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NEW AGGREGATION OPERATORS OF A-CUTS WITH NEUTROSOPHIC Z-NUMBERS AND THEIR PROPERTIES

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Abstract

The proposed model is a novel Multi-Criteria Decision-Making (MCDM) approach that modifies the neutrosophic Z-numbers set using the α-cut technique, referred to as α-cut Neutrosophic Z-numbers (αNZN) model. This method generalizes different classical fuzzy set approaches. This study also introduces the fundamental explanations of union, intersection, and complement of the α-cut sets of neutrinos called z-numbers, and explores their properties associated with arithmetic operations. This work presents a prosecution of relevant examples to enhance the clarity and the comprehension of the subject. Additionally, two new aggregation operators, namely; the α-cut Neutrosophic Z-numbers Weighted Average (αNZN-WA) and α-cut Neutrosophic Z-numbers Weighted Geometric (αNZN-WG) are proposed using the α-cut method to incorporate uncertainty represented as interval numbers. Furthermore, a scoring scheme is also formulated to rank alternatives more effectively.

Keywords: α-cut, Neutrosophic Set Theory, Z-numbers, α-cut Neutrosophic Z-numbers Weighted Averages, α-cut Neutrosophic Z-numbers Weighted Geometric

Introduction

In recent years, a wide variety of systematic procedures and techniques have been proposed to address complex and non-linear issues, bridging the gaps and overcoming fuzziness, challenges, incompleteness, and discontinuity. Every new method attempts to overcome the deficiencies of the old ones by narrowing the gap between uncertainty modelling and efficient decision-making procedures [1].

Fuzzy set theory, which was developed by Zadeh [2], is one of such contributions in this regard as it has played a significant role in different scientific and engineering fields because of its capability of making interpretations of inaccurate data. It is on this basis that Zadeh later developed the concept of z-numbers that he represents as 2 fuzzy numbers capturing a value constraint and a reliability measure [3]. This binary is a representation of human operations under uncertainty the numbers have been applicable in different areas such as sensor fusion and MCDM [4].

The introduction of z-numbers gave rise to the concept of neutrosophic sets which was developed by Smarandash [6] as a more descriptive tool to solve uncertain and

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incomplete choices. Neutrosophic set builds upon the fuzzy method by introducing three different functions of membership: Truth (*T*), Indeterminacy (*I*) and falsity (*F*). Traditional neutrosophic sets have more descriptive abilities, but still does not consider as an expression of the representation of reliability [7]. To deal with this limitation in a better manner, the reasoning behind the concept of z-numbers was merged with that of the neutrosophic system thereby coming up with neutrosophic z-numbers. The principle of reliability increases in each dimension with the aid of these hybrid representations through fuzzy ordered pairs that contribute to the interpretation precision associated with the modelling of uncertainty [8]. In addition, use of α-cutting, an important technique in fuzzy logic, has been adopted, which can further be used to interpret this framework by making an analysis of fuzzy sets according to the degree of confidence [9]. The use of alpha levels further enhances this method since it helps to eliminate the less reliable information by making the data less ambiguous and it also helps to reduce the process of extracting meaning when dealing with MCDM scenarios.

An additional key element that enhances the utility of NZNs is the concept of α-cutting. α-cuts, a well-known technique in fuzzy logic, allow for the filtering of data according to a chosen level of confidence, helping decision-makers to disregard unreliable or less certain information. By applying α-cuts to fuzzy sets, the data can be sliced at different confidence levels, allowing for more focused and less ambiguous analysis. In decision-making, especially in multi-criteria decision-making (MCDM) problems, α-cutting becomes an essential technique for narrowing down the decision space to include only those alternatives that meet the desired confidence level, thus eliminating less relevant or unreliable information [9]. This process reduces ambiguity and sharpens the decision-making process, ensuring that only the most relevant data is used in the final analysis.

Building on these developments, this study proposes a novel approach called αNZNs (α-cut Neutrosophic Z-numbers). This new model combines the expressiveness and flexibility of neutrosophic z-numbers with the filtering capabilities of α-cuts, offering a powerful tool for handling uncertain, incomplete, and contradictory data in decision-making processes. By slicing through fuzzy membership functions at different α-cut levels, the αNZN model enables the separation of data based on its reliability, ensuring that only the most reliable and relevant information is considered. This dual focus on reliability and uncertainty allows for multi-layered, multi-perspective analysis of ambiguous data, leading to more accurate and credible decisions under conditions of uncertainty.

The operationalization of this framework involves the formal definition of αNZNs, including the introduction of relevant arithmetic operations and a score function for ranking alternatives. Additionally, we propose two aggregation operators: the αNZN-WA (Weighted Averaging) and the αNZN-WG (Weighted Geometric) operators, which are designed to facilitate effective data aggregation within the αNZN environment. These operators allow for the synthesis of multiple uncertain data sources into a single, coherent result, making them especially useful in MCDM scenarios. The framework is then demonstrated through a comprehensive case study, showcasing the practicality and effectiveness of the αNZN model in real-world decision-making applications.

This study proposes a new integrative construction called αNZNs. It is a fusion of the expressiveness of the neutrosophic z-numbers and the sifting property of α-cuts. The model can separate out the data that fits certain criteria in reliability by cutting across

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fuzzy membership functions at present alpha levels. This leads to dividing information into layers and generating multi-perspective analysis of ambiguous and contradictory data resulting in more accurate and credible decisions under uncertainty.

To operationalize this framework, we formally define the concept of αNZNs, along with the associated arithmetic operations and a score function to facilitate ranking. Two aggregation operators' αNZN-WA and αNZN-WG operators are introduced to enable effective data aggregation within the αNZN environment. These tools are then applied to develop a robust MCDM methodology, which is demonstrated through a comprehensive case study to illustrate both its practicality and effectiveness.

The key contributions of this study are summarized as follows

This study introduces several significant contributions to the field of multi-criteria decision-making (MCDM) within uncertain and imprecise environments. The key contributions can be summarized as follows:

(a) The αNZN model allows for simultaneous expression of truth, falsity, and indeterminacy values along with their respective reliability measures using three ordered fuzzy number pairs then filter the values using α-cut level. This feature enables effective modelling in ambiguous and inconsistent environments.

(b) The defined operations and αNZN-WA and αNZN-NWG operators of αNZN s are to realize the aggregation problem soft the αNZN information and then the score function of αNZN is to rank NZNs, which provide the use full mathematical tools for MCDM problems in αNZN setting.

(c) The developed MCDM approach not only enhances the MCDM reliability but also provides a new effective way for MCDM problems in αNZN setting.

The study is organized as the following structures: Section "α-cut Neutrosophic Z-number"set presents the notion of a αNZN set, operations of αNZNs, and a score function of αNZN and their properties. Weighted aggregation operators of αNZN propose the αNZN-WA and αNZN-WG operators, lastly, conclusion.

The study is organized into several key sections. The section titled "α-cut Neutrosophic Z-number Set" provides a detailed introduction to the concept of the αNZN set. It covers its mathematical formulation, defines important operations applied to αNZN elements, and introduces the proposed score function for ranking purposes. Additionally, this section discusses the theoretical properties and implications of the αNZN constructs.

Following this, the section on weighted aggregation operators explores the development of the αNZN-WA and αNZN-WG operators. It includes formal definitions, practical examples, and an analysis of how these operators can be applied to real-world decision-making problems, highlighting their utility in the aggregation of imprecise data.

Literature Review

Handling uncertainty has long been a central concern in multi-criteria decision-making (MCDM). Traditional methods rely on sets of fuzzy, [2] where elements have partial

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membership levels in [0,1]. A pivotal technique in fuzzy computation is the α-cut, defined as the crisp which facilitates computation by transforming fuzzy sets into interval-valued representations [2]. α-cuts underpin stability and tractability in fuzzy inference, defuzzification, and especially in type-2 fuzzy or hierarchical aggregation methods [9].

Although fuzzy sets and their generalizations such as intuitionistic fuzzy sets [11] (At have been widely applied, they face challenges when handling information that is not only imprecise but also inconsistent or indeterminate. To address this, neutrosophic sets (NSs) were proposed by [6], specifying three separate membership degrees: truth (T), indeterminacy (I), and falsity (F). This triadic representation allows explicit modelling of vague, conflicting, or incomplete information scenarios, not possible with sole truth membership [6]. Single-Valued Neutrosophic Sets (SVNSs) constrain each component to [0,1], enabling practical application via defined aggregation, ranking, and similarity operators [11]. Extended forms like interval or trapezoidal neutrosophic sets have also been employed in MCDM with weighted arithmetic/geometric operators, integrating α-cuts for computational simplicity [12]

An additional dimension of uncertainty arises from reliability or credibility of information, which classical fuzzy frameworks do not capture. [13], a pair (A, B), This construct decouples what is stated from how certain it is and has been adopted in fuzzy MCDM, control systems, and computing-with-words methodologies. Various computational strategies such as expected utility-based reduction or conversion to fuzzy numbers enable practical usage of z-numbers in decision frameworks [13].

More recently, researchers have fused the conceptual strengths of neutrosophic sets and Z-numbers into Neutrosophic Z-Numbers (NZNs). In NZNs, the standard neutrosophic triplet (T, I, F) is each paired with a reliability measure, thereby modeling value, indeterminacy, falsity, and reliability simultaneously. [14] formalized NZNs, introducing aggregation operators the including NZN Weighted Arithmetic Average (NZNWAA) and NZN Weighted Geometric Average (NZNWGA) together with score functions tailored for MCDM [15] These models have been successfully applied in business-partner selection and other illustrative scenarios [16]. Additional extensions include similarity metrics specific to NZNs [17] and trapezoidal NZNs in industrial robot evaluation, as well as interval-valued NZNs for

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software development project selection and NZN aggregation using Aczél–Alsina operators for supplier choice [17].

Applications of NZNs have reported in different aspects. For instance, [18]. applied NZNs to the selection of business-partner, while [19] conducted similarity measures for NZNs in pattern recognition tasks. Other extensions such as trapezoidal NZNs for industrial robot evaluation [20], interval-valued NZNs for software enhancement project selection [21], and NZN aggregation using Aczél–Alsina operators for supplier choice. These studies indicate the flexibility of the NZN model across different decision-making situations.

Uncertainty, has long been a central issue in multi-criteria decision-making (MCDM), especially when treating imprecise, inconsistent, or indeterminate information. The theory of Fuzzy set has traditionally been utilized to handle the uncertainty, where elements are assigned partial membership levels between 0 and 1. One of the key techniques in fuzzy logic is the α-cut, which transforms fuzzy sets into interval-valued representations, enhancing computational efficiency and supporting stability in fuzzy inference and defuzzification processes [2]. It has been the foundational in type-2 fuzzy systems and hierarchical aggregation methods, offering stability and tractability [9].

However, while fuzzy sets and their extensions such as intuitionistic fuzzy sets, have been widely applied, there occur different challenges when dealing with inconsistent or indeterminate data. To highlight the challenges faced, neutrosophic sets (NSs) were proposed, incorporating with the mentioned three membership degrees. This triadic representation allows for more explicit modeling of vague, conflicting, or incomplete information, providing a crucial advancement over traditional fuzzy sets. Single-Valued Neutrosophic Sets (SVNSs) were introduced to further enhance practical applications, where each element is constrained to the range [0,1]. This form facilitates aggregation, ranking, and similarity measures for MCDM tasks [11]. Further extensions such as interval or trapezoidal neutrosophic sets, have incorporated α-cuts for simplified computations [12].

A significant aspect often overlooked in traditional fuzzy frameworks or models, is the reliability or credibility of the information being modeled. The concept of z-numbers was introduced to address it. A z-number is a pair (A, B), where A represents the value and B shows its associated reliability. This construct decouples provide a more complete representation of uncertainty. Z-numbers have been utilized in fuzzy MCDM, control

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systems, and computing-with-words methodologies, with strategies such as expected utility-based reductions or conversion into fuzzy numbers for practical decision-making [13].

In recent years, researchers have merged the advantages of neutrosophic sets and z-numbers, which results to the development of Neutrosophic Z-Numbers (NZNs). NZNs extend the neutrosophic triplet by pairing with the mentioned three membership degrees in which these degrees with reliability measures, enabling the simultaneous modeling of value, indeterminacy, falsity, and reliability [14]. formalized NZNs, introducing aggregation operators like NZN Weighted Arithmetic Average (NZNWAA) and NZN Weighted Geometric Average (NZNWGA), along with score functions specifically designed for MCDM applications [15]. These models have been successfully applied in fields like business-partner selection and other decision-making scenarios [16].

Further extensions of NZNs include the development of similarity metrics specific to NZNs [17], trapezoidal NZNs for industrial robot evaluation [20], and interval-valued NZNs for applications like software development project selection [21] and supplier choice using NZN aggregation with Aczél–Alsina operators [17]. These extensions highlight the versatility and applicability of the NZN framework in a variety of decision-making environments, demonstrating its robustness in handling complex, uncertain, and contradictory data.

PRELIMINARIES

This section provides the related definitions that will be used in the development of the proposed set.

Neutrosophic Z-numbers Operator [13]

The general formally define the NZN set as follows

Definition 1. Let X be a universe set. Then an NZN set in a universe set X is defined as the following form:

$$S_Z = \{x, \langle (T_V(x), T_R(x)), (I_V(x), I_R(x)), (F_V(x), F_R(x)) \rangle \backslash x \in X\}$$

where $z = (V,R)$, V is the role of a neutrosophic restriction, R is neutrosophic measures of reliability for V, S_Z is the integrate of NZN, and the order pairs of truth T, indeterminacy I and falsity fuzzy values F, along with the conditions

$$0 \leq T_V(x) + I_V(x) + F_V(x) \leq 3 \text{ and } 0 \leq T_R(x) + I_R(x) + F_R(x) \leq 3.$$

α-cut Technique[10]

The α-cut technique is a foundational method in fuzzy set theory, employed to convert fuzzy sets into precise intervals, thereby streamlining computational procedures and

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analytical processes. Initially proposed by Lotfi A. Zadeh within the framework of classical fuzzy set theory, the α-cut approach facilitates the examination of membership functions at designated confidence or belief levels, represented by $\alpha \in [0,1]$. Formally, the α-cut of a fuzzy set A, denoted as A^{α}, the α-cut A^{α} is given by $A^{\alpha} = \{x \in X \mid \mu_A(x) \geq \alpha\}$, Let A triangular α-cut fuzzy set is defined as $A^{\alpha} = (a, b, c)$ the membership of the fuzzy numbers at α-cut level are defined as follows

$$\mu_A(x) = \begin{cases} \dfrac{(x_1 - a)}{(b - a)}, & a < x_1 < b \\ \dfrac{(c - x_2)}{(c - b)}, & b < x_2 < c \\ 0 \quad , & otherwise \end{cases}$$

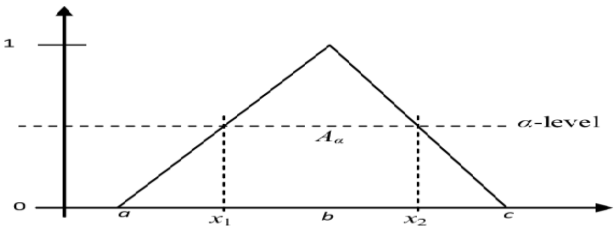

Figure 1. Fuzzy number and α- cut level

This transformation enables the abstraction of imprecise fuzzy data into interpretable interval-based formats, significantly enhancing the practical implementation of fuzzy systems in domains such as decision-making, optimization, and control applications [15].

The proposed α-cut Neutrosophic Z-numbers (αNZN) operator and its properties

This section shows the novel integration of the α-cut with NZN framework, which aims at improving the effectiveness and adaptability of MCDM processes. By leveraging the complementary strengths of the α-cut method known for its capacity to filter uncertain information based on confidence thresholds and the NZN model which encapsulates both uncertainty and reliability the proposed approach offers a robust and scalable decision-support mechanism. The α-cut of an NZN at a level α is defined as follows

$\check{S} = S_Z^{\alpha} = \{x, \langle (T_V^{\alpha}(x), T_R^{\alpha}(x)), (I_V^{\alpha}(x), I_R^{\alpha}(x)), (F_V^{\alpha}(x), F_R^{\alpha}(x)) \rangle \backslash x \in X\}$, where

$T_V^{\alpha}(x)$: Truth value at α-cut level, with $T_R^{\alpha}(x)$ as its associated reliability at α-level, $I_V^{\alpha}(x)$: Indeterminacy value at α-cut level, with $I_R^{\alpha}(x)$ as its associated reliability at α-level, $F_V^{\alpha}(x)$: Falsity value at α-cut level, with $F_R^{\alpha}(x)$ as its associated reliability at α-level. Each value component lies within [0,1] for a given $\alpha \in [0,1]$

$0 \leq T_V^{\alpha}(x) + I_V^{\alpha}(x) + F_V^{\alpha}(x) \leq 3$ and $0 \leq T_R^{\alpha}(x) + I_R^{\alpha}(x) + F_R^{\alpha}(x) \leq 3$

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Let A triangular α-cut neutrosophic z-number is defined as $A_V^{\alpha}(x) = (p_1, p_2, p_3,\ p_4, p_5, p_6, p_7)$ the truth membership, indeterminacy membership, and falsity membership of the neutrosophic set α-cut level are defined as follows

$$T_V^{\alpha}(x) = \begin{cases} \frac{(x-p_2)}{(p_4-p_2)} & , \quad p_2 < x < p_4 \\ 1 & , \quad x = p_4 \\ \frac{(p_6-x)}{(p_6-p_4)} & , p_4 < x < p_6 \\ 0 & , \quad otherwise \end{cases} \quad I_V^{\alpha}(x) = \begin{cases} \frac{(x-p_1)}{(p_3-p_1)}, & p_1 < x < p_3 \\ 0 \quad , & x = p_3 \\ \frac{(p_5-x)}{(p_5-p_3)}, & p_3 < x < p_5 \\ 1 \quad , & otherwise \end{cases} \quad F_V^{\alpha}(x) = \begin{cases} \frac{(x-p_3)}{(p_3-p_5)}, & p_3 < x < p_5 \\ 1 \quad , & x = p_5 \\ \frac{(p_7-x)}{(p_7-p_5)}, & p_5 < x < p_7 \\ 0 \quad , & otherwise \end{cases}$$

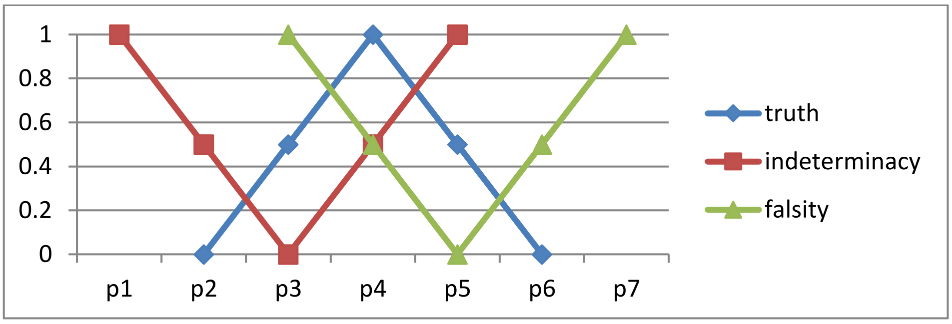

Figure 2: Triangular α-cut level neutrosophic z-numbers

Basic Operations of αNZN

In this subsection, we introduce some basic operations on αNZN, namely, complement, union and intersection, subset and equal, derive their properties, derive their properties and give some examples.

Definition 2. Let $\check{S} = \{x, \langle (T_V^{\alpha}(x), T_R^{\alpha}(x)), (I_V^{\alpha}(x), I_R^{\alpha}(x)), (F_V^{\alpha}(x), F_R^{\alpha}(x)) \rangle \backslash x \in X\}$

$\check{S}_1$=$\{x, \langle \left(T_{V_1}^{\alpha}(x), T_{R_1}^{\alpha}(x)\right), \left(I_{V_1}^{\alpha}(x), I_{R_1}^{\alpha}(x)\right), \left(F_{V_1}^{\alpha}(x), F_{R_1}^{\alpha}(x)\right) \rangle \backslash x \in X\}$ and

$\check{S}_2$= $\{x, \langle (T_{V_2}^{\alpha}(x), T_{R_2}^{\alpha}(x)), (I_{V_2}^{\alpha}(x), I_{R_2}^{\alpha}(x)), (F_{V_2}^{\alpha}(x), F_{R_2}^{\alpha}(x)) \rangle \backslash x \in X\}$, be three αNZN. Then,

The complement of $\check{S}$ is denoted by $(\check{S})^C$ and defined as follows:

$(\check{S})^C$=$\langle \cup_{T_V^{\alpha} \in \check{S}}\{1 - T_{(V,R)}^{\alpha}\}, \cup_{I_V^{\alpha} \in \check{S}}\{1 - I_{(V,R)}^{\alpha}\}, \cup_{F_V^{\alpha} \in \check{S}}\{1 - F_{(V,R)}^{\alpha}\} \rangle$

=$\langle (\cup_{T_V^{\alpha} \in \check{S}}\{1 - T_V^{\alpha}\}, \cup_{T_R^{\alpha} \in \check{S}}\{1 - T_R^{\alpha}\}), (\cup_{I_V^{\alpha} \in \check{S}}\{1 - I_V^{\alpha}\}, \cup_{I_R^{\alpha} \in \check{S}}\{1 - I_R^{\alpha}\}), (\cup_{F_V^{\alpha} \in \check{S}}\{1 - F_V^{\alpha}\}, \cup_{F_R^{\alpha} \in \check{S}}\{1 - F_R^{\alpha}\}) \rangle$

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i) Union

The union of tow αNZN is defined by

$\check{S}_1 \cup \check{S}_2 = \langle (\bigcup_{\substack{T^{\alpha}_{V_1}\in\check{S}_1,\\ T^{\alpha}_{V_2}\in\check{S}_2}} \max\{T^{\alpha}_{V_1}, T^{\alpha}_{V_2}\}, \bigcup_{\substack{T^{\alpha}_{R_1}\in\check{S}_1\\ T^{\alpha}_{R_2}\in\check{S}_2}} \max\{T^{\alpha}_{R_1}, T^{\alpha}_{R_2}\}), \langle (\bigcup_{\substack{I^{\alpha}_{V_1}\in\check{S}_1,\\ I^{\alpha}_{V_2}\in\check{S}_2}} \min\{I^{\alpha}_{V_1}, I^{\alpha}_{V_2}\},$
$\bigcup_{\substack{I^{\alpha}_{R_1}\in\check{S}_1\\ I^{\alpha}_{R_2}\in\check{S}_2}} \min\{I^{\alpha}_{R_1}, I^{\alpha}_{R_2}\}), \langle (\bigcup_{\substack{F^{\alpha}_{V_1}\in\check{S}_1,\\ F^{\alpha}_{V_2}\in\check{S}_2}} \min\{F^{\alpha}_{V_1}, F^{\alpha}_{V_2}\}, \bigcup_{\substack{F^{\alpha}_{R_1}\in\check{S}_1\\ F^{\alpha}_{R_2}\in\check{S}_2}} \min\{F^{\alpha}_{R_1}, F^{\alpha}_{R_2}\}) \rangle$

ii) Intersection

The intersection of tow αNZNis defined by

$\check{S}_1 \cap \check{S}_2 = \langle (\bigcup_{\substack{T^{\alpha}_{V_1}\in\check{S}_1,\\ T^{\alpha}_{V_2}\in\check{S}_2}} \min\{T^{\alpha}_{V_1}, T^{\alpha}_{V_2}\}, \bigcup_{\substack{T^{\alpha}_{R_1}\in\check{S}_1\\ T^{\alpha}_{R_2}\in\check{S}_2}} \min\{T^{\alpha}_{R_1}, T^{\alpha}_{R_2}\}), \langle (\bigcup_{\substack{I^{\alpha}_{V_1}\in\check{S}_1,\\ I^{\alpha}_{V_2}\in\check{S}_2}} \max\{I^{\alpha}_{V_1}, I^{\alpha}_{V_2}\},$
$\bigcup_{\substack{I^{\alpha}_{R_1}\in\check{S}_1\\ I^{\alpha}_{R_2}\in\check{S}_2}} \max\{I^{\alpha}_{R_1}, I^{\alpha}_{R_2}\}), \langle (\bigcup_{\substack{F^{\alpha}_{V_1}\in\check{S}_1,\\ F^{\alpha}_{V_2}\in\check{S}_2}} \max\{F^{\alpha}_{V_1}, F^{\alpha}_{V_2}\}, \bigcup_{\substack{F^{\alpha}_{R_1}\in\check{S}_1\\ F^{\alpha}_{R_2}\in\check{S}_2}} \max\{F^{\alpha}_{R_1}, F^{\alpha}_{R_2}\}) \rangle$

iii) Subset

The subset of tow αNZN is defined by

$\check{S}_1 \supseteq \check{S}_2 \Leftrightarrow T^{\alpha}_{V_1} \geq T^{\alpha}_{V_2}, T^{\alpha}_{R_1} \geq T^{\alpha}_{R}, I^{\alpha}_{V_1} \leq I^{\alpha}_{V_2}, I^{\alpha}_{R_1} \leq I^{\alpha}_{R_2}, F^{\alpha}_{V_1} \leq F^{\alpha}_{V_2}$, and $F^{\alpha}_{R_1} \leq F^{\alpha}_{R_2}$.

iv) Equal

The equal of tow αNZN is defined by

$\check{S}_1 = \check{S}_2 \Leftrightarrow \check{S}_1 \supseteq \check{S}_2$ and $\check{S}_2 \supseteq \check{S}_1$.

Example 1. Let $\check{S}_1$=⟨(0.8,0.9),(0.3,, 0.7),(0.2,0.8) ⟩and $\check{S}_2$=⟨(0.6,0.8),(0.4,0.7), (0.1,0.9)⟩. Are tow αNZN, then

i) Complement

$(\check{S}_1)^C$ = ⟨(0.2,0.1),(0.7,, 0.3),(0.8,0.2)⟩

ii) Union

$\check{S}_1 \cup \check{S}_2$ = ⟨(0.8, 0.9), (0.3, 0.7), (0.1, 0.8) ⟩

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iii) Intersection

$\check{S}_1 \cap \check{S}_2 = \langle$ ((0.6,0.8), (0.4,0.7), (0.2,0.8)) $\rangle$

Proposition 1. Given three αNZN, $\check{S}, \check{S}_1$ and $\check{S}_2$ then

1) $\check{S} \cup \check{S} = \check{S}$
2) $\check{S} \cap \check{S} = \check{S}$
3) $\check{S}_1 \cup \check{S}_2 = \check{S}_2 \cup \check{S}_1$
4) $\check{S}_1 \cap \check{S}_2 = \check{S}_2 \cap \check{S}_1$
5) $\check{S} \cup (\check{S}_1 \cap \check{S}_2) = (\check{S} \cup \check{S}_1) \cap (\check{S} \cup \check{S}_2)$
6) $\check{S} \cap (\check{S}_1 \cup \check{S}_2) = (\check{S} \cap \check{S}_1) \cup (\check{S} \cap \check{S}_2)$
7) $(\check{S})^C)^c = \check{S}$
8) $(\check{S})^C \cup (\check{S}_1)^C \cup \check{S}_2)^C) = (\check{S} \cap \check{S}_1 \cap \check{S}_2)^C$
9) $(\check{S})^C \cap (\check{S}_1)^C \cap (\check{S}_2)^C) = (\check{S} \cup \check{S}_1 \cup \check{S}_2)^C$

Proof 1): Using the set $\check{S}$ defined in Definition 2 and

Let $\check{S}_1 = \{x, \langle \left(T^{\alpha}_{V_1}(x), T^{\alpha}_{R_1}(x)\right), \left(I^{\alpha}_{V_1}(x), I^{\alpha}_{R_1}(x)\right), \left(F^{\alpha}_{V_1}(x), F^{\alpha}_{R_1}(x)\right)\rangle \backslash x \in X\}$ and

$\check{S}_2 = \{x, \langle \left(T^{\alpha}_{V_2}(x), T^{\alpha}_{R_2}(x)\right), \left(I^{\alpha}_{V_2}(x), I^{\alpha}_{R_2}(x)\right), \left(F^{\alpha}_{V_2}(x), F^{\alpha}_{R_2}(x)\right)\rangle \backslash x \in X\}$, be three αNZN Then,

$\check{S} \cup \check{S} = \langle (\cup_{T^{\alpha}_V \in \check{S}} \max\{T^{\alpha}_V, T^{\alpha}_V\}, \cup_{T^{\alpha}_R \in \check{S}} \max\{T^{\alpha}_R, T^{\alpha}_R\}), (\cup_{I^{\alpha}_V \in \check{S}} \min\{I^{\alpha}_V, I^{\alpha}_V\}, \cup_{I^{\alpha}_R \in \check{S}} \min\{I^{\alpha}_R, I^{\alpha}_R\}),$
$(\cup_{F^{\alpha}_V \in \check{S}} \min\{F^{\alpha}_V, F^{\alpha}_V\}, \cup_{F^{\alpha}_R \in \check{S}} \min\{F^{\alpha}_R, F^{\alpha}_R\}) \rangle$
$= \langle (\cup_{T^{\alpha}_V \in \check{S}} T^{\alpha}_V, \cup_{T^{\alpha}_R \in \check{S}} T^{\alpha}_R), (\cup_{I^{\alpha}_V \in \check{S}} I^{\alpha}_V, \cup_{I^{\alpha}_R \in \check{S}} I^{\alpha}_R), (\cup_{F^{\alpha}_V \in \check{S}} F^{\alpha}_V, \cup_{F^{\alpha}_R \in \check{S}} F^{\alpha}_R) \rangle$
$\Rightarrow \langle (T^{\alpha}_V, T^{\alpha}_R), (I^{\alpha}_V, I^{\alpha}_R), (F^{\alpha}_V, F^{\alpha}_R) \rangle = \check{S}$

Proof 2):Similar to the proof of (1).

Proof 3):): Using the set $\check{S}_1$, $\check{S}_2$ in previous proof

Then,

$\check{S}_1 \cup \check{S}_2 = \langle (\cup_{T^{\alpha}_{V_1} \in \check{S}_1, T^{\alpha}_{V_2} \in \check{S}_2} \max\left\{T^{\alpha}_{V_1}, T^{\alpha}_{V_2}\right\}, \cup_{T^{\alpha}_{R_1} \in \check{S}_1, T^{\alpha}_{R_2} \in \check{S}_2} \max\left\{T^{\alpha}_{R_1}, T^{\alpha}_{R_2}\right\}), (\cup_{I^{\alpha}_{V_1} \in \check{S}_1, I^{\alpha}_{V_2} \in \check{S}_2} \min\left\{I^{\alpha}_{V_1}, I^{\alpha}_{V_2}\right\},$
$\cup_{I^{\alpha}_{R_1} \in \check{S}_1, I^{\alpha}_{R_2} \in \check{S}_2} \min\left\{I^{\alpha}_{R_1}, I^{\alpha}_{R_2}\right\}), (\cup_{F^{\alpha}_{V_1} \in \check{S}_1, F^{\alpha}_{V_2} \in \check{S}_2} \min\left\{F^{\alpha}_{V_1}, F^{\alpha}_{V_2}\right\}, \cup_{F^{\alpha}_{R_1} \in \check{S}_1, F^{\alpha}_{R_2} \in \check{S}_2} \min\left\{F^{\alpha}_{R_1}, F^{\alpha}_{R_2}\right\})\rangle$

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$\check{S}_2 \cup$
$\check{S}_1 = \langle (\bigcup_{T^{\alpha}_{V_1}\in\check{S}_1, T^{\alpha}_{V_2}\in\check{S}_2} \max\{T^{\alpha}_{V_2}, T^{\alpha}_{V_1}\}, \bigcup_{T^{\alpha}_{R_1}\in\check{S}_1, T^{\alpha}_{R_2}\in\check{S}_2} \max\{T^{\alpha}_{R_2}, T^{\alpha}_{R_1}\}), (\bigcup_{I^{\alpha}_{V_1}\in\check{S}_1, I^{\alpha}_{V_2}\in\check{S}_2} \min\{I^{\alpha}_{V_2}, I^{\alpha}_{V_1}\},$
$\bigcup_{I^{\alpha}_{R_1}\in\check{S}_1, I^{\alpha}_{R_2}\in\check{S}_2} \min\{I^{\alpha}_{R_2}, I^{\alpha}_{R_1}\}), (\bigcup_{F^{\alpha}_{V_1}\in\check{S}_1, F^{\alpha}_{V_2}\in\check{S}_2} \min\{F^{\alpha}_{V_2}, F^{\alpha}_{V_1}\}, \bigcup_{F^{\alpha}_{R_1}\in\check{S}_1, F^{\alpha}_{R_2}\in\check{S}_2} \min\{F^{\alpha}_{R_2}, F^{\alpha}_{R_1}\}) \rangle$

$\therefore\ \check{S}_1 \cup \check{S}_2 = \check{S}_2 \cup \check{S}_1$

Proof 4):Similar to the proof of (3)

Proof 5): Let $x \in \check{S} \cup (\check{S}_1 \cap \check{S}_2)$.
If $x \in \check{S} \cup (\check{S}_1 \cap \check{S}_2)$, then x is either in $\check{S}$ or $(\check{S}_1 \cap \check{S}_2)$.
This means $x \in \check{S}$ or $x \in (\check{S}_1 \cap \check{S}_2)$.
If $x \in \check{S}$ or $\{x \in \check{S}_1$ and $x \in \check{S}_2\}$, then $\{x \in \check{S}$ or $x \in \check{S}_1\}$ and $\{x \in \check{S}$ or $x \in \check{S}_2\}$.
So we have, $x \in \check{S}$ or $\check{S}_1$ and $\check{S}_2$
$x \in (\check{S} \cup \check{S}_1)$ and $x \in (\check{S} \cup \check{S}_2)$
$x \in (\check{S} \cup \check{S}_1) \cap (\check{S} \cup \check{S}_2)$
Hence, $\check{S} \cup (\check{S}_1 \cap \check{S}_2) \Rightarrow (\check{S} \cup \check{S}_1) \cap (\check{S} \cup \check{S}_2)$
Therefore, $\check{S} \cup (\check{S}_1 \cap \check{S}_2) \subseteq (\check{S} \cup \check{S}_1) \cap (\check{S} \cup \check{S}_2)$
Let $x \in (\check{S} \cup \check{S}_1) \cap (\check{S} \cup \check{S}_2)$.
If $x \in (\check{S} \cup \check{S}_1) \cap (\check{S} \cup \check{S}_2)$, then x is in ($\check{S}$ or $\check{S}_1$) and ($\check{S}$ or $\check{S}_2$).
So we have, $x \in$ ($\check{S}$ or $\check{S}_1$) and $x \in$ ($\check{S}$ or $\check{S}_2$)
$\{x \in \check{S}$ or $x \in \check{S}_1\}$ and $\{x \in \check{S}$ or $x \in \check{S}_2\}$
$x \in \check{S}$or $\{x \in \check{S}_1$ and $x \in \check{S}_2\}$
$x \in \check{S} \cup \{x \in (\check{S}_1$ and $\check{S}_2)\}$
$x\, x \in \check{S} \cup \{x \in (\check{S}_1 \cap \check{S}_2)\}$
$x \in \check{S} \cup (\check{S}_1 \cap \check{S}_2)$
$\therefore\ \check{S} \cup (\check{S}_1 \cap \check{S}_2) = (\check{S} \cup \check{S}_1) \cap (\check{S} \cup \check{S}_2)$

Proof 6):Similar to the proof of (5).

Proof 7):Let $(\check{S})^C = \langle \bigcup_{T^{\alpha}_{V}\in\check{S}}\{1 - T^{\alpha}_{(V,R)}\}, \bigcup_{I^{\alpha}_{V}\in\check{S}}\{1 - I^{\alpha}_{(V,R)}\}, \bigcup_{F^{\alpha}_{V}\in\check{S}}\{1 - F^{\alpha}_{(V,R)}\} \rangle$ *We have,*
$((\check{S})^C)^c = \langle \bigcup_{T^{\alpha}_{V}, T^{\alpha}_{R}\in T^{\alpha}_{(V,R)}}\{1 - (1 - T^{\alpha}_{(V,R)})\}, \bigcup_{I^{\alpha}_{V}, I^{\alpha}_{R}\in I^{\alpha}_{(V,R)}}\{1 - (1 - I^{\alpha}_{(V,R)})\}, \bigcup_{F^{\alpha}_{V}, F^{\alpha}_{R}\in F^{\alpha}_{(V,R)}}\{1 - (1 - F^{\alpha}_{(V,R)})\} \rangle$
$= \langle (\bigcup_{T^{\alpha}_{V}, T^{\alpha}_{R}\in T^{\alpha}_{(V,R)}} T^{\alpha}_{(V,R)}), (\bigcup_{I^{\alpha}_{V}, I^{\alpha}_{R}\in I^{\alpha}_{(V,R)}} I^{\alpha}_{(V,R)}), (\bigcup_{F^{\alpha}_{V}, F^{\alpha}_{R}\in F^{\alpha}_{(V,R)}} F^{\alpha}_{(V,R)}) \rangle$
$\Rightarrow \langle (T^{\alpha}_{V_2}, T^{\alpha}_{R_2}), (I^{\alpha}_{V_2}, I^{\alpha}_{R_2}), (F^{\alpha}_{V_2}, F^{\alpha}_{R_2}) \rangle = \check{S}$
$\therefore ((\check{S})^C)^c = \check{S}$
Proof 8): Let $x \in (\check{S})^C \cup (\check{S}_1)^C \cup (\check{S}_2)^C)$
$\Rightarrow x \in (\check{S})^C \cup x \in (\check{S}_1)^C \cup x \in (\check{S}_2)^C$
$\Rightarrow x \notin \check{S} \cup x \notin \check{S}_1 \cup x \notin \check{S}_2$

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$\Rightarrow x \notin (\check{S} \cap \check{S}_1) \cap \check{S}_2$
$\Rightarrow x \notin \check{S} \cap \check{S}_1 \cap \check{S}_2$
$\Rightarrow x \in (\check{S} \cap \check{S}_1 \cap \check{S}_2)^C$
Since for all $x \in (\check{S})^C \cup \check{S}_1)^C \cup (\check{S}_2)^C$ such that $x \in (\check{S} \cap \check{S}_1 \cap \check{S}_2)^C$
$((\check{S})^C \cup (\check{S}_1)^C \cup (\check{S}_2)^C) = (\check{S} \cap \check{S}_1 \cap \check{S}_2)^C$

Proof 9):Similar to the proof of (8).

Arithmetic Operations for αNZN

In this subsection, we introduce some basic operations on αNZN, namely addition, multiplication, scalar multiplication and power, derive their properties and give some examples.

Definition 3. Let $\check{S} = \{x, \langle (T_V^\alpha(x), T_R^\alpha(x)), (I_V^\alpha(x), I_R^\alpha(x)), (F_V^\alpha(x), F_R^\alpha(x)) \rangle \backslash x \in X\}$

$\check{S}_1 = \{x, \langle (T_{V_1}^\alpha(x), T_{R_1}^\alpha(x)), (I_{V_1}^\alpha(x), I_{R_1}^\alpha(x)), (F_{V_1}^\alpha(x), F_{R_1}^\alpha(x)) \rangle \backslash x \in X\}$, and

$\check{S}_2 = \{x, \langle (T_{V_2}^\alpha(x), T_{R_2}^\alpha(x)), (I_{V_2}^\alpha(x), I_{R_2}^\alpha(x)), (F_{V_2}^\alpha(x), F_{R_2}^\alpha(x)) \rangle \backslash x \in X\}$, be three αNZN

Then,

i) Addition

$$\check{S}_1 \oplus \check{S}_2 = \langle (T_{V_1}^\alpha + T_{V_2}^\alpha - T_{V_1}^\alpha T_{V_2}^\alpha, T_{R_1}^\alpha + T_{R_2}^\alpha - T_{R_1}^\alpha T_{R_2}^\alpha), (I_{V_1}^\alpha I_{V_2}^\alpha, I_{R_1}^\alpha I_{R_2}^\alpha), (F_{V_1}^\alpha F_{V_2}^\alpha, F_{R_1}^\alpha F_{R_2}^\alpha) \rangle \quad (1)$$

ii) Multiplication

$$\check{S}_1 \otimes \check{S}_2 = \langle (T_{V_1}^\alpha T_{V_2}^\alpha, T_{R_1}^\alpha T_{R_2}^\alpha), (I_{V_1}^\alpha + I_{V_2}^\alpha - I_{V_1}^\alpha I_{V_2}^\alpha, I_{R_1}^\alpha + I_{R_2}^\alpha - I_{R_1}^\alpha I_{R_2}^\alpha), (F_{V_1}^\alpha + F_{V_2}^\alpha - F_{V_1}^\alpha F_{V_2}^\alpha, F_{R_1}^\alpha + F_{R_2}^\alpha - F_{R_1}^\alpha F_{R_2}^\alpha) \rangle \quad (2)$$

Scalar Multiplication

$$\lambda\check{S} = \langle \left(1 - (1 - T_V^\alpha)^\lambda, 1 - (1 - T_R^\alpha)^\lambda\right), \left((I_V^\alpha)^\lambda, (I_R^\alpha)^\lambda\right), \left((F_V^\alpha)^\lambda, (F_R^\alpha)^\lambda\right) \rangle \quad (3)$$

where $\lambda > 0$

iii) Power

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$$(\check{S})^{\lambda} = \langle\left((T_V^{\alpha})^{\lambda}, (T_R^{\alpha})^{\lambda}\right), \left(1-(1-I_V^{\alpha})^{\lambda}, 1-(1-I_R^{\alpha})^{\lambda}\right), (1-(1-F_V^{\alpha})^{\lambda}, 1-(1-F_R^{\alpha})^{\lambda}\rangle \quad (4)$$

where $\lambda > 0$

Example 2. Consider Example 1 and given $\lambda = 2$, then.

i. Addition

$\check{S}_1 \oplus \check{S}_2$= ⟨ ((0.92,0.98), (0.12,0.49), (0.2,0.72)) ⟩

ii. Multiplication

$\check{S}_1 \otimes \check{S}_2$= ⟨((0.48, 0.72), (0.58,0.91), (0.28,0.98)) ⟩

iii. Scalar Multiplication, $2S_{Z_1}^{\alpha}$

$2\check{S}_1$= ⟨ ((0.96, 0.99), (0.09, 0.49), (0.04, 0.64)) ⟩

iv. Power, $(\check{S}_1)^2$

$(\check{S}_1)^2$ = ⟨((0.64, 0.81), (0.51, 0.91), (0.36, 0.96)) ⟩

Proposition 2. Given three αNZN, $\check{S}$, $\check{S}_1$ and $\check{S}_2$, with $\lambda = 2$ then we have

1) $((\check{S})^{C})^{\lambda} = (\lambda\, \check{S}_2)^{c}$
2) $\lambda(\check{S})^{c} = ((\check{S})^{\lambda})^{c}$
3) $(\check{S}_1 \oplus \check{S}_2)^{c} = (\check{S}_1)^{c} \otimes (\check{S}_2)^{c}$
4) $(\check{S}_1 \otimes \check{S}_2)^{c} = (\check{S}_1)^{c} \oplus (\check{S}_2)^{c}$
5) $\lambda(\check{S}_1 \otimes \check{S}_2) = \lambda(\check{S}_1)^{c} \oplus \lambda(\check{S}_2)^{c}$
6) $(\check{S}_1 \otimes \check{S}_2)^{\lambda} = (\check{S}_1)^{\lambda} \otimes (\check{S}_2)^{\lambda}$

Proof 1)

Using the set $\check{S}$, in previous definition 3, we have,

$(\check{S})^{C}$ =⟨ $(\cup_{T_V^{\alpha}\in\check{S}}\{1-T_V^{\alpha}\}, \cup_{T_R^{\alpha}\in\check{S}}\{1-T_R^{\alpha}\}), (\cup_{I_V^{\alpha}\in\check{S}}\{1-I_V^{\alpha}\}, \cup_{I_R^{\alpha}\in\check{S}}\{1-I_R^{\alpha}\}), (\cup_{F_V^{\alpha}\in\check{S}}\{1-F_V^{\alpha}\}, \cup_{F_R^{\alpha}\in\check{S}}\{1-F_R^{\alpha}\})$⟩

$((\check{S})^{C})^{\lambda}$= ⟨$(\cup_{T_V^{\alpha}\in\check{S}}\{(1-T_V^{\alpha})^{\lambda}\}, \cup_{T_R^{\alpha}\in\check{S}}\{(1-T_R^{\alpha})^{\lambda}\}), (\cup_{I_V^{\alpha}\in\check{S}}\{1-(1-I_V^{\alpha})^{\lambda}\}, \cup_{I_R^{\alpha}\in\check{S}}\{1-(1-I_R^{\alpha})^{\lambda}\}), (\cup_{F_V^{\alpha}\in\check{S}}\{1-(1-F_V^{\alpha})^{\lambda}\}, \cup_{F_R^{\alpha}\in\check{S}}\{1-(1-F_R^{\alpha})^{\lambda}\})$⟩

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$\lambda\check{S}= \langle\ \big(1-(1-T_V^\alpha)^\lambda, 1-(1-T_R^\alpha)^\lambda\big), \big((I_V^\alpha)^\lambda, (I_R^\alpha)^\lambda\big), \big((F_V^\alpha)^\lambda, (F_R^\alpha)^\lambda\big)\ \rangle$

$(\lambda\check{S})^C = \langle\ (\cup_{T_V^\alpha\in\check{S}}\{(1-T_V^\alpha)^\lambda\}, \cup_{T_R^\alpha\in\check{S}}\{(1-T_R^\alpha)^\lambda\}), (\cup_{I_V^\alpha\in\check{S}}\{1-(1-I_V^\alpha)^\lambda\}, \cup_{I_R^\alpha\in\check{S}}\{1-(1-I_R^\alpha)^\lambda\}), (\cup_{F_V^\alpha\in\check{S}}\{1-(1-F_V^\alpha)^\lambda\}, \cup_{F_R^\alpha\in\check{S}}\{1-(1-F_R^\alpha)^\lambda\})\rangle$

$\therefore ((\check{S})^C)^\lambda = (\lambda\check{S})^c$

Proof 2): Similar to the proof of (1).

Proof 3) Let $(\check{S}_1 \oplus \check{S}_2) = \langle (T_{V_1}^\alpha + T_{V_2}^\alpha - T_{V_1}^\alpha T_{V_2}^\alpha, T_{R_1}^\alpha + T_{R_2}^\alpha - T_{R_1}^\alpha T_{R_2}^\alpha), (I_{V_1}^\alpha I_{V_2}^\alpha, I_{R_1}^\alpha I_{R_2}^\alpha), (F_{V_1}^\alpha F_{V_2}^\alpha, F_{R_1}^\alpha F_{R_2}^\alpha)\ \rangle$

$(\check{S}_1 \oplus \check{S}_2)^c = \langle (1-(T_{V_1}^\alpha + T_{V_2}^\alpha - T_{V_1}^\alpha T_{V_2}^\alpha), (1-(T_{R_1}^\alpha + T_{R_2}^\alpha - T_{R_1}^\alpha T_{R_2}^\alpha)), (1-I_{V_1}^\alpha I_{V_2}^\alpha), (1-I_{R_1}^\alpha I_{R_2}^\alpha), (1-F_{V_1}^\alpha F_{V_2}^\alpha, 1-F_{R_1}^\alpha F_{R_2}^\alpha)\ \rangle$

$(\check{S}_1)^C = \langle\ ((1-T_{V_1}^\alpha), (1-T_{R_1}^\alpha)), ((1-I_{V_1}^\alpha), (1-I_{R_1}^\alpha)), ((1-F_{V_1}^\alpha), (1-F_{R_1}^\alpha))\rangle$

$(\check{S}_2)^C = \langle\ ((1-T_{V_2}^\alpha), (1-T_{R_2}^\alpha)), ((1-I_{V_2}^\alpha), (1-I_{R_2}^\alpha)), ((1-F_{V_2}^\alpha), (1-F_{R_2}^\alpha))\rangle$

$(\check{S}_1)^C \otimes (\check{S}_2)^C = \langle ((1-T_{V_1}^\alpha)(1-T_{V_2}^\alpha), (1-T_{R_1}^\alpha)(1-T_{R_2}^\alpha)), ((1-I_{V_1}^\alpha)+(1-I_{V_2}^\alpha)-(1-I_{V_1}^\alpha)(1-I_{V_2}^\alpha)), ((1-I_{R_1}^\alpha)+(1-I_{R_2}^\alpha)-(1-I_{R_1}^\alpha)(1-I_{R_2}^\alpha)), \quad ((1-F_{V_1}^\alpha)+(1-F_{V_2}^\alpha)-(1-F_{V_1}^\alpha)(1-F_{V_2}^\alpha)), ((1-F_{R_1}^\alpha)+(1-F_{R_2}^\alpha)-(1-F_{R_1}^\alpha)(1-F_{R_2}^\alpha))\ \rangle$

$(\check{S}_1)^C \otimes (\check{S}_2)^C = \langle (1-(T_{V_1}^\alpha + T_{V_2}^\alpha - T_{V_1}^\alpha T_{V_2}^\alpha), (1-(T_{R_1}^\alpha + T_{R_2}^\alpha - T_{R_1}^\alpha T_{R_2}^\alpha)), ((1-I_{V_1}^\alpha I_{V_2}^\alpha),$

$(1-I_{R_1}^\alpha I_{R_2}^\alpha)), ((1-F_{V_1}^\alpha F_{V_2}^\alpha, 1-F_{R_1}^\alpha F_{R_2}^\alpha)\rangle$

$\therefore\ (\check{S}_1 \oplus S_{Z_2}^\alpha)^c = (\check{S}_1)^c\ \otimes (\check{S}_2)^c$

Proof 4), 5) and 6): Similar to the proof of (3).

Score function, weighted averaging and weighted geometricoperator for αNZN

Score function for αNZN [13]

The ranking method is essential in the decision making procedure. Hence, we give the definition of score function for αNZN as follows

Definition 4. Using the set $\check{S}$ in previous definition be αNZN, then the score function $Y_{(\check{S}_i)}$ is defined follows:

$$Y_{(\check{S}_i)} = \frac{\left(4 + T_{V_i}^\alpha + T_{R_i}^\alpha - I_{V_i}^\alpha - I_{R_i}^\alpha - F_{V_i}^\alpha - F_{R_i}^\alpha\right)}{6}, \text{ for } Y_{(\check{S}_i)} \in [0, 1] \qquad (5)$$

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Respectively, if $Y_{(\check{S}_1)} > Y_{(\check{S}_2)}$, then $\check{S}_1$ is bigger than $\check{S}_2$, represented by $\check{S}_1 > \check{S}_2$

Example 3. Let $\check{S}_1$and $\check{S}_2$ be two set of αNZN as $\check{S}_1$= ⟨(0.6,0.8),(0.3,, 0.9),(0.2,0.8)⟩, $\check{S}_2$= ⟨(0.4,0.7),(0.5,0.8),(0.1,0.9)⟩, then, their ranking is given as follows:

$$Y_{(\check{S}_1)}=\frac{(4 + 0.6+0.8 - 0.3-0.9-0.2-0.8)}{6} = 0.53, \quad Y_{(\check{S}_2)} =\frac{(4 + 0.4+0.7-0.5-0.8-0.1-0.9)}{6} = 0.47$$

Since $Y_{(\check{S}_1)} > Y_{(\check{S}_2)}$, their ranking is $\check{S}_1 > \check{S}_2$.

Definition 5. Let $\check{S}_j = \{x,\langle\left(T^{\alpha}_{V_j}(x), T^{\alpha}_{R_j}(x)\right), \left(I^{\alpha}_{V_j}(x), I^{\alpha}_{R_j}(x)\right), \left(F^{\alpha}_{V_j}(x), F^{\alpha}_{R_j}(x)\right)\rangle \backslash x \in X\}$ for $j = 1,2,\ldots,n$ be αNZN. A mapping αNZN-WA is called α-cut neutrosophic z-numbers weighted average operator if it satisfies

$$\alpha NZN\text{-}WA\ (\check{S}_1,\check{S}_2,\ldots,\check{S}_n) = \sum_{j=1}^{n} \omega_j \check{S}_i$$

$$=\langle \left(\cup_{T^{\alpha}_{V_j}\in\check{S}_j}\left\{1-\prod_{j=1}^{n}\left(1-T^{\alpha}_{V_j}\right)^{\omega_j}\right\}, \cup_{T^{\alpha}_{R_j}\in\check{S}_j}\left\{1-\prod_{j=1}^{n}\left(1-T^{\alpha}_{R_j}\right)^{\omega_j}\right\}\right), \left(\cup_{I^{\alpha}_{V_j}\in\check{S}_j}\left\{\prod_{j=1}^{n}\left(I^{\alpha}_{V_j}\right)^{\omega_j}\right\}, \cup_{I^{\alpha}_{R_j}\in\check{S}_j}\left\{\prod_{j=1}^{n}\left(I^{\alpha}_{R_j}\right)^{\omega_j}\right\}\right), \left(\cup_{F^{\alpha}_{V_j}\in\check{S}_j}\left\{\prod_{j=1}^{n}\left(F^{\alpha}_{V_j}\right)^{\omega_j}\right\}, \cup_{F^{\alpha}_{R_j}\in\check{S}_j}\left\{\prod_{j=1}^{n}\left(F^{\alpha}_{R_j}\right)^{\omega_j}\right\}\right)\rangle \quad (6)$$

where ω_j is the weight of the $\check{S}_j$ ($j = 1, 2,\ldots,n$) for $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Theorem 1. Using the set $\check{S}_j$ in previous definition 5

Then,

i) Idempotency

If $\check{S}_j = \check{S}$ for all $j = 1, 2,\ldots,n$,then $\alpha NZN\text{-}WA\ (\check{S}_1,\check{S}_2,\ldots,\check{S}_n) = \check{S}$

ii) Monotonicity

If $\check{S}_j \leq \check{S}_j^{\,*}$ for all $j = 1, 2,\ldots,n$,then

$\alpha NZN\text{-}WA\ (\check{S}_1,\check{S}_2,\ldots,\check{S}_n) \leq \alpha NZN\text{-}WA^{*}\ (\check{S}_1^{\,*},\check{S}_2^{\,*},\ldots,\check{S}_n^{\,*})$

iii) Boundedness

$\min_{j = 1,2,\ldots,n} \{\check{S}_j\} \leq \alpha NZN\text{-}WA\ (\check{S}_1,\check{S}_2,\ldots,\check{S}_n) \leq \max_{j = 1,2,\ldots,n} \{\check{S}_j\}$

Proof i) (Idempotency):

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Using the set $\check{S}_j$ in previous definition 5, we have

$\alpha NZN\text{-}WA = \sum_{j=1}^{n} \omega_j \check{S}_j = \langle\ (\cup_{T_V^\alpha \in \check{S}}\{1-\prod_{J=1}^{n}(1-T_V^\alpha)^{\omega_j}\},\ \cup_{T_R^\alpha \in \check{S}}\{1-\prod_{J=1}^{n}(1-T_R^\alpha)^{\omega_j}\}),$ $(\cup_{I_V^\alpha \in \check{S}}\{\prod_{J=1}^{n}(I_V^\alpha)^{\omega_j}\},\ \cup_{I_R^\alpha \in \check{S}}\{\prod_{J=1}^{n}(I_R^\alpha)^{\omega_j}\}),\ (\cup_{F_V^\alpha \in \check{S}}\{\prod_{J=1}^{n}(F_V^\alpha)^{\omega_j}\},\ \cup_{F_R^\alpha \in S_Z^\alpha}\{\prod_{J=1}^{n}(F_R^\alpha)^{\omega_j}\})\ \rangle$

$\Rightarrow \langle\ (\cup_{T_V^\alpha \in \check{S}}\{1-\prod_{J=1}^{n}(1-T_V^\alpha)^{\sum_{J=1}^{n}\omega_j}\},\ \cup_{T_R^\alpha \in S_Z^\alpha}\{1-\prod_{J=1}^{n}(1-T_R^\alpha)^{\sum_{J=1}^{n}\omega_j}\}),$ $(\cup_{I_V^\alpha \in \check{S}}\{\prod_{J=1}^{n}(I_V^\alpha)^{\sum_{J=1}^{n}\omega_j}\},\ \cup_{I_R^\alpha \in \check{S}}\{\prod_{J=1}^{n}(I_R^\alpha)^{\sum_{J=1}^{n}\omega_j}\}),\ (\cup_{F_V^\alpha \in \check{S}}\{\prod_{J=1}^{n}(F_V^\alpha)^{\sum_{J=1}^{n}\omega_j}\},$ $\cup_{F_R^\alpha \in \check{S}}\{\prod_{J=1}^{n}(F_R^\alpha)^{\sum_{J=1}^{n}\omega_j}\}\ )\rangle$

Since $\sum_{J=1}^{n}\omega_j$, we have

$\Rightarrow \langle\ (\cup_{T_V^\alpha \in \check{S}}\{1-(1-T_V^\alpha)\},\ \cup_{T_R^\alpha \in \check{S}}\{1-(1-T_R^\alpha)\}),\ (\cup_{I_V^\alpha \in \check{S}}\{(I_V^\alpha)\},\ \cup_{I_R^\alpha \in \check{S}}\{(I_R^\alpha)\}),$ $(\cup_{F_V^\alpha \in \check{S}}\{(F_V^\alpha)\},\ \cup_{F_R^\alpha \in \check{S}}\{(F_R^\alpha)\})\ \rangle$

$\Rightarrow \langle\ (\cup_{T_V^\alpha \in \check{S}}\{T_V^\alpha\},\ \cup_{T_R^\alpha \in \check{S}}\{T_R^\alpha\}),\ (\cup_{I_V^\alpha \in \check{S}}\{I_V^\alpha\},\ \cup_{I_R^\alpha \in \check{S}}\{I_R^\alpha\}),\ (\cup_{F_V^\alpha \in \check{S}}\{F_V^\alpha\},\ \cup_{F_R^\alpha \in \check{S}}\{F_R^\alpha\})\ \rangle$

$\Rightarrow \langle (T_V^\alpha(x), T_R^\alpha(x)), (I_V^\alpha(x), I_R^\alpha(x)), (F_V^\alpha(x), F_R^\alpha(x))\rangle = \check{S}$

which compleates the proof of Proposition (3. i).

Proof ii) (monotonicity):

Since $\check{S}_j \le \check{S}_j^*$ for all $j = 1, 2,..,n$,then we have

$T_{V_j}^{\alpha} \le T_{V_j}^{\alpha\ *}$, $1 - T_{V_j}^{\alpha} \ge 1 - T_{V_j}^{\alpha\ *}$

$\Rightarrow \prod_{J=1}^{n}\left(1-T_{V_j}^{\alpha}\right)^{\omega_j} \ge \prod_{J=1}^{n}\left(1-T_{V_j}^{\alpha\ *}\right)^{\omega_j}$

$\Rightarrow 1-\prod_{J=1}^{n}\left(1-T_{V_j}^{\alpha}\right)^{\omega_j} \le 1-\prod_{J=1}^{n}\left(1-T_{V_j}^{\alpha\ *}\right)^{\omega_j}$

$\Rightarrow \cup_{T_V^\alpha \in \check{S}}\left\{1-\prod_{J=1}^{n}\left(1-T_{V_j}^{\alpha}\right)^{\omega_j}\right\} \le \cup_{T_V^\alpha \in \check{S}}\left\{1-\prod_{J=1}^{n}\left(1-T_{V_j}^{\alpha\ *}\right)^{\omega_j}\right\}$

Since $\check{S}_j \le \check{S}_j^*$ for all $j = 1, 2,. . .,n$,then we have

$\Rightarrow \prod_{J=1}^{n}\left(I_{V_j}^{\alpha}\right)^{\omega_j} \le \prod_{J=1}^{n}\left(I_{V_j}^{\alpha\ *}\right)^{\omega_j}$

$\Rightarrow \cup_{I_V^\alpha \in \check{S}}\left\{\prod_{J=1}^{n}\left(I_{V_j}^{\alpha}\right)^{\omega_j}\right\} \le \cup_{I_V^\alpha \in \check{S}}\left\{\prod_{J=1}^{n}\left(I_{V_j}^{\alpha\ *}\right)^{\omega_j}\right\}$

Since $\check{S}_j \le \check{S}_j^*$ for all $j = 1, 2,. . .,n$,then we have

$\Rightarrow \prod_{J=1}^{n}\left(F_{V_j}^{\alpha}\right)^{\omega_j} \le \prod_{J=1}^{n}\left(F_{V_j}^{\alpha\ *}\right)^{\omega_j}$

$\Rightarrow \cup_{F_V^\alpha \in S_Z^\alpha}\left\{\prod_{J=1}^{n}\left(F_{V_j}^{\alpha}\right)^{\omega_j}\right\} \le \cup_{F_V^\alpha \in S_Z^\alpha}\left\{\prod_{J=1}^{n}\left(F_{V_j}^{\alpha\ *}\right)^{\omega_j}\right\}$

Since $\check{S}_j \le \check{S}_j^*$ for all $j = 1, 2,..,n$,then we have

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$T^{\alpha}_{R_j} \leq {T^{\alpha}_{R_j}}^{*}$, $1 - T^{\alpha}_{R_j} \geq 1 - {T^{\alpha}_{R_j}}^{*}$

$\Rightarrow \prod_{J=1}^{n} \left(1 - T^{\alpha}_{R_j}\right)^{\omega_j} \geq \prod_{J=1}^{n} \left(1 - {T^{\alpha}_{R_j}}^{*}\right)^{\omega_j}$

$\Rightarrow\ 1 - \prod_{J=1}^{n} \left(1 - T^{\alpha}_{R_j}\right)^{\omega_j} \leq 1 - \prod_{J=1}^{n} \left(1 - {T^{\alpha}_{R_j}}^{*}\right)^{\omega_j}$

$\Rightarrow \cup_{T^{\alpha}_{R} \in \check{S}} \left\{1 - \prod_{J=1}^{n} \left(1 - T^{\alpha}_{R_j}\right)^{\omega_j}\right\} \leq \cup_{T^{\alpha}_{R} \in \check{S}} \left\{1 - \prod_{J=1}^{n} \left(1 - {T^{\alpha}_{R_j}}^{*}\right)^{\omega_j}\right\}$

Since $\check{S}_j \leq {\check{S}_j}^{*}$ for all $j = 1, 2,. . .,n$,then we have

$\Rightarrow \prod_{J=1}^{n} \left(I^{\alpha}_{R_j}\right)^{\omega_j} \leq \prod_{J=1}^{n} \left({I^{\alpha}_{R_j}}^{*}\right)^{\omega_j}$

$\Rightarrow\ \cup_{I^{\alpha}_{R} \in \check{S}} \left\{\prod_{J=1}^{n} \left(I^{\alpha}_{R_j}\right)^{\omega_j}\right\} \leq \cup_{I^{\alpha}_{R} \in \check{S}} \left\{\prod_{J=1}^{n} \left({I^{\alpha}_{R_j}}^{*}\right)^{\omega_j}\right\}$

Since $\check{S}_j \leq {\check{S}_j}^{*}$ for all $j = 1, 2,. . .,n$,then we have

$\Rightarrow \prod_{J=1}^{n} \left(F^{\alpha}_{R_j}\right)^{\omega_j} \leq \prod_{J=1}^{n} \left({F^{\alpha}_{R_j}}^{*}\right)^{\omega_j}$

$\Rightarrow\ \cup_{F^{\alpha}_{R} \in \check{S}} \left\{\prod_{J=1}^{n} \left(F^{\alpha}_{R_j}\right)^{\omega_j}\right\} \leq \cup_{F^{\alpha}_{R} \in \check{S}} \left\{\prod_{J=1}^{n} \left({F^{\alpha}_{R_j}}^{*}\right)^{\omega_j}\right\}$

Based on the above analaysis, we have

αNZN-WA $(\check{S}_1,\check{S}_2,..., \check{S}_n) \leq$ *αNZN-WA** $({\check{S}_1}^{*},{\check{S}_2}^{*},..., {\check{S}_n}^{*})$

which compleates the proof of Proposition (3 ii).

Proof iii)(Boundedness): Similar to the proof of Proposition (3. ii).

Definition 6. Using the set $\check{S}_j$ in previous definition 5, we have for

$j = 1,2,\ldots,n$ be αNZN. A mapping αNZN-WG is called α-cut neutrosophic z-numbers weighted geometric operator if it satisfies

αNZN-WG $(\check{S}_1,\check{S}_2,..., \check{S}_n) = \prod_{j=1}^{m} (\check{S}_j)^{\omega_j} = \langle (\cup_{T^{\alpha}_{V_j} \in \check{S}_j} \left\{\prod_{J=1}^{n} \left(T^{\alpha}_{V_j}\right)^{\omega_j}\right\},$ $\cup_{T^{\alpha}_{R_j} \in \check{S}_j} \left\{\prod_{J=1}^{n} \left(T^{\alpha}_{R_j}\right)^{\omega_j}\right\}),$ $(\cup_{I^{\alpha}_{V_j} \in \check{S}_j} \left\{1 - \prod_{J=1}^{n} \left(I^{\alpha}_{V_j}\right)^{\omega_j}\right\},$ $\cup_{I^{\alpha}_{R_j} \in \check{S}_j} \left\{1 - \prod_{J=1}^{n} \left(I^{\alpha}_{R_j}\right)^{\omega_j}\right\}),$ $(\cup_{F^{\alpha}_{V_j} \in \check{S}_j} \left\{1 - \prod_{J=1}^{n} \left(F^{\alpha}_{V_j}\right)^{\omega_j}\right\},$ $\cup_{F^{\alpha}_{R_j} \in \check{S}_j} \left\{1 - \prod_{J=1}^{n} \left(F^{\alpha}_{R_j}\right)^{\omega_j}\right\}) \rangle$ (7)

where ω_j is the weight of the $\check{S}_j$ ($j = 1, 2,. . .,n$) for $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Theorem 2: Using the set $\check{S}_j$ in previous definition 5, we have

i) Idempotency

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If $\check{S}_j = \check{S}$ for all $j = 1, 2, . . ., n$,then $\alpha NZN\text{-}WG$ $(\check{S}_1, \check{S}_2, ..., \check{S}_n) = \check{S}$

ii) Monotonicity

If $\check{S}_j \le \check{S}_j^{\,*}$ for all $j = 1, 2, . . ., n$,then

$\alpha NZN\text{-}WG$ $(\check{S}_1, \check{S}_2, ..., \check{S}_n) \le \alpha NZN\text{-}WG^*(\check{S}_1^{\,*}, \check{S}_2^{\,*}, ..., \check{S}_n^{\,*})$

iii) Boundedness

$\min_{j=1,2,\dots,n} \{\check{S}_j\} \le \alpha NZN\text{-}WG\ (\check{S}_1, \check{S}_2, ..., \check{S}_n) \le \max_{j=1,2,\dots,n} \{\check{S}_j\}$

Proof i) (Idempotency):

Since $\check{S}_j = \check{S} = \{x, \langle (T^{\alpha}_{V}(x), T^{\alpha}_{R}(x)), (I^{\alpha}_{V}(x), I^{\alpha}_{R}(x)), (F^{\alpha}_{V}(x), F^{\alpha}_{R}(x)) \rangle \backslash x \in X\}$, we have

$\alpha NZN\text{-}WG = \prod_{j=1}^{m} (\check{S}_j)^{\omega_j} = \langle\ (\cup_{T^{\alpha}_{V} \in \check{S}} \{\prod_{J=1}^{n} (T^{\alpha}_{V})^{\omega_j}\},\ \cup_{T^{\alpha}_{R} \in \check{S}} \{\prod_{J=1}^{n} (T^{\alpha}_{R})^{\omega_j}\}),\ (\cup_{I^{\alpha}_{V} \in \check{S}} \{1 - \prod_{J=1}^{n} (1 - I^{\alpha}_{V})^{\omega_j}\},\ \cup_{I^{\alpha}_{R} \in \check{S}} \{1 - \prod_{J=1}^{n} (1 - I^{\alpha}_{R})^{\omega_j}\}),\ (\cup_{F^{\alpha}_{V} \in \check{S}} \{1 - \prod_{J=1}^{n} (1 - F^{\alpha}_{V})^{\omega_j}\},\ \cup_{F^{\alpha}_{R} \in \check{S}} \{1 - \prod_{J=1}^{n} (1 - F^{\alpha}_{R})^{\omega_j}\})\ \rangle$

$\Rightarrow \langle\ (\cup_{T^{\alpha}_{V} \in \check{S}} \left\{\prod_{J=1}^{n} (T^{\alpha}_{V})^{\sum_{J=1}^{n} \omega_j}\right\},\ \cup_{T^{\alpha}_{R} \in \check{S}} \left\{\prod_{J=1}^{n} (T^{\alpha}_{R})^{\sum_{J=1}^{n} \omega_j}\right\}),\ (\cup_{I^{\alpha}_{V} \in \check{S}} \left\{1 - \prod_{J=1}^{n} (1 - I^{\alpha}_{V})^{\sum_{J=1}^{n} \omega_j}\right\},\ \cup_{I^{\alpha}_{R} \in \check{S}} \left\{1 - \prod_{J=1}^{n} (1 - I^{\alpha}_{R})^{\sum_{J=1}^{n} \omega_j}\right\}),\ (\cup_{F^{\alpha}_{V} \in \check{S}} \left\{1 - \prod_{J=1}^{n} (1 - F^{\alpha}_{V})^{\sum_{J=1}^{n} \omega_j}\right\},\ \cup_{F^{\alpha}_{R} \in \check{S}} \left\{1 - \prod_{J=1}^{n} (1 - F^{\alpha}_{R})^{\sum_{J=1}^{n} \omega_j}\right\})\rangle$

Since $\sum_{J=1}^{n} \omega_j$, we have

$\Rightarrow \langle\ (\cup_{T^{\alpha}_{V} \in \check{S}} \{(T^{\alpha}_{V})\},\ \cup_{T^{\alpha}_{R} \in S^{\alpha}_{Z}} \{(T^{\alpha}_{R})\}),\ (\cup_{I^{\alpha}_{V} \in \check{S}} \{1 - (1 - I^{\alpha}_{V})\},\ \cup_{I^{\alpha}_{R} \in \check{S}} \{1 - (1 - I^{\alpha}_{R})\}),\ (\cup_{F^{\alpha}_{V} \in \check{S}} \{1 - (1 - F^{\alpha}_{V})\},\ \cup_{F^{\alpha}_{R} \in \check{S}} \{1 - (1 - F^{\alpha}_{R})\})\ \rangle$

$\Rightarrow \langle\ (\cup_{T^{\alpha}_{V} \in \check{S}} \{T^{\alpha}_{V}\},\ \cup_{T^{\alpha}_{R} \in S^{\alpha}_{Z}} \{T^{\alpha}_{R}\}),\ (\cup_{I^{\alpha}_{V} \in \check{S}} \{I^{\alpha}_{V}\},\ \cup_{I^{\alpha}_{R} \in \check{S}} \{I^{\alpha}_{R}\}),\ (\cup_{F^{\alpha}_{V} \in \check{S}} \{F^{\alpha}_{V}\},\ \cup_{F^{\alpha}_{R} \in \check{S}} \{F^{\alpha}_{R}\})\ \rangle$

$\Rightarrow \langle (T^{\alpha}_{V}(x), T^{\alpha}_{R}(x)), (I^{\alpha}_{V}(x), I^{\alpha}_{R}(x)), (F^{\alpha}_{V}(x), F^{\alpha}_{R}(x)) \rangle = \check{S}$

which compleates the proof of Proposition (4. i).

Proof ii) (monotonicity):

Since $\check{S}_j \le \check{S}_j^{\,*}$ for all $j = 1, 2, .., n$,then we have

$T^{\alpha}_{V_j} \le T^{\alpha}_{V_j}{}^{*}$

$\Rightarrow \prod_{J=1}^{n} \left(T^{\alpha}_{V_j}\right)^{\omega_j} \le \prod_{J=1}^{n} \left(T^{\alpha}_{V_j}{}^{*}\right)^{\omega_j}$

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$$\Rightarrow \bigcup_{T^{\alpha}_{V} \in \check{S}} \prod_{J=1}^{n} \left(T^{\alpha}_{V_j}\right)^{\omega_j} \bigcup_{T^{\alpha}_{V} \in \check{S}} \leq \prod_{J=1}^{n} \left(T^{\alpha}_{V_j}{}^{*}\right)^{\omega_j}$$

Since $\check{S}_j \leq \check{S}_j^{\ *}$ for all $j = 1, 2,. . .,n$,then we have

$$\mathrm{I}^{\alpha}_{\mathrm{V_j}} \leq \mathrm{I}^{\alpha}_{\mathrm{V_j}}{}^{*}$$

$$\Rightarrow (1 - I^{\alpha}_{V_j}) \geq (1 - I^{\alpha}_{V_j}{}^{*})$$

$$\Rightarrow \prod_{J=1}^{n} \left(1 - I^{\alpha}_{V_j}\right)^{\omega_j} \geq \prod_{J=1}^{n} \left(1 - I^{\alpha}_{V_j}{}^{*}\right)^{\omega_j}$$

$$\Rightarrow 1 - \prod_{J=1}^{n} \left(1 - I^{\alpha}_{V_j}\right)^{\omega_j} \leq 1 - \prod_{J=1}^{n} \left(1 - I^{\alpha}_{V_j}{}^{*}\right)^{\omega_j}$$

$$\Rightarrow \bigcup_{I^{\alpha}_{V_j} \in \check{S}} \left\{1 - \prod_{J=1}^{n} \left(1 - I^{\alpha}_{V_j}\right)^{\omega_j}\right\} \leq \bigcup_{I^{\alpha}_{V_j} \in \check{S}} \left\{1 - \prod_{J=1}^{n} \left(1 - I^{\alpha}_{V_j}{}^{*}\right)^{\omega_j}\right\}$$

Since $\check{S}_j \geq \check{S}_j^{\ *}$ for all $j = 1, 2,. . .,n$,then we have

$$F^{\alpha}_{V_j} \geq F^{\alpha}_{V_j}{}^{*}$$

$$\Rightarrow (1 - F^{\alpha}_{V_j}) \leq (1 - F^{\alpha}_{V_j}{}^{*})$$

$$\Rightarrow \prod_{J=1}^{n} \left(1 - F^{\alpha}_{V_j}\right)^{\omega_j} \leq \prod_{J=1}^{n} \left(1 - F^{\alpha}_{V_j}{}^{*}\right)^{\omega_j}$$

$$\Rightarrow 1 - \prod_{J=1}^{n} \left(1 - F^{\alpha}_{V_j}\right)^{\omega_j} \geq 1 - \prod_{J=1}^{n} \left(1 - F^{\alpha}_{V_j}{}^{*}\right)^{\omega_j}$$

$$\Rightarrow \bigcup_{I^{\alpha}_{V_j} \in \check{S}} \left\{1 - \prod_{J=1}^{n} \left(1 - F^{\alpha}_{V_j}\right)^{\omega_j}\right\} \geq \bigcup_{I^{\alpha}_{V_j} \in \check{S}} \left\{1 - \prod_{J=1}^{n} \left(1 - F^{\alpha}_{V_j}{}^{*}\right)^{\omega_j}\right\}$$

Since $\check{S}_j \leq \check{S}_j^{\ *}$ for all $j = 1, 2,. . .,n$,then we have

$$T^{\alpha}_{R_j} \leq T^{\alpha}_{R_j}{}^{*}$$

$$\Rightarrow (1 - T^{\alpha}_{R_j}) \geq (1 - T^{\alpha}_{R_{j_j}}{}^{*})$$

$$\Rightarrow \prod_{J=1}^{n} \left(1 - T^{\alpha}_{R_j}\right)^{\omega_j} \geq \prod_{J=1}^{n} \left(1 - T^{\alpha}_{R_j}{}^{*}\right)^{\omega_j}$$

$$\Rightarrow 1 - \prod_{J=1}^{n} \left(1 - T^{\alpha}_{R_j}\right)^{\omega_j} \leq 1 - \prod_{J=1}^{n} \left(1 - T^{\alpha}_{R_j}{}^{*}\right)^{\omega_j}$$

$$\Rightarrow \bigcup_{I^{\alpha}_{R_j} \in \check{S}} \left\{1 - \prod_{J=1}^{n} \left(1 - T^{\alpha}_{R_j}\right)^{\omega_j}\right\} \leq \bigcup_{I^{\alpha}_{R_j} \in \check{S}} \left\{1 - \prod_{J=1}^{n} \left(1 - T^{\alpha}_{R_j}{}^{*}\right)^{\omega_j}\right\}$$

Since $\check{S}_j \leq \check{S}_j^{\ *}$ for all $j = 1, 2,..,n$,then we have

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$I^{\alpha}_{R_j} \leq {I^{\alpha}_{R_j}}^*$

$\Rightarrow \prod_{J=1}^{n} \left(I^{\alpha}_{R_j}\right)^{\omega_j} \leq \prod_{J=1}^{n} \left({I^{\alpha}_{R_j}}^*\right)^{\omega_j}$

$$\Rightarrow \bigcup_{I^{\alpha}_{R_j} \in \check{S}} \prod_{J=1}^{n} \left(I^{\alpha}_{R_j}\right)^{\omega_j} \leq \bigcup_{I^{\alpha}_{R_j} \in \check{S}} \prod_{J=1}^{n} \left({I^{\alpha}_{R_j}}^*\right)^{\omega_j}$$

Since $\check{S}_j \leq \check{S}_j^*$ for all $j = 1, 2, \ldots, n$,then we have

$I^{\alpha}_{V_j} \leq {I^{\alpha}_{V_j}}^*$

$\Rightarrow (1 - I^{\alpha}_{V_j}) \geq (1 - {I^{\alpha}_{V_j}}^*)$

$\Rightarrow \prod_{J=1}^{n} \left(1 - I^{\alpha}_{V_j}\right)^{\omega_j} \geq \prod_{J=1}^{n} \left(1 - {I^{\alpha}_{V_j}}^*\right)^{\omega_j}$

$\Rightarrow 1 - \prod_{J=1}^{n} \left(1 - I^{\alpha}_{V_j}\right)^{\omega_j} \leq 1 - \prod_{J=1}^{n} \left(1 - {I^{\alpha}_{V_j}}^*\right)^{\omega_j}$

$$\Rightarrow \bigcup_{I^{\alpha}_{V_j} \in \check{S}} \left\{1 - \prod_{J=1}^{n} \left(1 - I^{\alpha}_{V_j}\right)^{\omega_j}\right\} \leq \bigcup_{I^{\alpha}_{V_j} \in \check{S}} \left\{1 - \prod_{J=1}^{n} \left(1 - {I^{\alpha}_{V_j}}^*\right)^{\omega_j}\right\}$$

Since $\check{S}_j \leq \check{S}_j^*$ for all $j = 1, 2, \ldots, n$,then we have

$F^{\alpha}_{V_j} \leq {F^{\alpha}_{V_j}}^*$

$\Rightarrow (1 - F^{\alpha}_{V_j}) \geq (1 - {F^{\alpha}_{V_j}}^*)$

$\Rightarrow \prod_{J=1}^{n} \left(1 - F^{\alpha}_{V_j}\right)^{\omega_j} \geq \prod_{J=1}^{n} \left(1 - {F^{\alpha}_{V_j}}^*\right)^{\omega_j}$

$\Rightarrow 1 - \prod_{J=1}^{n} \left(1 - F^{\alpha}_{V_j}\right)^{\omega_j} \leq 1 - \prod_{J=1}^{n} \left(1 - {F^{\alpha}_{V_j}}^*\right)^{\omega_j}$

$$\Rightarrow \bigcup_{I^{\alpha}_{V_j} \in \check{S}} \left\{1 - \prod_{J=1}^{n} \left(1 - F^{\alpha}_{V_j}\right)^{\omega_j}\right\} \leq \bigcup_{I^{\alpha}_{V_j} \in \check{S}} \left\{1 - \prod_{J=1}^{n} \left(1 - {F^{\alpha}_{V_j}}^*\right)^{\omega_j}\right\}$$

Based on the above analaysis, we have

$\alpha NZN\text{-}WG\ (\check{S}_1, \check{S}_2, \ldots, \check{S}_n) \leq \alpha NZN\text{-}WG^*\ (\check{S}_1^*, \check{S}_2^*, \ldots, , \check{S}_n^*)$

which compleates the proof of Proposition (4. ii).

The proof of iii) (Boundedness)

It is similar to proof of Proposition (4. ii).

In what follows, a decision-making approach was developed under α-cut t neutrosophic with z-numbers information.

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Conclusions

This study successfully introduces the concept of α-cut neutrosophic with z-numbers (αNZN) by combining the α-cut technique and the neutrosophic set with z-numbers. The proposed set provides different notable features. Firstly, it enables a more flexible judgment process, replacing rigid, conventional, and definitive evaluations with the more adaptable approach of the proposed αNZN. Secondly, fundamental properties of αNZN such as operational laws, union, intersection, and complement, are all examined. A proposal concerning the proposed properties is examined and validated. Thirdly, this article introduces two aggregation operators, namely, the αNZN-WA and αNZN-WG operators, which are used for aggregating α-cut neutrosophic sets with z-numbers (αNZN). In addition, several key desirable properties of the aggregation operators have been demonstrated and validated.

The proposed αNZN model not only provides a new method for dealing with uncertainty but also offers significant improvements in handling inconsistent and incomplete information, which are commonly encountered in complex decision-making environments. The introduction of aggregation operators tailored to this framework further strengthens its applicability, making it a valuable tool for researchers and practitioners in fields ranging from business and management to engineering and computing. Future work can explore additional extensions to the αNZN framework, such as the incorporation of dynamic weighting schemes, adaptive α-cut levels, and the application of these models in real-time decision systems. Additionally, further empirical studies are needed to validate the effectiveness of the αNZN framework in diverse decision-making contexts, ensuring its practical utility and widespread adoption.

Conflicts of Interest

The researchers declare that there is no any conflict of interest regarding publication of this paper.

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