq m-1$, and the final edge d''_m completes the cycle by joining d'_m to d'_1.

Under the G^{---} transformation, the vertex set of the resulting graph is formed by combining the original vertex and edge set, which is formally expressed as

$$V(G^{---}(C_m)) = V(C_m) \cup \{d''_n \mid 1 \leq n \leq m\}$$

Consider a color function $\vartheta: V[G^{---}(C_m)] \to C$, where the color set is defined as $C = \{1,2,\ldots, m - \lfloor \frac{m}{3} \rfloor\}$

The function ϑ maps each vertex of $G^{---}(C_m)$ to a color set C according to the following rules.

For $1 \leq n \leq m$

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$$\vartheta(d'_n) = \begin{cases} \left\lfloor \dfrac{2n}{3} \right\rfloor, & \text{if } n \equiv 0 \pmod 3, \\ \left\lfloor \dfrac{2n+1}{3} \right\rfloor, & \text{if } n \equiv 1 \pmod 3, \\ n - \left\lfloor \dfrac{n+2}{3} \right\rfloor, & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

$$\vartheta(d''_n) = n - \left\lfloor \frac{n}{3} \right\rfloor$$

Thus, the vertices of $G^{---}(C_m)$ are equitably colored by the coloring function defined above. This immediately yields the upper bound

$$\chi_{=}(G^{---}(C_m)) \leq m - \left\lfloor \frac{m}{3} \right\rfloor, m \geq 4$$

Furthermore, by Proposition 2.1, we obtain the lower bound

$$\chi_{=}(G^{---}(C_m)) \geq \chi(G^{---}(C_m)) = m - \left\lfloor \frac{m}{3} \right\rfloor, m \geq 4$$

Combining the upper and lower bounds, it follows that

$$\chi_{=}(G^{---}(C_m)) = m - \left\lfloor \frac{m}{3} \right\rfloor, m \geq 4$$

Theorem 3.3. Let $s \geq 4$ be an integer. Then the equitable coloring of the G^{---} transformation of the wheel W_s is given by

$$\chi_{=}(G^{---}(W_s)) = \begin{cases} s-1, & \text{if } s \equiv 1 \pmod 3, \\ s, & \text{if } s \equiv 0,2 \pmod 3 \end{cases}$$

Proof. Let the vertex set of the wheel graph W_s be defined as $V(W_s) = \{e\} \cup \{e'_n \mid 1 \leq n \leq s-1\}$, and its edge set as $E(W_s) = \{e''_n, e'''_n \mid 1 \leq n \leq s-1\}$, where e is central hub vertex, each edge e''_n corresponds to the connection between consecutive vertices e'_n and e'_{n+1} for $1 \leq n \leq s-2$, and the final edge e''_{s-1} completes the cycle by joining e'_{s-1} to e'_1, and e'''_n corresponds to the connection between vertices e and e'_n.

Under the G^{---} transformation, the vertex set of the resulting graph is formed by combining the original vertex and edge set, which is formally expressed as

$$V[G^{---}(W_s)] = V(W_s) \cup \{e''_n, e'''_n \mid 1 \leq n \leq s-1\}$$

Consider a color function $\vartheta: V[G^{---}(W_s)] \to C$, where the color set is defined as $C = \left\{1,2,\dots, \begin{cases} s-1, & \text{if } s \equiv 1 \pmod 3, \\ s, & \text{if } s \equiv 0,2 \pmod 3 \end{cases}\right\}$. The function ϑ maps each vertex of $G^{---}(W_s)$ to a color set C according to the following rule

- If $s = 4$

$$\vartheta(e) = \vartheta(e'_n) = 1, 1 \leq n \leq s-1$$

$$\vartheta(e''_n) = 2, 1 \leq n \leq s-1$$

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$$\vartheta(e_n''') = 3, 1 \leq n \leq s-1$$

- For $s \geq 5$

$$\vartheta(e) = \vartheta(e_{s-1}''')$$

1. $1 \leq n \leq s-1$

$$\vartheta(e_n') = \begin{cases} \left\lfloor \frac{2n}{3} \right\rfloor, & \text{if } n \equiv 0 \text{ (mod 3)}, \\ \left\lfloor \frac{2n+1}{3} \right\rfloor, & \text{if } n \equiv 1 \text{ (mod 3)}, \\ n - \left\lfloor \frac{n+2}{3} \right\rfloor, & \text{if } n \equiv 2 \text{ (mod 3)}, \end{cases}$$

$$\vartheta(e_n'') = n - \left\lfloor \frac{n}{3} \right\rfloor$$

2. If $s \equiv 1,2$ (mod 3) & $1 \leq n \leq s-1$

$$\vartheta(e_n''') = \left\lfloor \frac{n-1}{3} \right\rfloor + s - \left\lfloor \frac{s-1}{3} \right\rfloor$$

3. If $s \equiv 0$ (mod 3)

$$\vartheta(e_1''') = \vartheta(e_{s-1}'')$$

$$\vartheta(e_n''') = \left\lceil \frac{n-1}{3} \right\rceil + 2\left(\frac{s}{3}\right), 2 \leq n \leq s-1$$

Thus, the vertices of $G^{---}(W_s)$ are equitably colored by the coloring function defined above. This immediately yields the upper bound

$$\chi_{=}(G^{---}(W_s)) \leq \begin{cases} s-1, & \text{if } s \equiv 1 \text{ (mod 3)}, \\ s, & \text{if } s \equiv 0,2 \text{ (mod 3)} \end{cases}$$

We establish the lower bound by contradiction.

Claim: $\chi(G^{---}(W_s)) = \begin{cases} s-2, & \text{if } s \equiv 1 \text{ (mod 3)}, \\ s-1, & \text{if } s \equiv 0,2 \text{ (mod 3)} \end{cases}$

Let us attempt to color the graph using the coloring function defined earlier, up to the claim. After applying this coloring scheme, a subset of vertices remains uncolored:

- If $s \equiv 0,2$ (mod 3): the uncolored vertices are $\{e, e_{s-1}'''\}$.
- If $s \equiv 1$ (mod 3): the uncolored vertices are $\{e, e_{s-1}''', e_{s-2}''', e_{s-3}'''\}$.

Now, observe that each color used so far has already been assigned to either exactly 3 vertices or exactly 2 vertices. Since equitable coloring requires that no color class exceeds the allowed size, none of the existing colors can be reused for the remaining uncolored vertices. Consequently, we are forced to introduce additional distinct colors to complete the coloring. This contradicts the assumption.

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Illustrative Example: Consider the graph W_{11}, where $s = 11 \equiv 2 \pmod 3$. Suppose we attempt to equitably color all vertices using only $s - 1 = 10$ colors, following the previously defined coloring function. After coloring, the vertices $\{e, e_{10}'''\}$ remain uncolored. In the current coloring, each of the 10 colors has been used exactly three times, except for color 7, which appears only twice at $\{e_{10}', e_{10}''\}$. Since e is adjacent to e_{10}''', assigning color 7 to e would violate the proper coloring condition. Assigning any other existing color would exceed the equitable limit of three vertices per color class. Therefore, an additional color is required to complete the equitable coloring, implying that $\chi_{=}(G^{---}(W_{11})) = 11$.

Hence, we obtain the lower bound

$$\chi_{=}(G^{---}(W_s)) \geq \begin{cases} s-1, & \text{if } s \equiv 1 \pmod 3, \\ s, & \text{if } s \equiv 0,2 \pmod 3, \end{cases} \forall s \geq 4$$

Combining the upper and lower bounds, it follows that

$$\chi_{=}(G^{---}(W_s)) = \begin{cases} s-1, & \text{if } s \equiv 1 \pmod 3, \\ s, & \text{if } s \equiv 0,2 \pmod 3, \end{cases} \forall s \geq 4$$

Theorem 3.4. Let $a, b \geq 2$ be integers. Then the equitable coloring of the G^{---} transformation of umbrella $U_{a,b}$ is given by

$$\chi_{=}\big(G^{---}(U_{a,b})\big) = \begin{cases} a + b - \left\lfloor \frac{a}{3} \right\rfloor - 1, & \text{if } b \geq 2 \text{ and } a \equiv 0,1 \pmod 3, \\ a + b - \left\lceil \frac{a}{3} \right\rceil, & \text{if } b \geq 2 \text{ and } a \equiv 2 \pmod 3. \end{cases}$$

Proof. Let the vertices of $U_{a,b}$ be defined as $V(U_{a,b}) = \{v_i, w_j \mid 1 \leq i \leq a, 1 \leq j \leq b\}$, where v_i be the vertices of path graph and w_j be the vertices of fan subgraph. And its edge set as $E(U_{a,b}) = \{v_{i'}' \mid 1 \leq i' \leq a-1\} \cup \{w_{j'}' \mid 1 \leq j' \leq b-1\} \cup \{x_j \mid 1 \leq j \leq b\}$, where each edge $v_{i'}'$ corresponds to the connection between consecutive vertices v_i and v_{i+1} for $1 \leq i \leq a-1$, the edge $w_{j'}'$ corresponds to the connection between vertices w_j and w_{j+1} and the edge x_j connects the vertices v_q and $w_{b-(j-1)}$.

Under the G^{---} transformation, the vertex set of the resulting graph is formed by combining the original vertex and edge set, which is formally expressed as

$$\chi_{=}\big(G^{---}(U_{a,b})\big) = \begin{cases} a + b - \left\lfloor \frac{a}{3} \right\rfloor - 1, & \text{if } b \geq 2 \text{ and } a \equiv 0,1 \pmod 3, \\ a + b - \left\lceil \frac{a}{3} \right\rceil, & \text{if } b \geq 2 \text{ and } a \equiv 2 \pmod 3. \end{cases}$$

Consider a color function $\vartheta: V[G^{---}(U_{a,b})] \to C$, where the color set is defined as

$$C = \{1,2, \dots, \begin{cases} a + b - \left\lfloor \frac{a}{3} \right\rfloor - 1, & \text{if } b \geq 2 \text{ and } a \equiv 0,1 \pmod 3, \\ a + b - \left\lceil \frac{a}{3} \right\rceil, & \text{if } b \geq 2 \text{ and } a \equiv 2 \pmod 3. \end{cases} \}$$

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The function ϑ maps each vertex of $G^{---}(U_{a,b})$ to a color set C according to the following rules.

- If $a \equiv 1 \pmod 3$, $b \geq 2$

$$\vartheta(v_i) = \begin{cases} i - \left\lfloor \frac{i}{3} \right\rfloor, & \text{if } i \equiv 0,1 \pmod 3, \\ i - \left\lceil \frac{i}{3} \right\rceil, & \text{if } i \equiv 2 \pmod 3 \end{cases} \quad \forall 1 \leq i \leq a-1$$

$$\vartheta(v'_{i'}) = i' - \left\lfloor \frac{i'}{3} \right\rfloor, \forall 1 \leq i' \leq a-1$$

$$\vartheta(v_i) = \vartheta(w_b), \text{if } i = a, b \equiv 1 \pmod 3$$

$$\vartheta(v_i) = \vartheta(x_b), \text{if } i = a, b \equiv 0,2 \pmod 3$$

1. If $b \equiv 2 \pmod 3$, $1 \leq j \leq b$

$$\vartheta(x_j) = \vartheta(w_b) + \left\lceil \frac{j}{3} \right\rceil$$

2. If $b \equiv 0,1 \pmod 3$, $1 \leq j \leq b-1$

$$\vartheta(x_1) = \vartheta(w_b)$$

$$\vartheta(x_{j+1}) = \vartheta(w_b) + \left\lceil \frac{j}{3} \right\rceil$$

- If $a \equiv 2 \pmod 3$

$$\vartheta(v_i) = \begin{cases} i - \left\lfloor \frac{i}{3} \right\rfloor, & \text{if } i \equiv 0,1 \pmod 3, \\ i - \left\lceil \frac{i}{3} \right\rceil, & \text{if } i \equiv 2 \pmod 3 \end{cases} \quad \forall 1 \leq i \leq a$$

$$\vartheta(v'_{i'}) = i' - \left\lfloor \frac{i'}{3} \right\rfloor, \forall 1 \leq i' \leq a-1$$

1. If $b \equiv 2 \pmod 3$, $1 \leq j \leq b$

$$\vartheta(x_j) = \vartheta(w_b) + \left\lceil \frac{j}{3} \right\rceil$$

2. If $b \equiv 0,1 \pmod 3$, $1 \leq j \leq b-1$

$$\vartheta(x_1) = \vartheta(w_b)$$

$$\vartheta(x_{j+1}) = \vartheta(w_b) + \left\lceil \frac{j}{3} \right\rceil$$

- If $a \equiv 0 \pmod 3$

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$$\vartheta(v_i) = \begin{cases} i - \left\lfloor \frac{i}{3} \right\rfloor, & \text{if } i \equiv 0,1 \pmod 3, \\ i - \left\lceil \frac{i}{3} \right\rceil, & \text{if } i \equiv 2 \pmod 3 \end{cases} \quad \forall 1 \le i \le a-1$$

$$\vartheta(v'_{i'}) = i' - \left\lfloor \frac{i'}{3} \right\rfloor, \forall 1 \le i' \le a-2$$

$$\vartheta(v_i) = \vartheta(w_b), \text{if } i = a, b \equiv 1 \pmod 3$$

$$\vartheta(v'_{i'}) = \vartheta(x_b), \text{if } i' = a-1, b \equiv 1 \pmod 3$$

$$\vartheta(v_i) = \vartheta(v'_{i'}) = \vartheta(x_b), \text{if } i = a \text{ \textbackslash\& } i' = a-1, b \equiv 0,2 \pmod 3$$

1. If $b \equiv 2 \pmod 3$, $1 \le j \le b$

$$\vartheta(x_j) = \vartheta(w_b) + \left\lceil \frac{j}{3} \right\rceil$$

2. If $b \equiv 0,1 \pmod 3$, $1 \le j \le b-1$

$$\vartheta(x_1) = \vartheta(w_b)$$

$$\vartheta(x_{j+1}) = \vartheta(w_b) + \left\lceil \frac{j}{3} \right\rceil$$

- For $a, b \ge 2$, $1 \le j \le b$

$$\vartheta(w_j) = j + \left\lfloor \frac{a + \left\lfloor \frac{a-2}{3} \right\rfloor}{2} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor, \text{if } j \equiv 0,1 \pmod 3$$

$$\vartheta(w_j) = j + \left\lfloor \frac{a + \left\lfloor \frac{a-2}{3} \right\rfloor}{2} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor - 1, \text{if } j \equiv 2 \pmod 3$$

- For $a, b \ge 2$

$$\vartheta(w'_{j'}) = j' + \left\lfloor \frac{a + \left\lfloor \frac{a-2}{3} \right\rfloor}{2} \right\rfloor - \left\lfloor \frac{j'}{3} \right\rfloor, 1 \le j \le b-1$$

Thus, the vertices of $G^{---}(U_{a,b})$ are equitably colored by the coloring function defined above. This immediately yields the upper bound

$$\chi_{=}\left(G^{---}(U_{a,b})\right) \le \begin{cases} a + b - \left\lfloor \frac{a}{3} \right\rfloor - 1, & \text{if } b \ge 2 \text{ and } a \equiv 0,1 \pmod 3, \\ a + b - \left\lceil \frac{a}{3} \right\rceil, & \text{if } b \ge 2 \text{ and } a \equiv 2 \pmod 3. \end{cases}$$

We establish the lower bound by contradiction.

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Claim: $\chi_{=}\left(G^{---}(U_{a,b})\right) = \begin{cases} a+b-\left\lfloor\frac{a}{3}\right\rfloor-2, & \text{if } b \geq 2 \text{ and } a \equiv 0,1 \pmod 3, \\ a+b-\left\lceil\frac{a}{3}\right\rceil-1, & \text{if } b \geq 2 \text{ and } a \equiv 2 \pmod 3. \end{cases}$

Let us attempt to color the graph using only the number of colors specified in the claim. After applying this coloring scheme, a subset of vertices remains uncolored:

- If $s \equiv 0 \pmod 3$: there are 3 or 4 vertices being uncolored.
- If $s \equiv 1 \pmod 3$: there are 3 or 4 vertices being uncolored.
- If $s \equiv 2 \pmod 3$: there are 2 or 3 vertices being uncolored.

Now, observe that each color used so far has already been assigned to either exactly 3 vertices or exactly 2 vertices. Since equitable coloring requires that no color class exceeds the allowed size, none of the existing colors can be reused for the remaining uncolored vertices. Consequently, we are forced to introduce additional distinct colors to complete the coloring. This contradicts the assumption.

Illustrative Example: Consider the graph $U_{7,5}$. Suppose we attempt to equitably color all vertices using only $a+b-\lfloor\frac{a}{3}\rfloor-2 = 7+5-2-2 = 8$ colors, following the previously defined coloring function. After coloring, the vertices $\{v_7, x_4, x_5\}$ remain uncolored. In the current coloring, each of the 8 colors has been used exactly three times. Due to adjacency, we cannot repeat any existing colors. Therefore, an additional color is required to complete the equitable coloring, implying that $\chi_{=}(G^{---}(U_{7,5})) = 9$.

Hence, we obtain the lower bound

$$\chi_{=}\left(G^{---}(U_{a,b})\right) \geq \begin{cases} a+b-\left\lfloor\frac{a}{3}\right\rfloor-1, & \text{if } b \geq 2 \text{ and } a \equiv 0,1 \pmod 3, \\ a+b-\left\lceil\frac{a}{3}\right\rceil, & \text{if } b \geq 2 \text{ and } a \equiv 2 \pmod 3. \end{cases}$$

Combining the upper and lower bounds, it follows that

$$\chi_{=}\left(G^{---}(U_{a,b})\right) = \begin{cases} a+b-\left\lfloor\frac{a}{3}\right\rfloor-1, & \text{if } b \geq 2 \text{ and } a \equiv 0,1 \pmod 3, \\ a+b-\left\lceil\frac{a}{3}\right\rceil, & \text{if } b \geq 2 \text{ and } a \equiv 2 \pmod 3. \end{cases}$$

Theorem 3.5. Let $s \geq 3$, $t \geq 1$ be integers. Then the equitable coloring of the G^{---} transformation of the tadpole $T_{s,t}$ is given by

$$\chi_{=}\left(G^{---}(T_{s,t})\right) = \begin{cases} s+t-\left\lceil\frac{s+t}{3}\right\rceil, & \text{if } s \equiv 0 \pmod 3, t \equiv 2 \pmod 3 \\ s+t-\left\lfloor\frac{s+t}{3}\right\rfloor, & \text{otherwise} \end{cases}$$

Proof. Let the vertices of tadpole $T_{s,t}$ be defined as $V(T_{s,t}) = \{f_i, g_j \mid 1 \leq i \leq s, 1 \leq j \leq t\}$, where f_i represents the vertices of cycle subgraph and g_j represents the vertices of path subgraph. And its edge set as $E(T_{s,t}) = \{f_i', g_j' \mid 1 \leq i \leq s, 1 \leq j \leq t\}$, where g_t' be the edge

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that connects the first vertex f_1 of cycle with the last vertex of path g_t, remaining g_j' corresponds to the connection between consecutive vertices g_j and g_{j+1}, each edge f_i' corresponds to the connection between vertices f_i and f_{i+1} and the final edge f_s' connects the vertices f_s and f_1 respectively.

Under the G^{---} transformation, the vertex set of the resulting graph is formed by combining the original vertex and edge set, which is formally expressed as

$$V[G^{---}(T_{s,t})] = V(T_{s,t}) \cup \{f_i', g_j' \mid 1 \le i \le s, 1 \le j \le t\}$$

Consider a color function $\vartheta: V[G^{---}(T_{s,t})] \rightarrow C$, where the color set is defined as

$$C = \{1,2,\dots, \begin{cases} s+t-\left\lceil \frac{s+t}{3} \right\rceil, & \text{if } s \equiv 0 \text{ (mod 3)}, t \equiv 2 \text{ (mod 3)} \\ s+t-\left\lfloor \frac{s+t}{3} \right\rfloor, & \text{otherwise} \end{cases} \}$$

The function ϑ maps each vertex of $G^{---}(T_{s,t})$ to a color set C according to the following rules.

- If $s \ge 3, t \ge 1$

$$\vartheta(g_j) = \begin{cases} j-\left\lfloor \frac{j}{3} \right\rfloor, & \text{if } j \equiv 0,1 \text{ (mod 3)}, \\ j-\left\lceil \frac{j}{3} \right\rceil, & \text{if } j \equiv 2 \text{ (mod 3)} \end{cases} \quad \forall 1 \le j \le t$$

$$\vartheta(g_j') = j-\left\lfloor \frac{j}{3} \right\rfloor, 1 \le j \le t$$

- For all $s \ge 3, t \equiv 2$ (mod 3)

1. If $s \equiv 0$ (mod 3)

$$\vartheta(f_1) = \vartheta(f_1') = \vartheta(f_s') = \vartheta(g_t')$$

$$\vartheta(f_{i+1}) = \vartheta(f_1) + \begin{cases} i-\left\lfloor \frac{i}{3} \right\rfloor, & \text{if } i \equiv 0,1 \text{ (mod 3)}, \\ i-\left\lceil \frac{i}{3} \right\rceil, & \text{if } i \equiv 2 \text{ (mod 3)} \end{cases} \quad \forall 1 \le i \le s-1$$

$$\vartheta(f_{i+1}') = \vartheta(f_1) + i-\left\lfloor \frac{i}{3} \right\rfloor, 1 \le i \le s-2$$

2. If $s \equiv 1,2$ (mod 3)

$$\vartheta(f_1) = \vartheta(f_1') = \vartheta(g_t')$$

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$$\vartheta(f_{i+1}) = \vartheta(f_1) + \begin{cases} i - \left\lfloor \frac{i}{3} \right\rfloor, & \text{if } i \equiv 0,1 \pmod 3, \\ i - \left\lceil \frac{i}{3} \right\rceil, & \text{if } i \equiv 2 \pmod 3 \end{cases} \quad \forall 1 \leq i \leq s-1$$

$$\vartheta(f'_{i+1}) = \vartheta(f_1) + i - \left\lfloor \frac{i}{3} \right\rfloor, 1 \leq i \leq s-1$$

- For $s \geq 3$, $t \equiv 0 \pmod 3$

1. If $s \equiv 0,1 \pmod 3$

$$\vartheta(f_i) = \vartheta(g_t) + \begin{cases} i - \left\lfloor \frac{i}{3} \right\rfloor, & \text{if } i \equiv 0,1 \pmod 3, \\ i - \left\lceil \frac{i}{3} \right\rceil, & \text{if } i \equiv 2 \pmod 3 \end{cases} \quad \forall 1 \leq i \leq s$$

$$\vartheta(f_i') = \vartheta(g_t) + i - \left\lfloor \frac{i}{3} \right\rfloor, 1 \leq i \leq s$$

2. If $s \equiv 2 \pmod 3$

$$\vartheta(f_i) = \vartheta(g_t) + \begin{cases} i - \left\lfloor \frac{i}{3} \right\rfloor, & \text{if } i \equiv 0,1 \pmod 3, \\ i - \left\lceil \frac{i}{3} \right\rceil, & \text{if } i \equiv 2 \pmod 3 \end{cases} \quad \forall 1 \leq i \leq s-1$$

$$\vartheta(f_i') = \vartheta(g_t) + i - \left\lfloor \frac{i}{3} \right\rfloor, 1 \leq i \leq s$$

$$\vartheta(f_s) = \vartheta(f_s')$$

- If $s \geq 3$, $t \equiv 1 \pmod 3$

1. If $s \equiv 0,2 \pmod 3$

$$\vartheta(f_1) = \vartheta(g_t')$$

$$\vartheta(f_{i+1}) = \vartheta(f_1) + i - \left\lfloor \frac{i}{3} \right\rfloor, 1 \leq i \leq s-1$$

$$\vartheta(f_i') = \vartheta(f_1) + \begin{cases} i - \left\lfloor \frac{i}{3} \right\rfloor, & \text{if } i \equiv 0,1 \pmod 3, \\ i - \left\lceil \frac{i}{3} \right\rceil, & \text{if } i \equiv 2 \pmod 3 \end{cases} \quad \forall 1 \leq i \leq s$$

2. If $s \equiv 1 \pmod 3$

$$\vartheta(f_i) = \vartheta(g_t) + \begin{cases} i - \left\lfloor \frac{i}{3} \right\rfloor, & \text{if } i \equiv 0,1 \pmod 3, \\ i - \left\lceil \frac{i}{3} \right\rceil, & \text{if } i \equiv 2 \pmod 3 \end{cases} \quad \forall 1 \leq i \leq s$$

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$$\vartheta(f_i') = \vartheta(g_t) + i - \left\lfloor \frac{i}{3} \right\rfloor, 1 \leq i \leq s$$

Thus, the vertices of $G^{---}(T_{s,t})$ are equitably colored by the coloring function defined above. This immediately yields the upper bound

$$\chi_{=}\big(G^{---}(T_{s,t})\big) \leq \begin{cases} s+t-\left\lceil \dfrac{s+t}{3} \right\rceil, & \text{if } s \equiv 0 \text{ (mod 3)}, t \equiv 2 \text{ (mod 3)} \\ s+t-\left\lfloor \dfrac{s+t}{3} \right\rfloor, & \text{otherwise} \end{cases}$$

We establish the lower bound by contradiction.

Claim: $\chi_{=}\big(G^{---}(T_{s,t})\big) = \begin{cases} s+t-\left\lceil \frac{s+t}{3} \right\rceil - 1, & \text{if } s \equiv 0 \text{ (mod 3)}, t \equiv 2 \text{ (mod 3)} \\ s+t-\left\lfloor \frac{s+t}{3} \right\rfloor - 1, & \text{otherwise} \end{cases}$

Let us attempt to color the graph using only the specified number of colors. After applying this coloring scheme, a subset of vertices remains uncolored:

- If $s \equiv 0$ (mod 3), $t \equiv 2$ (mod 3): the uncolored vertices are $\{g_t', f_1, f_1', f_s'\}$.
- If $s \not\equiv 0$ (mod 3), $t \not\equiv 2$ (mod 3): the uncolored set is either $\{f_{s-1}, f_s, f_s'\}$ or $\{f_s, f_s'\}$.

In each of the preceding scenarios, the neighborhood of every uncolored vertex intersects all existing color classes. This means that for each uncolored vertex, every color currently in use appears in its neighborhood. Assigning any of the existing colors to these vertices would violate the proper coloring condition. Consequently, it is impossible to color the remaining vertices without introducing at least one new, distinct color. This necessity directly contradicts our initial assumption that the claimed number of colors would suffice to produce an equitable coloring of the graph.

Illustrative Example: Consider the graph $T_{8,5}$. Suppose we attempt to equitably color all the vertices using only $s+t-\lfloor\frac{s+t}{3}\rfloor - 1 = 8+5-4-1 = 8$ colors, in accordance with the previously defined coloring function. Upon completion of this coloring scheme, the vertices $\{f_8, f_8'\}$ remain uncolored. Moreover, each of these uncolored vertices is adjacent to vertices from all existing color classes, implying that every color currently in use appears in their respective neighborhoods. At this point, each of the 8 colors has been assigned to exactly three vertices, thereby satisfying the equitable constraint. However, since f_8 and f_8' are adjacent to all existing color classes, no color can be reassigned to them without violating either the proper coloring condition or the equitable distribution. Consequently, an additional color is required to complete the equitable coloring. To resolve this, a new color—say, color 9—must be introduced and assigned to the pair $\{f_8, f_8'\}$. This necessity confirms that the graph cannot be equitably colored with only 8 colors. Therefore, we conclude that $\chi_{=}[G^{---}(T_{8,5})] = 9$.

Hence, we obtain the lower bound

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$$\chi_{=}\big(G^{---}(T_{s,t})\big) \geq \begin{cases} s+t-\left\lceil \dfrac{s+t}{3} \right\rceil, & \text{if } s \equiv 0 \ (\text{mod } 3), t \equiv 2 \ (\text{mod } 3) \\ s+t-\left\lfloor \dfrac{s+t}{3} \right\rfloor, & \text{otherwise} \end{cases}$$

Combining the upper and lower bounds, it follows that

$$\chi_{=}\big(G^{---}(T_{s,t})\big) = \begin{cases} s+t-\left\lceil \dfrac{s+t}{3} \right\rceil, & \text{if } s \equiv 0 \ (\text{mod } 3), t \equiv 2 \ (\text{mod } 3) \\ s+t-\left\lfloor \dfrac{s+t}{3} \right\rfloor, & \text{otherwise} \end{cases}$$

Theorem 3.6. Let $h \geq 3$ be an integer. Then the equitable coloring of the G^{---} transformation of the sundat St_h is given by

$$\chi_{=}(G^{---}(St_h)) = 2h$$

Proof. Let the vertices of St_h be defined as $V(St_h) = \{i\} \cup \{k_n, l_n, m_n \mid 1 \leq n \leq h\}$, and its edge set as $E(St_h) = \{p_{n'}, q_{n'}, r_n \mid 1 \leq n' \leq 2h, 1 \leq n \leq h\}$. Here, for $1 \leq n' \leq h$, $p_{n'}$ represents the edge ik_n, while for $h+1 \leq n' \leq 2h$, $p_{n'}$ represents the edge il_n. Similarly, for $1 \leq n' \leq h$, $q_{n'}$ represents the edge $k_n m_n$, and for $h+1 \leq n' \leq 2h$, $q_{n'}$ represents the edge $l_n m_n$. Finally, for $1 \leq n \leq h$, r_n represents the edge im_n.

Under the G^{---} transformation, the vertex set of the resulting graph is formed by combining the original vertex and edge set, which is formally expressed as

$$V(G^{---}(St_h)) = V(St_h) \cup \{p_{n'}, q_{n'}, r_n \mid 1 \leq n' \leq 2h, 1 \leq n \leq h\}$$

Consider a color function $\vartheta: V[G^{---}(St_h)] \rightarrow C$, where the color set is defined as $C = \{1,2,\ldots,2h\}$

The function ϑ maps each vertex of $G^{---}(St_h)$ to a color set C according to the following rules.

$$\vartheta(m_n) = \vartheta(r_n) = n, 1 \leq n \leq h$$

$$\vartheta(k_n) = \vartheta(l_n) = h+n, 1 \leq n \leq h$$

$$\vartheta(q_{n'}) = \vartheta(q_{n'+h}) = n', 1 \leq n' \leq h$$

$$\vartheta(p_{n'}) = \vartheta(p_{n'+h}) = h+n', 1 \leq n' \leq h$$

$$\vartheta(i) = \vartheta(p_1)$$

Thus, the vertices of $G^{---}(St_h)$ are equitably colored by the coloring function defined above. This immediately yields the upper bound

$$\chi_{=}(G^{---}(St_h)) \leq 2h, \forall h \geq 3$$

Furthermore, by Proposition 2.1, we obtain the lower bound

$$\chi_{=}(G^{---}(St_h)) \geq \chi(G^{---}(St_h)) = 2h, \forall h \geq 3$$

Combining the upper and lower bounds, it follows that

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$$\chi_{=}(G^{---}(St_h)) = 2h, \forall h \geq 3$$

Theorem 3.7. Let $c, d \geq 2$ be integers. Then the equitable coloring of the G^{---} transformation of the octopus $O_{c,d}$ is given by

$$\chi_{=}\left(G^{---}(O_{c,d})\right) = c + d$$

Proof. Let the vertices of $O_{c,d}$ be defined as $V(O_{c,d}) = \{a\} \cup \{b_i, s_j \mid 1 \leq i \leq c, 1 \leq j \leq d\}$, where b_i $(1 \leq i \leq c)$ are the leaf vertices of a star subgraph, s_j $(1 \leq j \leq d)$ are the vertices of a fan subgraph, and a is the central hub vertex. And its edge set be $E(O_{c,d}) = \{b_i', s_j', s_{j'}'' \mid 1 \leq i \leq c, 1 \leq j \leq d, 1 \leq j' \leq d-1\}$ where b_i' denotes the edge connecting a and b_i, s_j' corresponds to the connection between the vertices a and $s_{d-(j-1)}$, and $s_{j'}''$ corresponds to the connection between the consecutive vertices $s_{j'}$ and $s_{j'+1}$.

Under the G^{---} transformation, the vertex set of the resulting graph is formed by combining the original vertex and edge set, which is formally expressed as

$$V\left(G^{---}(O_{c,d})\right) = V(O_{c,d}) \cup \{b_i', s_j', s_{j'}'' \mid 1 \leq i \leq c, 1 \leq j \leq d, 1 \leq j' \leq d-1\}$$

Consider a color function $\vartheta: V[G^{---}(O_{c,d})] \to C$, where the color set is defined as $C = \{1,2, \dots, c+d\}$

The function ϑ maps each vertex of $G^{---}(O_{c,d})$ to a color set C according to the following rules.

$$\vartheta(a) = \vartheta(b_1)$$

$$\vartheta(b_i) = \vartheta(b_i') = i, 1 \leq i \leq c$$

$$\vartheta(s_j) = \vartheta(b_c) + \begin{cases} \left\lfloor \frac{2j}{3} \right\rfloor, & \text{if } j \equiv 0 \pmod 3, \\ \left\lfloor \frac{2j+1}{3} \right\rfloor, & \text{if } j \equiv 1 \pmod 3, \ \forall 1 \leq j \leq d \\ j - \left\lfloor \frac{j+2}{3} \right\rfloor, & \text{if } j \equiv 2 \pmod 3, \end{cases}$$

$$\vartheta(s_{j'}'') = \vartheta(b_c) + j' - \left\lfloor \frac{j}{3} \right\rfloor, \forall 1 \leq j' \leq d-1$$

- If $d \equiv 0,2 \pmod 3$

$$\vartheta(s_j') = \vartheta(s_d) + \left\lfloor \frac{j-1}{3} \right\rfloor, 1 \leq j \leq d$$

- If $d \equiv 1 \pmod 3$

$$\vartheta(s_1') = \vartheta(s_d)$$

$$\vartheta(s_{j+1}') = \vartheta(s_1') + \left\lfloor \frac{j-1}{3} \right\rfloor + 1, 1 \leq j \leq d-1$$

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Thus, the vertices of $G^{---}(O_{c,d})$ are equitably colored by the coloring function defined above. This immediately yields the upper bound

$$\chi_{=}\big(G^{---}(O_{c,d})\big) \leq c+d, \forall c,d \geq 2$$

We establish the lower bound by contradiction.

Claim: $\chi_{=}\big(G^{---}(O_{c,d})\big) = c+d-1, \forall c,d \geq 2$

Let us attempt to color the graph using the coloring function defined earlier, up to the specified claim $c+d-1$ colors. After applying this coloring scheme, a subset of vertices remains uncolored:

- If $d \equiv 0,1 \pmod 3$: the uncolored vertices $\{s'_{d-2}, s'_{d-1}, s'_d\}$.
- If $d \equiv 2 \pmod 3$: the uncolored vertices $\{s'_{d-1}, s'_d\}$

These vertices are adjacent to all the existing colors. But s'_j are not adjacent with each other, so one can give same color to all of them. But, color 2 is repeated twice. So, no other color can be repeated thrice. It violates the equitable coloring condition. One cannot give any colors that are used in s'_j vertices. So, a new color $c+d$ has to be given to $\{s'_{d-2}, s'_{d-1}, s'_d\}$ or $\{s'_{d-1}, s'_d\}$.

Hence, we obtain the lower bound

$$\chi_{=}\big(G^{---}(O_{c,d})\big) \geq c+d, \forall c,d \geq 2$$

Combining the upper and lower bounds, it follows that

$$\chi_{=}\big(G^{---}(O_{c,d})\big) = c+d, \forall c,d \geq 2$$

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