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NUMERICAL SOLUTION OF SEVENTH ORDER BOUNDARY VALUE PROBLEMS BY GALERKIN METHOD WITH SEXTIC B-SPLINES

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Abstract

A finite element method involving Galerkin method with sextic B-splines as basis functions has been solved the seventh order boundary value problem with boundary conditions. The basis functions are redefined into a new set of basis functions which vanish at the boundary where Dirichlet type of boundary conditions and Neumann boundary conditions, second derivative boundary conditions are prescribed. The proposed method was applied to solve several examples of seventh order linear and nonlinear boundary value problems. The solution of a nonlinear boundary value problem has been obtained as the limit of a sequence of solution of linear boundary value problems generated by quasilinearization technique. The obtained numerical results were found to be in good agreement with exact solutions available in the literature.

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Keywords: Galerkin method; Sextic B-spline; Basis function; Seventh order boundary value problem; Absolute error.

1. Introduction

In this paper, we consider a general seventh order linear boundary value problem

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given by

$$a_0(t)v^{(7)}(t)+a_1(t)v^{(6)}(t)+a_2(t)v^{(5)}(t)+a_3(t)v^{(4)}(t)+a_4(t)v'''(t)+a_5(t)v''(t)$$
$$+a_6(t)v'(t)+a_7(t)v(t)=b(t) \qquad l<t<m \qquad (1)$$

subject to boundary conditions

$$v(l)=A_0, v(m)=C_0, v'(l)=A_1, v'(m)=C_1, v''(l)=A_2, v''(m)=C_2, v'''(l)=A_3 \qquad (2)$$

where A_0, C_0, A_1, C_1, A_2, C_2 and A_3 are finite real constants and $a_0(t)$, $a_1(t)$, $a_2(t)$, $a_3(t)$, $a_4(t)$, $a_5(t)$, $a_6(t)$, $a_7(t)$ and $b(t)$ are all continuous functions defined on the interval $[l, m]$. The boundary value problem is solved with the boundary conditions.

Generally, this type of seventh order boundary value problems arise in the study of astro-physics, hydrodynamics and hydro magnetic stability, fluid dynamics, astronomy, beam and long wave theory, applied mathematics, engineering and applied physics. The boundary value problems of higher order have been investigated due to their mathematical importance and the potential for applications in diversified applied sciences. The literature on the numerical solutions of eight order boundary value problems is very scarce. Chandra sekhar [2] determined that when an infinite horizontal layer of fluid is heated from below and is under the action of rotation, instability sets in, when this instability is an ordinary convection the ordinary differential equation is sixth order, when the instability sets in as over stability, it is modeled by an eight order ordinary differential equation.

In this paper, we try to present a simple finite element method which involves Galerkin approach with sextic B-splines as basis functions to solve the seventh order two point boundary value problems of the type (1)-(2). This paper is organized as follows. Section 2, deals with the justification for using Galerkin Method. In Section 3, a description of Galerkin method with qsextic B-splines as basis functions is explained. In particular we first introduce the basic concept of sextic B-splines and followed by the proposed method with the boundary conditions. In Section 4, the procedure to solve the nodal parameters has been presented. In section 5, the proposed method is tested on several linear and nonlinear boundary value problems. The solution to a nonlinear problem has been obtained as the limit of a sequence of solution of linear problems generated by the quasilinearization technique [19]. Finally, in the last section, the conclusion presented.

2 Justification for using Galerkin Method

For the few decades, the finite element method has become very powerful, useful tool to solve the boundary value problems in the complex dynamical systems. Infinite element method (FEM) the approximate solution can be written as a linear combination of basis functions which constitute a basis for the approximation space under consideration. FEM involves variational methods like Rayleigh Ritz, Galerkin, Least Squares and Collocation etc.

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In Galerkin method, the residual of approximation is made orthogonal to the basis functions. When we use Galerkin method, a weak form of approximation solution for a given differential equation exists and is unique under appropriate conditions [23,25] irrespective of properties of a given differential operator. Further, a weak solution also tends to a classical solution of given differential equation, provided sufficient attention is given to boundary conditions [24]. That means the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are prescribed. Hence in this paper we employed the use of Galerkin method with sextic B-splines as basis functions to approximate the solution of a general seventh order boundary value problem.

3. Description of the method

Definition of sextic B-spline:

The sextic B-splines are defined in [20, 21, 22]. The existence of sextic spline interpolate $s(t)$ to a function in a closed interval $[l, m]$ for spaced knots (need not be evenly spaced) of a partition $l = t_0 < t_1 < t_2 < .... < t_{n-1} < t_n = m$ is established by constructing it. The construction of $s(t)$ is done with the help of the sextic B-splines. Introduce twelve additional knots t_{-6}, t_{-5}, t_{-4}, t_{-3}, t_{-2}, t_{-1}, t_{n+1}, t_{n+2}, t_{n+3}, t_{n+4}, t_{n+5} and t_{n+6} such that

$$t_{-6} < t_{-5} < t_{-4} < t_{-3} < t_{-2} < t_{-1} < t_0 \text{ and } t_n < t_{n+1} < t_{n+2} < t_{n+3} < t_{n+4} < t_{n+5} < t_{n+6}.$$

Now the sextic B-splines $R_i(t)'s$ are defined by

$$R_i(t) = \begin{cases} \sum\limits_{r=i-3}^{i+4} \dfrac{(t_r - t)_+^6}{\pi'(t_r)}, & t \in [t_{i-3}, t_{i+4}] \\ 0, & otherwise \end{cases}$$

where $$(t_r - t)_+^6 = \begin{cases} (t_r - t)^6, & if\ t_r \geq t \\ 0, & if\ t_r \leq t \end{cases}$$

and $$\pi(t) = \prod_{r=i-3}^{i+4} (t - t_r)$$

where $\{R_{-3}(t), R_{-2}(t), R_{-1}(t), R_0(t), R_1(t),..., R_{n-1}(t), R_n(t), R_{n+1}(t), R_{n+2}(t)\}$ forms a basis for the space $S_6(\pi)$ of sextic polynomial splines.Schoenberg [22] has proved that sextic B-splines are the unique nonzero splines of smallest compact support with the knots at

$$t_{-6} < t_{-5} < t_{-4} < t_{-2} < t_{-1} < t_0 < t_1 < \ldots < t_{n+1} < t_{n+2} < t_{n+3} < t_{n+4} < t_{n+5} < t_{n+6}.$$

International Journal of Applied Mathematics

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To solve the boundary value problem (1) and (2) by the Galerkin method with sextic B-splines as basis functions, we define the approximation for $v(t)$ as

$$v(t)=\sum_{j=-3}^{n+2}\alpha_j R_j(t) \tag{3}$$

where α_j's are the nodal parameter to be determined. In Galerkin method, the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are specified.In the set of sextic B-splines $\{R_{-3}(t), R_{-2}(t), R_{-1}(t), R_0(t), R_1(t), R_2(t), \ldots\ldots\ldots,$

$R_{n-1}(t), R_n(t), R_{n+1}(t), R_{n+2}(t)\}$ the basis functions $R_{-3}(t), R_{-2}(t), R_{-1}(t), R_0(t), R_1(t), R_2(t), R_{n-3}(t), R_{n-2}(t), R_{n-1}(t), R_n(t), R_{n+1}(t)$ and $R_{n+2}(t)$ do not vanish at one of the boundary points. So, there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions are specified. The procedure for redefining is as follows.

Using the sextic B-splines and the Dirichlet boundary conditions of (2), we get the approximate solution at the boundary points as

$$A_0=v(l)=v(t_0)=\alpha_{-3}R_{-3}(t_0)+\alpha_{-2}R_{-2}(t_0)+\alpha_{-1}R_{-1}(t_0)+\alpha_0R_0(t_0)+\alpha_1R_1(t_0)+\alpha_2R_2(t_0) \tag{4}$$

$$\begin{aligned}C_0=v(m)=v(t_n)&=\alpha_{n-3}R_{n-3}(t_n)+\alpha_{n-2}R_{n-2}(t_n)+\alpha_{n-1}R_{n-1}(t_n)\\&+\alpha_nR_n(t_n)+\alpha_{n+1}R_{n+1}(t_n)+\alpha_{n+2}R_{n+2}(t_n)\end{aligned} \tag{5}$$

Eliminating α_{-3} and α_{n+2} from the equations (3),(4) and (5),we get

$$v(t)=w_1(t)+\sum_{j=-2}^{n+1}\alpha_j S_j(t) \tag{6}$$

where
$$w_1(t)=\frac{A_0}{R_{-3}(t_0)}R_{-3}(t)+\frac{C_0}{R_{n+2}(t_n)}R_{n+2}(t) \tag{7}$$

and
$$S_j(t)=\begin{cases}R_j(t)-\dfrac{R_j(t_0)}{R_{-3}(t_0)}R_{-3}(t), & j=-2,-1,0,1,2\\ R_j(t), & j=3,\ldots,n-4\\ R_j(t)-\dfrac{R_j(t_n)}{R_{n+2}(t_n)}R_{n+2}(t), & j=n-3,n-2,n-1,n,n+1.\end{cases} \tag{8}$$

Using the Neumann boundary conditions of (2) to the approximate solution $v(t)$ in (6), we get

$$A_1=v'(l)=v'(t_0)=w_1'(t_0)+\alpha_{-2}S_{-2}'(t_0)+\alpha_{-1}S_{-1}'(t_0)+\alpha_0S_0'(t_0)+\alpha_1S_1'(t_0)+\alpha_2S_2'(t_0) \tag{9}$$

Volume 38 No. 12s, 2025

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$$C_1 = v'(m) = v'(t_n) = w_1'(t_n) + \alpha_{n-3}S_{n-3}'(t_n) + \alpha_{n-2}S_{n-2}'(t_n) + \alpha_{n-1}S_{n-1}'(t_n) + \alpha_n S_n'(t_n) + \alpha_{n+1}S_{n+1}'(t_n) \quad (10)$$

Eliminating α_{-2} and $\alpha_{n+1}$from the equations (6), (9) and (10), we get approximation for $v(t)$ as

$$v(t) = w_2(t) + \sum_{j=-1}^{n} \alpha_j T_j(t) \quad (11)$$

where $$w_2(t) = w_1(t) + \frac{A_1 - w_1'(t_0)}{S_{-2}'(t_0)} S_{-2}(t) + \frac{C_1 - w_1'(t_n)}{S_{n+1}'(t_n)} S_{n+1}(t) \quad (12)$$

and $$T_j(t) = \begin{cases} S_j(t) - \dfrac{S_j'(t_0)}{S_{-2}'(t_0)} S_{-2}(t), & j = -1,0,1,2 \\ S_j(t), & j = 3,\ldots,n-4 \\ S_j(t) - \dfrac{S_j'(t_n)}{S_{n+1}'(t_n)} S_{n+1}(t), & j = n-3, n-2, n-1, n. \end{cases} \quad (13)$$

Using the second order derivative boundary conditions of (2) to the approximate solution $v(t)$ in (11), we get

$$A_2 = v''(l) = v''(t_0) = w_2''(t_0) + \alpha_{-1}T_{-1}''(t_0) + \alpha_0 T_0''(t_0) + \alpha_1 T_1''(t_0) + \alpha_2 T_2''(t_0) \quad (14)$$

$$C_2 = v''(m) = v''(t_n) = w_2''(t_n) + \alpha_{n-3}T_{n-3}''(t_n) + \alpha_{n-2}T_{n-2}''(t_n) + \alpha_{n-1}T_{n-1}''(t_n) + \alpha_n T_n''(t_n) \quad (15)$$

Eliminating α_{-1} and α_n from the equations (11),(14) and (15),we get approximation for $v(t)$ as

$$v(t) = w(t) + \sum_{j=0}^{n-1} \alpha_j \tilde{R}_j(t) \quad (16)$$

where

$$w(t) = w_2(t) + \frac{A_2 - w_2''(t_0)}{T_{-1}''(t_0)} T_{-1}(t) + \frac{C_2 - w_2''(t_n)}{T_n''(t_n)} T_n(t) \quad (17)$$

and

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$$\tilde{R}_j(t)=\begin{cases} T_j(t)-\dfrac{T_j''(t_0)}{T_{-1}''(t_0)}T_{-1}(t), & j=0,1,2 \\ T_j(t), & j=2,3,...,n-3 \\ T_j(t)-\dfrac{T_j''(t_n)}{T_n''(t_n)}T_n(t), & j=n-3,n-2,n-1 \end{cases} \tag{18}$$

Now, the new set of basis functions for the approximation $v(t)$ are$\{\tilde{R}_j(t)$, j=1,2,... n–1$\}$.Applying the Galerkin method to (1) with the new set of basis functions, we get

$$\int_{t_0}^{t_n}\Big[a_0(t)v^{(7)}(t)+a_1(t)v^{(6)}(t)+a_2(t)v^{(5)}(t)+a_3(t)v^{(4)}(t)+a_4(t)v'''(t)+a_5(t)v''(t)+a_6(t)v'(t)$$

$$+a_7(t)v(t)\Big]\tilde{R}_i(t)\,dt=\int_{t_0}^{t_n}b(t)\tilde{R}_i(t)dt \qquad for \quad i=0,1,2,.......,n-2,n-1 \tag{19}$$

Integrating by parts terms the first two terms on the left hand side of (19), we get term the Neumann, second order derivative boundary conditions prescribed in (2), we get

$$\int_{t_0}^{t_n}a_0(t)\tilde{R}_i(t)v^{(7)}(t)\,dt=-\frac{d^3}{dt^3}\big[a_0(t)\tilde{R}_i(t)\big]v'''(t)\big|_{t_n}+A_3\frac{d^3}{dt^3}\big[a_0(t)\tilde{R}_i(t)\big]\big|_{t_0}$$
$$+\int_{t_0}^{t_n}\frac{d^4}{dt^4}\big[a_0(t)\tilde{R}_i(t)\big]v'''(t)\,dt \tag{20}$$

$$\int_{t_0}^{t_n}a_1(t)\tilde{R}_i(t)v^{(6)}(t)\,dt=-\int_{t_0}^{t_n}\frac{d}{dt}\big[a_1(t)\tilde{R}_i(t)\big]v^{(5)}(t)\,dt \tag{21}$$

Substituting (20), (21) in (19) and using the approximation for $v(t)$ given in (16),and after rearranging the terms for resulting equations, we get a system of equations in the matrix form as

Aα=**B** (22)

Where **A**= $[a_{ij}]$;

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$$a_{ij}=\int_{t_0}^{t_n}\{[(a_2(t)\tilde{R}_i(t))-\frac{d}{dt}(a_1(t)\tilde{R}_i(t)]\tilde{R}_j^{(5)}(t)+a_3(t)\tilde{R}_i(t)\tilde{R}_j^{(4)}(t)+[\frac{d^4}{dt^4}(a_0(t)\tilde{R}_i(t)]$$

$$+a_4(t)\tilde{R}_i(t)]\tilde{R}_j'''(t)+a_5(t)\tilde{R}_i(t)\tilde{R}_j''(t)+a_6(t)\tilde{R}_i(t)\tilde{R}_j'(t)+a_7(t)\tilde{R}_i(t)\tilde{R}_j(t)\}dt$$

$$-\frac{d^3}{dt^3}[a_0(t)\tilde{R}_i(t)]\tilde{R}_j'''(t)|_{t_n} \qquad \text{for} \quad \mathrm{i}=0,1,2,3,\ldots,\mathrm{n}-1;\mathrm{j}=0,1,2,3,\ldots,\mathrm{n}-1. \qquad (23)$$

B= $[b_i]$;

$$b_i=\int_{t_0}^{t_n}\{b(t)\tilde{R}_i(t)+[a_2(t)\tilde{R}_i(t)-\frac{d}{dt}[a_1(t)\tilde{R}_i(t)]\tilde{R}_j^{(5)}(t)+a_3(t)\tilde{R}_i(t)\tilde{R}_j^{(4)}(t)$$

$$+[\frac{d^4}{dt^4}(a_0(t)\tilde{R}_i(t)+a_4(t)\tilde{R}_i(t)]\tilde{R}_j'''(t)+a_5(t)\tilde{R}_i(t)\tilde{R}_j''(t)+a_6(t)\tilde{R}_i(t)\tilde{R}_j'(t)$$

$$+a_7(t)\tilde{R}_i(t)\tilde{R}_j(t)\}dt+\frac{d^3}{dt^3}[a_0(t)\tilde{R}_i(t)]\tilde{R}_j'''(t)|_{t_n}-A_3\frac{d^3}{dt^3}[a_0(t)\tilde{R}_i(t)]|_{t_0}$$

$$\text{for} \quad \mathrm{i}=0,1,2,3,\ldots,\mathrm{n}-1. \qquad (24)$$

and $\alpha=[\alpha_1,\alpha_2,.....,\alpha_{n-1}]^T$

4. Procedure to find a solution for nodal parameters

A typical integral element in the matrix **A** is

$$\sum_{p=0}^{n-1} I_p$$

where $I_p=\int_{t_p}^{t_{p+1}} r_i(t)r_j(t)Z(t)dt$ and $r_i(t)$,$r_j(t)$ are the sextic B-spline basis functions or their derivatives .It may be noted that $I_p=0$ if $(t_{i-3}, t_{i+4}) \cap (t_{j-3}, t_{j+4}) \cap (t_p, t_{p+1}) = \theta$. To evaluate each I_p , we employed 7-point Gauss-Legendre quadrature formula .Thus the stiff matrix **A** is a thirteen diagonal band matrix. The nodal parameter vector α has been obtained from the system **A**α = **B** using a band matrix solution package. We have used FORTRAN-90 program to solve the boundary value problems (1)-(2) by the proposed method

5. Numerical Results

To demonstrate the applicability of the proposed method for solving the seventh order boundary value problems of the types (1) and (2), we considered four linear boundary value problems and two nonlinear boundary value problems. Numerical results for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

Example 1: Consider the linear boundary value problem

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$$v^{(7)} + tv = -\left(35 + 13t + t^3\right)e^t, \qquad 0 < t < 1 \qquad (25)$$

subject to $v(0) = v(1) = 0,\ v'(0) = 1,\ v'(1) = -e,\ v''(0) = 0,\ v''(1) = -4e,\ v'''(0) = -3.$

The exact solution for the above problem is $v(t) = t(1-t)e^x$. The proposed method is tested on this problem where the domain [0, 1] is divided into10 equal subintervals. The obtained numerical results for this problem are given in Table 1. The maximum absolute error obtained by the proposed method is 9.596348×10^{-06}.

Example 2: Consider the linear boundary value problem

$$v^{(7)} - 4v''' + 2v' + tv = \left(6 + 2t - t^2\right)e^t, \qquad 0 \le t \le 1 \qquad (26)$$

subject to $v(0) = 1, v(1) = 0, v'(0) = 0, v'(1) = -e, v''(0) = -1, v''(1) = -2e,\ v'''(0) = -2.$

The exact solution is $v = (1-t)e^t$. The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 2. The maximum absolute error obtained by the proposed method is 9.23872×10^{-6}.

Example 3: Consider the nonlinear boundary value problem

$$v^{(7)} = 6!\left[e^{-7v} - \frac{2}{(1+t)^7}\right], \qquad 0 < t < e^{\frac{1}{2}} - 1 \qquad (27)$$

subject to $v(0) = 0, v(e^{\frac{1}{2}} - 1) = \frac{1}{2}, v'(0) = 1, v'(e^{\frac{1}{2}} - 1) = e^{-\frac{1}{2}}, v''(0) = -1, v''(e^{\frac{1}{2}} - 1) = \frac{-1}{e}, v'''(0) = 2.$

The exact solution for the above problem is $v(t) = lnt$. The nonlinear boundary value problem (27) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [19] as

$$v_{(n+1)}^{(7)} + \left[7!e^{-7v_{(n)}}\right]v_{(n+1)} = \left[7!v_{(n)} + 6!\right]e^{-7v_{(n)}} - \frac{2\times 6!}{(1+t)^7} \qquad n = 0,1,2,3,.... \qquad (28)$$

subject to $v_{(n+1)}(0) = 0, v_{(n+1)}(e^{\frac{1}{2}} - 1) = \frac{1}{2}, v'_{(n+1)}(0) = 1, v'_{(n+1)}(e^{\frac{1}{2}} - 1) = e^{-\frac{1}{2}}, v''_{(n+1)}(0) = -1,$
$v''_{(n+1)}(e^{\frac{1}{2}} - 1) = \frac{-1}{e}, v'''_{(n+1)}(0) = 2.$

International Journal of Applied Mathematics

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Here $v_{(n+1)}$ is the $(n+1)^{th}$ approximation for $v(t)$.The domain $\left[0, e^{\frac{1}{2}} - 1\right]$ is divided into 10 equal subintervals and the proposed method is applied to the sequence of a linear problems (28). The obtained numerical results for this problem are presented in Table 3. The maximum absolute error obtained by the proposed method is 2.884865×10^{-5}.

Table 1: Numerical results for Example 1

t	Exact solution	Absolute error by Proposed method
0.1	9.946539E-02	7.450581E-08
0.2	1.954244E-01	1.549721E-06
0.3	2.834704E-01	3.874302E-07
0.4	3.580379E-01	1.490116E-06
0.5	4.121803E-01	3.635883E-06
0.6	4.373085E-01	3.278255E-06
0.7	4.228881E-01	5.811453E-06
0.8	3.560865E-01	8.076429E-06
0.9	2.213642E-01	9.596348E-06

Table 2: Numerical results for Example 2

t	Exact Solution	Absolute error by proposed method
0.1	9.946538E-01	0.000000E+00
0.2	9.771222E-01	5.304813E-06
0.3	9.449012E-01	3.576279E-06
0.4	8.950948E-01	7.629395E-06
0.5	8.243606E-01	9.238720E-06
0.6	7.288475E-01	1.549721E-06
0.7	6.041259E-01	5.960464E-06
0.8	4.451082E-01	8.016825E-06
0.9	2.459602E-01	8.419156E-06

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Example 4: Consider the nonlinear boundary value problem

$$v^{(7)} - e^{-t}v^2 = -e^{-t} - e^{-3t}, \qquad 0<t<1 \qquad (29)$$

subject to $v(0)=1, v(1)=\frac{1}{e}, v'(0)=-1, v'(1)=\frac{-1}{e}, v''(0)=1, v''(1)=\frac{1}{e}, v'''(0)=-1$.

The exact solution for the above problem is $v(t)=e^{-t}$. The nonlinear boundary value problem (29) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [19] as

$$v^{7}_{(n+1)} - \left[2v_n e^{-t}\right]v_{(n+1)} = -v^{2}_{n}e^{-t} - e^{-t} - e^{-3t}, \qquad n=0,1,2,3\ldots\ldots \qquad (30)$$

subject to

$$v_{(n+1)}(0)=1, v_{(n+1)}(1)=\frac{1}{e}, v'_{(n+1)}(0)=-1, v'_{(n+1)}(1)=\frac{-1}{e}, v''_{(n+1)}(0)=1, v''_{(n+1)}(1)=\frac{1}{e}, v'''_{(n+1)}(0)=-1$$

Here $v_{(n+1)}$ is the $(n+1)^{th}$ approximation for $v(t)$. The domain [0,1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of a linear problems (30). The obtained numerical results for this problem are presented in Table 4. The maximum absolute error obtained by the proposed method is 1.1146×10^{-5}.

Table 3: Numerical results for Example 3

t	Exact solution	Absolute error by proposed method
6.487213E-02	6.285473E-02	2.376735E-06
1.297443E-01	1.219913E-01	1.117587E-06
1.946164E-01	1.778251E-01	2.905726E-06
2.594885E-01	2.307057E-01	1.069903E-05
3.243607E-01	2.809298E-01	2.148747E-05
3.892328E-01	3.287517E-01	2.884865E-05
4.541049E-01	3.743905E-01	2.470613E-05
5.189770E-01	4.180371E-01	2.032518E-05
5.838492E-01	4.598581E-01	1.251698E-05

Table 4: Numerical results for Example 4

t	Exact solution	Absolute error by proposed method
0.1	9.048374E-01	3.576279E-07

Volume 38 No. 12s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

0.2	8.187308E-01	1.370907E-06
0.3	7.408182E-01	8.940697E-06
0.4	6.703200E-01	9.953976E-06
0.5	6.065307E-01	7.331371E-06
0.6	5.488116E-01	1.114607E-05
0.7	4.965853E-01	5.692244E-06
0.8	4.493290E-01	2.413988E-06
0.9	4.065697E-01	2.175570E-06

6. Conclusions

In this paper, we have deployed a Galerkin method with sextic B-splines as basis functions to solve seventh order boundary value problems with boundary conditions. The sextic B-spine basis set has been redefined into a new set of basis functions which vanish on the boundary where the Dirichlet and Neumann, secondary order boundary conditions are prescribed. The proposed method has been tested on four linear and three nonlinear seventh order boundary value problems. The numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature. The objective of this paper is to present a simple, efficient method to solve a seventh order boundary value problem.

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