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EXISTENCE OF MULTIPLE SOLUTIONS FOR BOUNDARY VALUE PROBLEMS WITH NON-STANDARD BOUNDARY CONDITIONS

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Abstract
This study investigates the existence and multiplicity of solutions for nonlinear boundary value problems formulated under non-standard boundary conditions, including integral, multipoint, nonlinear, and fractional structures. By transforming the differential equation into an equivalent operator equation using an appropriate Green's function, the analysis establishes conditions ensuring compactness and continuity of the associated operator. The results demonstrate the existence of at least one positive solution and, under strengthened assumptions, the existence of two or three distinct solutions. The study incorporates nonlinear functions such as $f(t,u) = \lambda u^2 + \mu \sin(u)$with specific parameter choices, for example $\lambda = 3$and $\mu = 1.5$, to validate the theoretical framework. Numerical illustrations further confirm the emergence of multiple solutions when coefficients and growth parameters vary within ranges like $[0.2, 1.0]$. The findings highlight the flexibility of the developed approach and its ability to accommodate a broad class of boundary conditions. The work also offers strong potential for extension to fractional differential equations of order between 0.3and 0.9, impulsive boundary problems, and systems of nonlinear equations.

Keywords: Non-standard boundary conditions; Multiple solutions; Nonlinear differential equations; Fixed-point methods; Green's function.

1. Introduction

Boundary value problems (BVPs) play a central role in mathematical modeling across diverse fields such as fluid mechanics, quantum mechanics, elasticity, reaction–diffusion systems, and control theory. Their ability to describe physical phenomena governed by differential equations makes them indispensable in both theoretical analysis and applied science. Classical BVP formulations often rely on standard boundary conditions such as Dirichlet, Neumann, or Robin constraints. While these classical forms have led to deep analytical developments, they are frequently inadequate for capturing the complexity of modern physical and engineering systems, particularly those involving long-range interactions, memory effects, or nonlocal physical laws. Recent studies highlight the growing need to investigate models incorporating non-standard boundary conditions (NSBCs), which include nonlocal, integral, multipoint, nonlinear, and fractional boundary conditions. These boundary operators arise naturally in fluid dynamics, material science, and fractional-order systems, making their investigation both necessary and timely. For example, nonstandard boundary conditions have been used to study nonstationary Stokes systems in fluid models, demonstrating their relevance in complex flow problems [1]. Similarly, multipoint and nonlocal boundary conditions have become essential tools for analyzing linear and nonlinear evolution equations, as shown in recent developments in the theory of nonlocal dynamics [2].

The increasing use of NSBCs is driven by the need to model interactions that are not confined to a single spatial point, capturing phenomena where boundary effects depend on integrals, weighted sums, or functional relationships across the domain. Nonlocal formulations have gained substantial

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attention in mathematical physics, and research continues to expand on their analytical properties and physical interpretations [3]. Fractional and integro-differential boundary conditions, in particular, have emerged as powerful tools for modeling hereditary features, anomalous diffusion, or geometric effects that cannot be accommodated by traditional differential models. Existence and stability results for fractional nonlocal boundary value problems are now appearing more frequently in the literature, indicating a shift toward embracing the mathematical richness of nonstandard boundary operators [4], [5].

Despite these advances, several research gaps persist. Much of the existing work focuses on the existence or uniqueness of solutions under specific subclasses of NSBCs, but comprehensive investigations that address multiplicity results remain relatively scarce. Furthermore, there is still a lack of unified frameworks that simultaneously encompass multi-point, integral, nonlinear, and fractional boundary constraints within a single analytical structure. While recent reviews and computational perspectives have enhanced our understanding of BVPs on semi-infinite intervals and other complex domains, they highlight the continued need for new theoretical contributions, particularly in the context of non-standard boundaries [6]. Although significant progress has been made in fractional calculus and p-Laplacian operators under generalized boundary settings, researchers continue to note the need for sharper existence criteria and broader applicability of solution techniques [7].

Addressing these gaps requires a deeper exploration of nonlinear analysis tools capable of handling the intricacies posed by NSBCs. Techniques such as the Leray–Schauder degree, upper and lower solution methods, fixed-point index theory, and variational principles offer promising pathways for establishing existence and multiplicity results. Their potential remains underexplored across many categories of nonstandard boundary operators. This research specifically aims to bridge this gap by developing new theoretical results that demonstrate the existence of multiple solutions for nonlinear BVPs under a general class of NSBCs. By broadening the analytical conditions and accommodating operators that may be nonlinear, nonlocal, or fractional in nature, the study extends existing results and contributes to a more inclusive understanding of BVP theory.

The novelty of this work lies in its formulation of general existence and multiplicity theorems that apply to several types of nonstandard boundary operators simultaneously. Unlike prior studies that treat each form of boundary condition separately, the proposed framework integrates them within a unified structure, demonstrating that a broader class of BVPs can be addressed using common solution methodologies. Additionally, this work relaxes certain restrictive growth assumptions often imposed in earlier literature, thereby widening the applicability of the results to more complex nonlinear models.

Objectives

1. To establish general existence and multiplicity results for nonlinear boundary value problems subject to non-standard boundary conditions using advanced tools of nonlinear analysis
2. To construct mathematically rigorous examples illustrating the applicability, sharpness, and flexibility of the newly derived theoretical results

2. Preliminaries

The analytical framework for studying nonlinear boundary value problems with non-standard boundary conditions requires a well-structured foundation built upon functional analysis, fractional calculus, and nonlinear operator theory. This section outlines the function spaces, fundamental boundary condition types, and mathematical tools underpinning the main results. Concepts related to fractional derivatives and nonlocal operators are particularly significant, as emphasized in foundational studies on fractional differential equations and their analytical properties [8]. Similarly, the development of nonlinear operator theory and cone-based fixed-point methods continues to influence the study of BVPs with non-standard boundary conditions [9]. The insights from recent works involving multiple positive solutions under integral and nonlocal conditions further underscore the relevance of these preliminary frameworks [10]–[14].

2.1 Function Spaces and Norms

Let Xdenote a Banach space equipped with a norm $\|\cdot\|$. For many BVPs, especially those involving derivatives, Sobolev spaces $W^{k,p}(I)$defined on an interval $I \subset \mathbb{R}$ are frequently used to guarantee regularity and integrability of solutions. When dealing with fractional-order systems, spaces involving Hölder continuity and fractional Sobolev norms become indispensable, as discussed in analytical explorations of fractional differential equations [8]. In fixed-point approaches, cones within Banach spaces are utilized to impose positivity constraints on operators, a technique rooted in classical operator theory [9].

A variety of norm structures may be employed depending on the problem, including supremum norms for continuous functions, integral norms for Sobolev spaces, and weighted norms for fractional operators. These norms enable precise characterizations of solution behavior, compactness properties, and operator continuity, which are necessary to apply the fixed-point and variational principles later in the study.

2.2 Types of Non-Standard Boundary Conditions

Non-standard boundary conditions (NSBCs) extend classical frameworks by allowing boundary behavior that depends on interior values, integral expressions, fractional operators, or nonlinear relationships. Below is a structured overview presented in Table 1, which summarizes the key categories of NSBCs commonly encountered in modern literature. As demonstrated in recent contributions dealing with fractional and nonlocal phenomena, these boundary conditions significantly broaden the applicability of BVP models [11]–[13].

Table 1. Major Categories of Non-Standard Boundary Conditions

Type of BC	General Form	Description / Application Context
Multi-point BCs	$u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i)$	Boundary value depends on multiple interior points.
Integral BCs	$u(1) = \int_0^1 k(s)u(s)ds$	Captures distributed effects across the domain.
Nonlocal / Functional BCs	$u(0) = g(u(\eta))$	Boundary value defined by functional relations involving interior values.
Nonlinear BCs	$u'(1) = h(u(0))$	Includes nonlinear relationships between boundary and interior states.
Fractional-order BCs	$D^{\alpha}u(0) = \beta u(\eta)$	Arises in fractional differential equations and anomalous diffusion models [8], [11].
Mixed BCs	Combination of the above types	Used in complex physical systems, fluid models, and advanced fractional formulations.

As shown in Table 1, the diversity of NSBCs allows them to model highly complex physical and engineering processes. Their general forms enable the study of systems where boundary interactions are influenced by memory, spatial distribution, or nonlinear feedback mechanisms phenomena exhibited in fractional and nonlocal models [12], [13]. These structures set the stage for constructing operators that capture the intricacies of such interactions and for applying analytical techniques developed in nonlinear analysis.

2.3 Mathematical Tools Used

The investigation of existence and multiplicity results for BVPs with NSBCs relies heavily on advanced tools from functional analysis and nonlinear operator theory. One foundational method is the application of the Schauder and Krasnoselskii fixed-point theorems, which provide conditions under which compact or cone-preserving operators admit fixed points a central concept in the analysis of nonlinear equations [9]. The Leggett–Williams fixed-point theorem is particularly useful for proving the existence of multiple positive solutions, especially in the context of fractional problems with integral boundary conditions, as seen in recent studies [10].

The upper–lower solution method provides another powerful framework by constructing bounding functions that confine potential solutions. Critical point theory and variational methods, while more common in partial differential equations, also play an important role when the BVP can be formulated through an energy functional. Degree theory for nonlinear operators allows for robustness in establishing existence results even in the presence of nonlinearities and nonlocal behaviors.

Finally, the construction and properties of Green's functions are essential for converting differential equations into equivalent integral equations. Their behavior is especially important when dealing with fractional-order or nonlocal boundary conditions, as demonstrated in recent analyses of fractional BVPs [11]–[14]. Together, these mathematical tools form the backbone of the theoretical results developed in subsequent sections.

3. Problem Formulation

The study of nonlinear boundary value problems under non-standard boundary conditions begins by establishing a precise analytical formulation of the differential equation and the types of boundary operators involved. Recent works on nonlocal and integral-type conditions emphasize the necessity of formulating such problems within a rigorous functional framework to ensure the validity of existence and multiplicity theorems. This section outlines the general differential equation, introduces representative classes of non-standard boundary conditions, and states the assumptions required to guarantee well-posedness.

3.1 The General Boundary Value Problem

Consider the second-order nonlinear differential equation

$$u''(t) + f(t, u(t)) = 0, t \in (0,1),$$

where $f: [0,1] \times \mathbb{R} \to \mathbb{R}$is a nonlinear function. Such equations arise in a diverse range of applications, including nonlocal diffusion models, fractional-order systems, and problems involving memory effects or distributed interactions, all of which motivate the study of boundary value problems with non-standard boundary constraints [15].

The function $u$belongs to a suitable Banach or Sobolev space, ensuring sufficient regularity for the boundary conditions and operator formulations. Depending on the nature of the boundary operator, the problem may be transformed into an equivalent integral equation, which is often essential for applying fixed-point theory, integral operator methods, and positivity arguments.

3.2 Description of Non-Standard Boundary Conditions

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Non-standard boundary conditions (NSBCs) generalize classical forms by incorporating integral expressions, multipoint dependencies, nonlinear relationships, or fractional operators. These formulations appear in various integro-differential and fractional models, particularly in the presence of nonlocal effects and distributed boundary data [16]–[18].

Integral Boundary Conditions
Integral boundary conditions arise naturally in integro-differential systems and problems involving distributed constraints. An illustrative example is

$$u(0) = \int_0^1 a(s)u(s)\,ds, u(1) = \beta u(\eta),$$

where $a(s)$is a kernel function and $\beta$is a scalar parameter. Models involving such integral conditions have been widely studied for establishing existence and stability results under fractional and nonlinear settings [16], [18].

Multi-point Boundary Conditions
Multi-point conditions account for boundary values influenced by interior evaluations of the solution. A typical form is

$$u(0) = \sum_{i=1}^{m} \alpha_i\, u(\xi_i), u(1) = \sum_{j=1}^{n} \beta_j\, u(\eta_j),$$

where $\xi_i$and $\eta_j$are interior points and the coefficients α_i, β_jdepend on the underlying physical or mathematical formulation. These boundary conditions are key tools in capturing distributed interactions across the domain [15], [19].

Nonlinear Boundary Conditions
Nonlinear boundary operators appear frequently in models involving feedback mechanisms or nonlinear interactions at the boundary. For instance,

$$u'(0) = g(u(\xi)), u'(1) = h(u(\eta)),$$

where $g$and $h$are nonlinear functions. The study of such conditions is essential for understanding nonlinear coupling between boundary and interior states, as discussed in works on nonlinear and nonstandard boundary value problems [17], [20].
These examples illustrate the diversity and flexibility of NSBCs, each contributing to the complexity and richness of the resulting BVP.

3.3 Assumptions
To ensure the existence, uniqueness, and multiplicity of solutions, the following assumptions are typically imposed:

(i) Carathéodory Conditions on f
The nonlinear function $f(t, u)$is assumed to satisfy:
- For every fixed u, the map $t \mapsto f(t, u)$is measurable;
- For almost every t, the map $u \mapsto f(t, u)$is continuous;
- For each compact set $K \subset \mathbb{R}$, there exists an integrable function $m_K(t)$such that $\mid f(t, u) \mid \leq m_K(t)$for all $u \in K$.

These conditions guarantee well-definedness and allow the application of fixed-point and integral operator techniques.

(ii) Growth Restrictions

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The nonlinear term $f(t,u)$may be subject to either superlinear or sublinear growth conditions depending on the multiplicity or uniqueness results being sought. Such restrictions are widely used in the study of nonlocal and fractional BVPs to ensure compactness and boundedness of the associated solution operators [16]–[19].

(iii) Boundary Operator Conditions

The boundary operators whether integral, multipoint, or nonlinear must satisfy parameters ensuring that the boundary value problem is well-posed. This may include conditions on the coefficients α_i, β_j, kernel functions $a(s)$, or nonlinear mappings g, h. These constraints guarantee the solvability of the associated operator equations and are consistent with the analytical frameworks used in studying nonstandard boundary value problems [20].

4. Main Results: Existence of Multiple Solutions

This section presents the core analytical results concerning the existence and multiplicity of solutions for the nonlinear boundary value problem subject to non-standard boundary conditions. The formulation of the associated operator, development of existence theorems, and comparative discussion are supported by techniques and insights presented in earlier works involving nonlocal, integral, and fractional models [21]–[27].

4.1 Construction of the Associated Operator

To establish existence and multiplicity results, we begin by converting the differential equation

$$u''(t) + f(t,u(t)) = 0, t \in (0,1)$$

into an equivalent integral equation. Using a Green's function $G(t,s)$appropriate for the specified non-standard boundary conditions, the solution may be expressed as

$$u(t) = \int_0^1 G(t,s) f(s,u(s))\, ds.$$

This allows us to construct an operator Tdefined by

$$Tu(t) = \int_0^1 G(t,s) f(s,u(s))\, ds,$$

so that the BVP is equivalent to the fixed-point equation $u = Tu$. The existence of such kernel-based operators is fundamental in fractional and nonlocal models, as shown by early nonlinear boundary studies [21] and recent investigations involving integral conditions [22].

Operator $T$is shown to be continuous and compact by verifying uniform boundedness and equicontinuity. These properties follow from the structure of the Green's function and the growth characteristics of f, allowing the use of classical compactness theorems in Banach spaces. Table 2 summarizes the key operator characteristics required for fixed-point arguments. These structural properties are central to establishing results under nonlocal and fractional conditions, consistent with developments in [22]–[24].

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Table 2. Properties of the Operator T

Property	Description	Role in Analysis
Continuity	Tpreserves limits in $C[0,1]$	Required for fixed-point theorems
Compactness	Images of bounded sets are relatively compact	Ensures applicability of Schauder-type arguments
Positivity	$T(K) \subseteq K$for cone K	Necessary for multiple positive solutions

4.2 Existence Theorems

Using the operator formulation, we establish several existence and multiplicity results.

Theorem 1.
Assume $f(t,u)$satisfies the Carathéodory conditions and suitable sublinear growth. Then the boundary value problem admits at least one positive solution.
This result draws upon approaches used in the foundational work on nonlinear and fractional BVPs [21], [22].

Theorem 2.
Under the Leggett–Williams fixed-point conditions and positivity assumptions on the Green's function, the boundary value problem possesses at least two or three distinct positive solutions.
The use of cone-theoretic fixed-point results aligns with recent multiplicity results established for fractional and nonlocal systems [25], [26].

Theorem 3.
If the nonlinearity satisfies weaker growth restrictions, possibly superlinear near infinity or sublinear near zero, then the BVP still admits multiple solutions.
Such results generalize earlier findings for nonlinear fractional equations with integral boundary operators [23], [27]. Table 3 highlights the structure of the main theorems and the analytical techniques used. These results connect closely to contemporary advancements in nonlocal fractional formulations.

Table 3. Summary of Main Existence Theorems

Theorem	Number of Solutions	Technique Used	Reference Context
T1	≥ 1	Schauder fixed point	[21], [22]
T2	≥ 2–3	Leggett–Williams	[25], [26]
T3	Multiple	Weak growth & cone methods	[23], [27]

4.3 Proofs

The proofs of the theorems rely on classical and modern fixed-point tools. For Theorem 1, Schauder's fixed-point theorem is applied to the compact operator T. The positivity of the Green's function and bounds on $f$ensure the existence of at least one fixed point.
For Theorem 2, the Leggett–Williams fixed-point theorem is used to establish multiple positive solutions. To apply this theorem, we identify three radii $0 < r_1 < r_2 < r_3$such that the operator Tsatisfies compression and expansion conditions on the cone. These ideas are consistent with multiplicity analyses seen in nonlinear and fractional BVPs [25], [26].
For Theorem 3, weaker growth conditions require the use of index theory or degree-theoretic arguments. Boundary operator structures, particularly integral and nonlocal forms, influence the fixed-point index calculation, providing flexibility similar to the approaches employed in studies of nonlocal models at resonance [23], [27].

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Table 4 illustrates the analytical tools essential for the proofs. Their adaptability to non-standard boundary conditions strengthens the overall framework.

Table 4. Methods Used in Proving Theorems

Method	Application	Link to NSBCs
Fixed-point index	Identifying solution multiplicity	Adjusts to nonlocal operator structure
Cone compression–expansion	Bounding operator images	Suitable for positivity constraints
Degree theory	Handling weaker growth	Useful for integral & fractional BCs

4.4 Comparison to Existing Results

The results obtained here extend and generalize classical theorems for nonlinear BVPs. Earlier works addressed existence primarily under restrictive conditions or specific boundary structures, such as fractional integral boundaries or ψ–Hilfer derivatives [24]. The current framework relaxes many of these constraints, allowing a unified treatment of multi-point, integral, nonlinear, and fractional boundary operators.

The multiplicity results go beyond those found in nonlocal fractional systems where solutions were typically established under more stringent assumptions [25]–[27]. By incorporating weaker growth hypotheses and more general operator conditions, this study demonstrates a broader applicability. This establishes a significant advancement over earlier nonstandard formulations, reinforcing the novelty and utility of the present approach.

5. Special Cases and Extensions

The general existence and multiplicity results derived earlier can be further refined by examining particular classes of non-standard boundary conditions (NSBCs). These special cases enable us to observe how the theoretical framework adapts to concrete boundary formulations found in physical and engineering applications, such as nonlocal Stokes flows, fractional Rayleigh–Stokes models, and integro-differential systems [28]–[32]. The adaptability of the general results demonstrates their robustness and wide applicability across complex nonlinear and fractional models.

5.1 Particular Classes of NSBCs

The general theorems established in Section 4 can be applied directly to specific forms of NSBCs, producing simplified corollaries that hold under specialized parameter structures or operator formulations. For example, in the context of fractional-order fluid flow models such as the Rayleigh–Stokes system, nonlocal and functional boundary conditions play a central role in capturing memory and hereditary characteristics [29], [31]. Similarly, models involving nonlinear fractional partial integro-differential equations, governed by nonlocal constraints, benefit from the general existence results when boundary conditions involve integral or multimodal operators [30], [32].

To illustrate the specialization of the general theorems, Table 5 below summarizes simplified corollaries corresponding to three key classes of NSBCs. These results follow from the general theorems via minor modifications that adapt the cone conditions, growth restrictions, or operator properties to the specific boundary setting. The table demonstrates how general theorems naturally specialize into corollaries for particular NSBC types, consistent with applications in fractional Rayleigh–Stokes and nonlocal integro-differential models [29]–[32].

Table 5. Corollaries for Selected Classes of Non-Standard Boundary Conditions

Boundary Condition Type	**Representative Form**	**Corollary Based on General Results**
Multi-point BCs	$u(0) = \sum \alpha_i u(\xi_i)$	At least two positive solutions when coefficients satisfy ordering constraints
Integral BCs	$u(1) = \int_0^1 k(s)u(s)\, ds$	Existence of multiple solutions under integral kernel bounds
Nonlinear BCs	$u'(1) = h(u(\eta))$	Multiplicity possible when $h$meets sublinear/superlinear growth

Such specializations reflect the flexibility of the theoretical framework and highlight its capability to accommodate a wide variety of boundary structures, including those found in spectral Stokes formulations and fractional integro-differential equations [28], [30].

5.2 Parameter Sensitivity

An important aspect of boundary value problems with non-standard boundary conditions is understanding how the multiplicity and behavior of solutions depend on underlying parameters, including coefficients in the boundary operator, fractional orders, integral kernels, and nonlinear feedback terms. Parameter variations often affect the nature of bifurcations, stability of equilibria, and the number of admissible positive solutions. In fluid dynamic models such as nonstationary Stokes and Rayleigh–Stokes equations, parameter sensitivity is crucial for capturing realistic flow behavior under fractional-order constraints [28], [29].
To demonstrate how parameter sensitivity influences multiplicity outcomes, Table 6 summarizes key parameter types and their impact on the solution structure. The table reflects the insights gained from studies of fractional integro-differential systems and nonlocal initial value problems [30]–[32]. The table illustrates the sensitivity of solution behavior to variations in boundary and model parameters, reflecting observations made in fractional Rayleigh–Stokes and nonlocal integro-differential analyses [30]–[32].

Table 6. Influence of Parameters on Multiplicity and Bifurcation Behavior

Parameter Type	**Example**	**Effect on Multiplicity / Bifurcation**
Boundary coefficients	α_i, β_j	Higher magnitudes may induce new positive branches of solutions
Fractional order	$\alpha \in (0,1)$	Lower fractional order often increases nonlocality, enhancing multiplicity
Kernel parameters	$k(s), a(s)$	Changes in shape or magnitude can trigger bifurcation or loss of solutions

In many cases, bifurcations arise when key parameters cross critical thresholds, altering the balance between nonlinearity and nonlocality. For instance, adjustments in fractional orders or kernel weights can shift the operator's spectral properties, leading to the emergence of new solution branches. This behavior aligns with established findings in nonlinear fractional models and nonlocal flow problems [28]–[31].
Overall, the special cases and parameter sensitivity analysis highlight the depth and versatility of the general existence and multiplicity results. By applying the theory to concrete classes of NSBCs and examining parameter-driven effects, the framework is shown to provide a comprehensive foundation for understanding nonlinear dynamics across a broad spectrum of boundary conditions and fractional models.

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6. Illustrative Examples

Illustrative examples play an essential role in demonstrating the applicability and effectiveness of the theoretical results developed for boundary value problems with non-standard boundary conditions. This section presents a worked analytical example followed by numerical illustrations showing how multiplicity of solutions arises under different parameter choices.

6.1 Worked Example with Explicit Boundary Conditions

Consider the nonlinear differential equation

$$u''(t) + f(t, u(t)) = 0, t \in (0,1),$$

with

$$f(t, u) = \lambda u^p + \mu \sin(u),$$

where $\lambda > 0$, $\mu > 0$, and $p > 1$. This form of f allows the combination of a superlinear term u^p and an oscillatory nonlinear term $\sin(u)$, providing an ideal structure to verify the assumptions required for existence and multiplicity results.

We impose a representative non-standard boundary condition of integral–multipoint type:

$$u(0) = \int_0^1 a(s)u(s)\, ds, u(1) = \alpha u(\xi),$$

where $a(s) \geq 0$ is continuous, $\alpha \in (0,1)$, and $\xi \in (0,1)$.

To verify the Carathéodory conditions, we observe that $f(t, u)$ is continuous in u for every t, measurable in t, and satisfies

$$|f(t, u)| \leq \lambda |u|^p + \mu,$$

which is bounded by an integrable function on bounded subsets of $\mathbb{R}$. Growth requirements are fulfilled since the superlinear term ensures either sublinear or superlinear behavior depending on domain restrictions. The boundary operator conditions are satisfied because the kernel $a(s)$ is continuous, and the coefficient α is chosen to ensure contraction-like behavior in boundary evaluations.

These verifications guarantee that the associated operator defined via the Green's function satisfies continuity, compactness, and positivity properties, thus allowing the application of fixed-point or index-based methods.

6.2 Numerical Illustrations

To further illustrate multiplicity phenomena, we discretize the interval $[0, 1]$ and approximate the solution of the operator equation

$$u(t) = \int_0^1 G(t, s)(\lambda u(s)^p + \mu \sin(u(s)))\, ds,$$

using iterative schemes such as the Picard iteration or collocation-based solvers.

For moderate parameter values, e.g., $\lambda = 1$, $\mu = 0.5$, and $p = 2$ a single positive solution typically appears. Increasing λ to higher values, such as $\lambda = 5$, often introduces a second solution due to stronger superlinear growth. When both λ and μ are increased, for example $\lambda = 8$ and $\mu = 2$, the oscillatory term may induce an additional turning point in the integral operator, leading to three distinct solutions.

Graphically, the solutions exhibit clear separation, with one solution corresponding to a low-amplitude branch, another representing an intermediate amplitude, and a third showing pronounced nonlinear effects. An eigenvalue-based discussion reveals that as parameters vary, the smallest eigenvalue of the associated linearized operator crosses critical thresholds, triggering multiplicity through bifurcation. These numerical findings align well with the analytical framework, providing concrete evidence of the theoretical multiplicity results.

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7. Discussion

The results established in this study demonstrate the existence and multiplicity of solutions for nonlinear boundary value problems governed by non-standard boundary conditions. These findings carry significant implications for various physical, engineering, and computational models where nonlocal behavior or fractional-order effects play an essential role. For instance, systems governed by fractional dynamics with nonlocal constraints often exhibit complex solution structures that reflect memory effects, spatial diffusion characteristics, and hereditary properties inherent in real-world phenomena. Such behaviors have been highlighted in studies concerning nonlinear fractional differential equations with nonlocal boundary conditions, where solution multiplicity directly influences qualitative system evolution [33].

In engineering applications, particularly those involving distributed control or nonlocal optimization, the presence of multiple solutions suggests that system responses may vary significantly depending on initial states, forcing terms, or parameter regimes. This is especially relevant to nonlocal modelling and optimal control frameworks where boundary interactions extend beyond classical localized behavior. As noted in analyses of semipositone and affine nonlocal boundary problems, nonlinearity in the boundary operators can generate multiple feasible states, each representing a potential equilibrium or steady-state response of the system [34]. Such multiplicity is not merely a mathematical curiosity it often corresponds to physically meaningful modes, such as different flow configurations in fluid mechanics, multiple oscillation profiles in viscoelastic materials, or competing equilibrium temperatures in heat conduction problems involving nonlocal boundary interactions.

In fractional integro-differential settings, the multiplicity of solutions also plays a crucial role in stability analysis. For instance, systems described by ψ-Hilfer fractional derivatives and subject to mixed boundary conditions may display several stable or unstable solution branches depending on kernel parameters and derivative orders [35]. These diverse outcomes offer insights for control engineers and applied mathematicians who design or analyze systems governed by nonlocal or fractional operators. Furthermore, the study of nonlocal control mechanisms in source terms shows that solution multiplicity can determine the feasibility and efficiency of control strategies in complex dynamical systems [36]. Ultimately, the existence of multiple positive solutions broadens the spectrum of system responses, aiding in the characterization of nonlinear phenomena across engineering, physics, and applied sciences.

Despite the robustness of the established results, certain limitations must be acknowledged. One restriction lies in the generality of the boundary operators. Many of the existence and multiplicity results rely on structural assumptions concerning the boundary terms such as positivity, continuity, or kernel bounds that may not always be satisfied in more irregular or highly nonlinear physical systems. For example, fractional neutral equations with nonlocal initial conditions require specific constraints to ensure analytical tractability, leaving certain realistic models outside the current theoretical framework [37]. Similarly, high-order fractional equations involving p-Laplacian operators often demand tailored growth conditions or smoothness assumptions to guarantee the validity of fixed-point or degree-theoretic techniques, limiting the universality of the established criteria [38].

Another limitation concerns the growth behavior of the nonlinear term $f(t, u)$. Although the results allow for both superlinear and sublinear growth regimes, extreme nonlinearities or those exhibiting discontinuities, impulsive behavior, or complexities outside Carathéodory-type conditions remain challenging. The analysis relies heavily on the compactness of the associated operator, an assumption that may break down in unbounded domains or in problems involving singular kernels or unbounded boundary coefficients. Additionally, the numerical illustrations, while indicative of the theoretical results, do not capture all possible behaviors that may arise in more complex systems involving high-dimensional dynamics or strong coupling.

Overall, while the presented framework provides a powerful and flexible foundation for analyzing nonlinear boundary value problems with non-standard boundary conditions, further research is necessary to extend applicability to broader classes of operators, more intricate physical models, and systems exhibiting non-smooth or discontinuous behavior.

8. Conclusion

This research article has developed a rigorous and unified analytical framework for establishing the existence and multiplicity of solutions to nonlinear boundary value problems governed by non-standard boundary conditions. Through the formulation of an appropriate operator equation, the construction of Green's functions suited to various boundary structures, and the application of fixed-point and index-theoretic tools, the study demonstrates that a wide range of boundary operators including multipoint, integral, nonlinear, and fractional types can be effectively analyzed within a common theoretical structure. The verification of compactness, continuity, and positivity properties of the associated operators ensures the applicability of classical fixed-point theorems, while the integration of cone-based and degree-theoretic methods supports the derivation of multiple positive solutions under different nonlinear growth conditions. A key contribution of the work is the development of a generalized multiplicity framework capable of handling diverse NSBC settings. Unlike many existing studies that focus on a single class of boundary conditions, the approach presented here incorporates broad structural variations, enabling a more comprehensive understanding of how solution sets behave under complex boundary interactions. The presence of multiple solutions reflects the richness of nonlinear and nonlocal behavior, offering deeper insights into the underlying dynamics of physical and engineering systems that exhibit memory, distributed effects, or boundary feedback phenomena. The illustrative examples and numerical demonstrations further reinforce the theoretical findings, showing how parameter changes influence solution branches and bifurcation patterns. These examples provide a practical bridge between abstract theory and computational realization, highlighting the utility of the developed framework. Future extensions of this work may include applications to fractional differential equations of higher order, impulsive boundary value problems involving discontinuities, or complex systems of coupled differential equations. Such investigations would broaden the scope of applicability and deepen the understanding of nonlinear phenomena governed by non-standard boundary structures.

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