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ABOUT GEHMAN DENDRITE G_3 AS GENERALIZED INVERSE LIMIT SPACE

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Abstract

This paper introduces a new uncountable family of upper semi-continuous set-valued functions whose generalized inverse limits are homeomorphic to the Gehman dendrite G_3 , a key object in continuum theory characterized by ramification points of order three and a Cantor set of endpoints. While previous work by Farhan and Mena established one such family, our construction extends their results, providing a distinct class of bonding functions defined on the unit interval using specific parameters and graph structures. Our main results prove that for any function in these families, the inverse limit space is G_3, which we establish by showing its ramification points are of order three and its endpoints form a Cantor set. We provide illustrative examples and counterexamples that demonstrate the necessity of our conditions. This work broadens the catalog of set-valued dynamical systems that produce classic topological structures and highlights the nuanced relationship between bonding function graphs and their inverse limits.

Keywords: inverse limit space, generalized inverse limits, upper semi-continuous function, and Gehman dendrites.

INTRODUCTION AND DEFINITIONS

The initial ideas of generalized inverse limits were first introduced by Mahavier [7], where the continuous bonding functions of traditional inverse limits were replaced by upper semi-continuous bonding functions from the factor space I_{n+1} to the hyperspace of closed subsets 2^{I_n} . The works of Mahavier and Ingram, including their book [6], [5], and paper [4] brought significant developments to the subject and made it more prominent in continuum theory and topology. Since then, hundreds of articles have been published on the topic. Recently, Charatonik, Mena, and Roe [3] proved that certain continuum properties can be preserved from the factor spaces and the bonding function graphs to the inverse limit space, provided that one of the projection maps from the bonding function graphs onto either its domain or codomain is atomic. These properties include irreducibility, tree-likeness, hereditary decomposability, hereditary incoherence, discohernce, in decomposability, and arc-likeness.

In [5, Example 2.22], Ingram demonstrated that when the graph of a bonding function resembles the letter "H" on its side, the corresponding inverse limit space is a dendroid whose ramification points

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all have order 3, and whose set of endpoints forms a Cantor set. It follows from [2] that the inverse limit space of Ingram's example is locally connected. Hence, it is a dendrite, specifically the Gehman dendrite G_3. Recently, Farhan and Mena obtained the Gehman dendrite G_3 as a generalized inverse limit where the bonding functions are elements of an uncountable family of inverse sequences [1], in their family, the graphs of the bonding functions consist of a full vertical and a full horizontal line segment at the top and bottom, while the remaining parts are half-horizontal lines. In this paper, we obtain the Gehman dendrite G_3 from a new family of upper semi-continuous set-valued bonding functions, which differs from the family introduced by Farhan and Mena. We prove that, under a certain condition, even if one of the horizontal lines in the graph of a bonding function is shorter than a half-line, the inverse limit is still homeomorphic to G_3. A topological space X is a continuum if X is a non-empty, connected, compact, metric space[8]. The term" subcontinuum of X" refers to a subset of the continuum X that is a continuum. If, $\forall x \in X$ and every neighborhood $N \in N(x)$ in a continuum X, the component of N containing x is a neighborhood of x, then X is said to be a locally connected continuum or LC[9,10].If X is a topological space, we denote the family of all non-empty closed subsets of X by 2^X and we called the hyperspaces of X. Let $f : X \to 2^Y$ be a set-valued function, where X and Y are continua. The function f is upper semi-continuous or u.s.c. at a point $x \in X$ if, for every open set $V \subseteq Y$ containing $f(x)$, there exists an open set U.A subset of X containing the point x such that when $t \in U$, we have $f(t) \subseteq V$ We refer to a set value function $f : X \to 2^Y$ as u.s.c. if it is u.s.c. at all $x \in X$ [4].If both X and Y are compact metric spaces and g is a set-valued function between X and2^Y , then g is u.s.c. function if and only if in the graph $\{(x,y): y \in g(x)\}$, is closed [5]. Let $\{X_k\}_{k\in N}$ be a sequence of continua and let $f_k : X_{k+1} \to 2^{X_k}$ be the u.s.c. function. The generalized inverse limit or GIF of $\{X_k , f_k\}$ is defined by $\varprojlim \{X_k , f_k\} = \{(x_k)_{k=1}^{\infty} \in \prod_{k=1}^{\infty} X_k : x_k \in f_k(x_{k+1})\ for\ all\ k \in N$ The topology of $\varprojlim \{X_k , f_k\}$ is the subspace of the product space$\prod_{k=1}^{\infty} X_k$ Let C denote the Cantor set.

1. Methodology

The study is conducted in a purely theoretical and constructive framework within continuum theory and generalized inverse limits.

The work begins by fixing the unit interval $I = [0,1]$ and recalling the basic notions of continua, dendrites, ramification points, endpoints, upper semi-continuous (u.s.c.) set-valued maps, and generalized inverse limits $\lim_{\leftarrow}\{X_k, f_k\}$ with u.s.c. bonding maps $f_k: X_{k+1} \to 2^{X_k}$. Distinct parameters $d, a, \alpha \in (0,1)$ with $a < \alpha$ are chosen, and the finite set $A = \{0, a, d, 1\}$ is defined. A u.s.c. bonding map $g: I \to 2^I$ is then specified piecewise by

$$g(x) = \begin{cases} A \setminus \{d\} & 0 \le x < a, \\ A & a \le x < \alpha, \\ [0,1] & x = \alpha, \\ \{0,1\} & \alpha < x \le 1, \end{cases}$$

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and G is taken to be the uncountable family of all such maps obtained from different choices of d, a, α with $a < \alpha$.
For a fixed $g \in G$, the generalized inverse limit $M = \lim_{\leftarrow}\{I, g\}$ is formed as a subspace of the Hilbert cube $\prod_{k=1}^{\infty} I$ with coordinates satisfying $x_k \in g(x_{k+1})$ for all k. Using known results on inverse limits with u.s.c. bonding maps, it is first established (by comparison with Farhan–Mena's work) that M is a dendrite. An analogous uncountable family H is built using parameters $d, a, \alpha \in (0,1)$ with $a > \alpha$ and a different but similar piecewise definition of a u.s.c. map $g: I \to 2^I$, again with ranges alternating between subsets of A, the full interval [0,1], and {0,1}. For each $g \in H$, the inverse limit $H = \lim_{\leftarrow}\{I, g\}$ is constructed and decomposed into sets $O(H)$, $R(H)$, and $E(H)$ in exactly the same manner. By repeating the neighborhood and case analysis for points in $O(H)$ and $R(H)$, and using the same argument to show that $E(H)$ is a Cantor set of endpoints, the inverse limit for every $g \in H$ is also shown to be homeomorphic to the Gehman dendrite G_3. Several explicit choices of parameters in the definitions of g are given to visualize the corresponding bonding graphs and inverse limits (Examples 3.2–3.3), confirming that the construction yields spaces homeomorphic to G_3. Additional examples (Examples 3.4–3.5) modify the bonding graphs (e.g., by including an extra horizontal level or changing inequalities between parameters) to produce dendrites whose ramification points still have order 3, but with infinitely many isolated endpoints so that the resulting inverse limits are not homeomorphic to G_3.**Result**

1.1. MAIN THEOREMS

Recall that the Gehman dendrite G_3 is a dendrite whose ramification points all have order 3, and whose set of endpoints is equivalent to the C. In the following, we provide a family of USC functions that yield an inverse limit space equivalent to the Gehman dendrite G_3 under certain necessary conditions.

Let d, a *and* α be distinct elements in $(0, 1)$ such that $a < \alpha$. Let $A = \{0, a, d, 1\}$. Let g be the USC function $g : [0, 1] \to 2^{[0,1]}$ defined by $g(x) = A \setminus \{d\}$ *if* $0 \le x < a$; $g(x) = A$ *if* $a \le x < \alpha$; $g(x) = [0, 1]$ *if* $x = \alpha$; $g(x) = \{0, 1\}$ *if* $\alpha < x \le 1$. Let G be the family of USC functions that we defined above. The following is the main theorem:

1.1.1.* Theorem: *Let $g \in G$. Then $\lim\limits_{\leftarrow}\{I, g\} = G_3$

Proof. Let $M = \lim\limits_{\leftarrow}\{I, g\}$. As in [1, Theorem 2.1], M is a dendrite. To prove the M *is* G_3 All is left to show is that all of its $R(M)$ points are of order 3 and the set $E(M)$ is equivalent to C.

Since $g(A) \subseteq A$ *and* $g^{-1}(\alpha) = \{\alpha\}$, all points in the inverse limit M must be the union of the following:

$$O(M) = \{(x_1, x_2, \dots, x_n, t, \alpha, \alpha, \dots): n \in N, \quad x_i \in A \text{ and } t \notin A \cup \{\alpha\}\} \cup \{(x_1, x_2, \dots, x_n, t, \alpha, \alpha, \dots) : x_i \in A, \text{with } n > 1, x_n = a \text{ and } x_{n-1} = d\} \cup \{(\alpha, \alpha, \dots)\}.$$

$$R(M) = \{(x_1, x_2, \dots, x_n, \alpha, \alpha, \dots) : n > 1, x_i \in A \text{ with } x_n = a$$

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$$and\ x_{n-1} \neq d\} \cup \{(x_1, x_2, \ldots, x_n, \alpha, \alpha, \ldots) : n \in N, x_i \in A\ and$$
$$x_n \neq a\} \cap \{(x_1, \alpha, \alpha, \ldots) : x_1 \in A\}.$$
$$E(M) = \{(x_1, x_2, \ldots) : x_i \in A\ for\ 1 \leq i\}.$$

Note that if $x_{n-1} = d$ then $x_n \in$ [a, α]; if $x_{n-1} = a$ then $x_n \in [0, \alpha]$; and if $x_{n-1} \in \{0, 1\}$ then $x_n \in$ I. We use notation

$$\epsilon_i = min\{|x_i - y|, y \in A \cup \{\alpha\}, y \neq x_i\}\ and\ \epsilon = min\{|y - \alpha|, y \in A\}.$$

We show that O(L) is the set of ordinary points of L. Let $p \in O(L)$.

First, suppose that $p = (x_1, x_2, \ldots, x_n, x, \alpha, \alpha, \ldots) \in O(L), x \notin A \cup \{\alpha\}$. The point p is contained in the segment $l = (x_1, x_2, \ldots, x_n, t, \alpha, \alpha, \ldots)$, t ∈ (x − ϵ, x + ϵ) where $\epsilon = min\{|x - y| : y \in A \cup \{\alpha\}\}$. The set

$$N = \prod_{k=1}^{n}(x_k - \epsilon_{k,} x_k + \epsilon_k) \times (x - \epsilon, x + \epsilon) \times \prod_{k=n+2}^{\infty} I$$

in an open neighborhood of p in the Hilbert cube. One can see that $N \cap M = l$.

Thus, in this case, $p \in O(L)$.Similarly, $(x, \alpha, \alpha, \ldots), x \in (\alpha - \epsilon, \alpha + \epsilon)$ is a segment containing $(\alpha, \alpha, \ldots)$.

Second, suppose $p = (x_1, x_2, \ldots, x_n, \alpha, \alpha, \ldots) \in O(L)$, and $x_n = a$. By definition of O(L), x_{n-1} = d. The point $p = (x_1, x_2, \ldots, x_{n-1} = d, x_n = a, \alpha, \alpha, \ldots)$ is the common endpoint of the rays $l_1 = (x_1, x_2, \ldots, d, a, t, \alpha, \alpha, \ldots)$, t ∈ [0, α],

$(x_1, x_2, \ldots, x_{n-1} = d, x_n = a, \alpha, \alpha, \ldots)$, s ∈ [a, α]. As before $N = \prod_{k=1}^{n-2}(x_k - \epsilon_{k,} x_k + \epsilon_k) \times (d - \epsilon_{n-1}, d + \epsilon_{n-1}) \times (a - \epsilon_n, a + \epsilon_n) \times (\alpha - \epsilon', \alpha + \epsilon') \times \prod_{i=n+2}^{\infty} I$ is an open set

the Hilbert cube contains p i

that contains l1 ∪ l2 with N ∩ M = l1 ∪ l2.

Next, we show that R (M) is the set of ramification points of L, and all ramification points are of order three. Let p ∈ R (M). We have several cases:

Let $p = (x_1, x_2, \ldots, x_{n-1}, x_n, \alpha, \alpha, \ldots) \in R(L)$, and $x_i \in A$ and $x_n \neq a$. We have several cases:

Suppose $x_{n-1}, x_n \in \{0, 1\}$. As in [1, Theorem 2.1, Case 1], the point p is a ramification point of order 3.

Suppose $x_{n-1} \in \{0, 1\}\ and\ x_n = d$. Suppose WLG $x_{n-1} =$ 0. The point $p = (x_1, x_2, \ldots, x_{n-2}, 0, d, \alpha, \alpha, \ldots) \in R(M)$, is the end point of the segment $l_1 = (x_1, x_2, \ldots, x_{n-2}, 0, d, t, \alpha, \alpha, \ldots)$, $x_i \in A, t \in [a, \alpha]$ and an interior point of $l_2 = (x_1, x_2, \ldots, x_{n-2}, 0, s, \alpha, \alpha, \ldots)$, $x_i \in A, s \in [0, 1]$. Moreover, the sets

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$$N_1 = \prod_{k=1}^{n-2} (x_k - \epsilon_{k,} x_k + \epsilon_k) \times (0 - \epsilon_{n-1}, 0 + \epsilon_{n-1}) \times (d - \epsilon_n, d + \epsilon_n) \times (\alpha - \epsilon', \alpha + \epsilon') \times \prod_{i=n+2}^{\infty} I$$

And

$$N_2 = \prod_{k=1}^{n-2} (x_k - \epsilon_{k,} x_k + \epsilon_k) \times (0 - \epsilon_{n-1}, 0 + \epsilon_{n-1}) \times (d - \epsilon_n, d + \epsilon_n) \times \prod_{i=n+2}^{\infty} I$$

are open in the Hilbert
cube such that p ∈ H1 ∪ H2 and (N1 ∪ N2) ∩ M = l1 ∪ l2.
Thus, in this case p is a ramification point of order 3.
Suppose $x_n \in \{0,1\}$ and $x_{n-1} = a$. Suppose WLG $x_n = 0$.
The point $p = (x_1, x_2, \dots, x_{n-2}, a, 0, \alpha, \alpha, \dots) \in R(M)$, is
the interior point of the segment $l_1 = (x_1, x_2, \dots, x_{n-2}, a, 0, t, \alpha, \alpha, \dots)$
$x_i \in A$, $t \in [0,1]$ and an end point of the segment $l_2 = (x_1, x_2, \dots, x_{n-2}, a, s, \alpha, \alpha, \dots)$, $x_i \in A, s \in [0, \alpha]$. Moreover the sets

$$N_1 = \prod_{k=1}^{n-2} (x_k - \epsilon_{k,} x_k + \epsilon_k) \times (a - \epsilon_{n-1}, a + \epsilon_{n-1}) \times (0 - \epsilon_n, 0 + \epsilon_n) \times (\alpha - \epsilon', \alpha + \epsilon') \times \prod_{i=n+2}^{\infty} I$$

And

$$N_2 = \prod_{k=1}^{n-2} (x_k - \epsilon_{k,} x_k + \epsilon_k) \times (a - \epsilon_{n-1}, a + \epsilon_{n-1}) \times (0 - \epsilon_n, 0 + \epsilon_n) \times \prod_{i=n+1}^{\infty} I$$

are open in the Hilbert
cube such that p ∈ H1 ∪ H2 and (N1 ∪ N2) ∩ M = l1 ∪ l2.
Thus, in this case p is a ramification point with order 3.

Suppose $x_{n-1} = d = x_n \in A$. The case where $a < \alpha < d$ will not happen because of the definition of the u.s.c. function g. We assume $a < d < \alpha$. The point $p = (x_1, x_2, \dots, x_{n-2}, d, d, \alpha, \alpha, \dots) \in R(M)$, is the end point of

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the segment $l_1 = (x_1, x_2, \dots, x_{n-2}, d, d, t, \alpha, \alpha, \dots)$, $x_i \in A$, $t \in [a, \alpha]$ and an interior point of $l_2 = (x_1, x_2, \dots, x_{n-2}, d, s, \alpha, \alpha, \dots)$, $x_i \in A, s \in [a, \alpha]$.Moreover the sets $N_1 = \prod_{k=1}^{n-2}(x_k - \epsilon_{k,} x_k + \epsilon_k) \times (d - \epsilon_{n-1}, d + \epsilon_{n-1}) \times (d - \epsilon_n, d + \epsilon_n) \times (\alpha - \epsilon', \alpha + \epsilon') \times \prod_{i=n+2}^{\infty} I$

$$N_2 = \prod_{k=1}^{n-2}(x_k - \epsilon_{k,} x_k + \epsilon_k) \times (d - \epsilon_{n-1}, d + \epsilon_{n-1}) \times (d - \epsilon_n, d + \epsilon_n) \times \prod_{i=n+1}^{\infty} I$$

are open in the Hilbert cube such that $p \in N_1 \cup N_2$ and $(N_1 \cup N_2) \cap M = l_1 \cup l_2$.

Thus, in this case p is a ramification point with order 3.

Suppose $x_{n-1} = a$ and $x_n = d$. We assume $a < d < \alpha$. The point $p = (x_1, x_2, \dots, x_{n-2}, a, d, \alpha, \alpha, \dots) \in R(M)$, is the end point of the segment $l_1 = (x_1, x_2, \dots, x_{n-2}, a, d, t, \alpha, \alpha, \dots)$,

$x_i \in A, t \in [a, \alpha]$ and an interior point of $l_2 = (x_1, x_2, \dots, x_{n-2}, a, s, \alpha, \alpha, \dots), x_i \in A, s \in [0, \alpha]$. Moreover the sets $N_1 = \prod_{k=1}^{n-2}(x_k - \epsilon_{k,} x_k + \epsilon_k) \times (a - \epsilon_{n-1}, a + \epsilon_{n-1}) \times (d - \epsilon_n, d + \epsilon_n) \times (\alpha - \epsilon', \alpha + \epsilon') \times \prod_{i=n+2}^{\infty} I$ and $N_2 = \prod_{k=1}^{n-2}(x_k - \epsilon_{k,} x_k + \epsilon_k) \times (a - \epsilon_{n-1}, a + \epsilon_{n-1}) \times (d - \epsilon_n, d + \epsilon_n) \times \prod_{i=n+1}^{\infty} I$
,

are open in the Hilbert cube such that $p \in N_1 \cup N_2$ and $(N_1 \cup N_2) \cap M = l_1 \cup l_2$.

Thus, in this case p is a ramification point with order 3.

Let $p = (x_1, x_2, \dots, x_{n-1}, x_n, \alpha, \alpha, \dots) \in R(L), and$ $x_i \in A, x_n = a, x_{n-1} \neq d$. We have several cases:

Suppose $x_{n-1} \in \{0, 1\}$ and $x_n = a$. As in [1, Theorem 2.1, Case 3], the point p is a ramification point and has the order 3.

Suppose$x_{n-1} = a = x_n$.The point $p = (x_1, x_2, \dots, x_{n-2}, a, a, \alpha, \alpha, \dots) \in R(M)$, is the end point of the segment $l_1 = (x_1, x_2, \dots, x_{n-2}, a, a, t, \alpha, \alpha, \dots)$, $x_i \in A, t \in [0, \alpha]$ and p ∈ int(l2) where l2 is the segment $l_2 = (x_1, x_2, \dots, x_{n-2}, a, s, \alpha, \alpha, \dots), x_i \in A, s \in [0, \alpha]$. Moreover the sets $N_1 = \prod_{k=1}^{n-2}(x_k - \epsilon_{k,} x_k + \epsilon_k) \times (a - \epsilon_{n-1}, a + \epsilon_{n-1}) \times (a - \epsilon_n, a + \epsilon_n) \times (\alpha - \epsilon', \alpha + \epsilon') \times \prod_{i=n+2}^{\infty} I$ and $N_2 = \prod_{k=1}^{n-2}(x_k - \epsilon_{k,} x_k + \epsilon_k) \times (a - \epsilon_{n-1}, a + \epsilon_{n-1}) \times (a - \epsilon_n, a + \epsilon_n) \times \prod_{i=n+1}^{\infty} I$

are open sets in the Hilbert cube such?

that $p \in N_1 \cup N_2$ and $(N_1 \cup N_2) \cap M = l_1 \cup l_2$. Thus, in this case p It is a ramification point and it has of order 3.
The case where $x_n = a$ and $x_{n-1} = d$ dose not belong to the set (R).

From above cases, we proved that any point in R(L) is a ramification point and its of order three.

The remaining part is to show that E(D) is equivalent to C. We do not present the proof here because the idea of the proof is identical to the proof given in [1, Theorem 2.1]. □

In the following, we provided another family of u.s.c. function that gives an inverse limit homeomorphic to the Gehman dendrite G3. We are not going providing the proof here because the proof is similar that we presented in the Theorem 2.1.

Let d, a and α be distinct elements in $(0,1)$ such that $a > \alpha$. Let $A = \{0, a, d, 1\}$. Let g be the u.s.c. function $g : [0,1] \to 2^{[0,1]}$ $defined\ by$ $g(x) = A \setminus \{d\}$ if $0 \le x < \alpha$; $g(x) = [0,1]$ if $x = \alpha$; $g(x) = A \setminus \{a\}$ if $\alpha < x < a$; $g(x) = \{0,1\}$ if $a \le x \le 1$. Let H be the family of all such function. The second main theorem in this article is the following.

1.1.2. **Theorem: Let $g \in H$. Then $\varprojlim\{I, g\} = G_3$.**

Proof. The proof is identical to the proof of Theorem 2.1. □

Examples

In this section, we provide a few examples as applications of Theorem 2.1, as well as some examples that demonstrate that the conditions imposed in the previous theorems are necessary.

In the following examples, we show that under the assumptions of the above theorem, the generalized inverse limit space is homeomorphic to the Gehman dendrite G3.

Example 3.2. Let $f_1 : I \to 2^I$ be the u.s.c. function as in Theorem 3.1.1 where $d = \alpha_1 = \frac{1}{10}$ and $a = \alpha_2 = \frac{2}{10}$. Let $S = \varprojlim\{I, f_1\}$.
Using Theorem 3.1.1, the space S is equivalent to G3, and the picture representation of S is shown in Figure 1.

Example 3.3. Let $f_2 \in G$ and let $f_2 : I \to 2^I$be the u.s.c. function as in Theorem 3.1.1, where $a = \alpha_1 = \frac{2}{10}$ and $d = \alpha_2 = \frac{7}{10}$. Let $L = \varprojlim\{I, f_2\}$.
By using Theorem 2.1, The space L is equivalent to G3, and the picture representation of L is shown in Figure 2.

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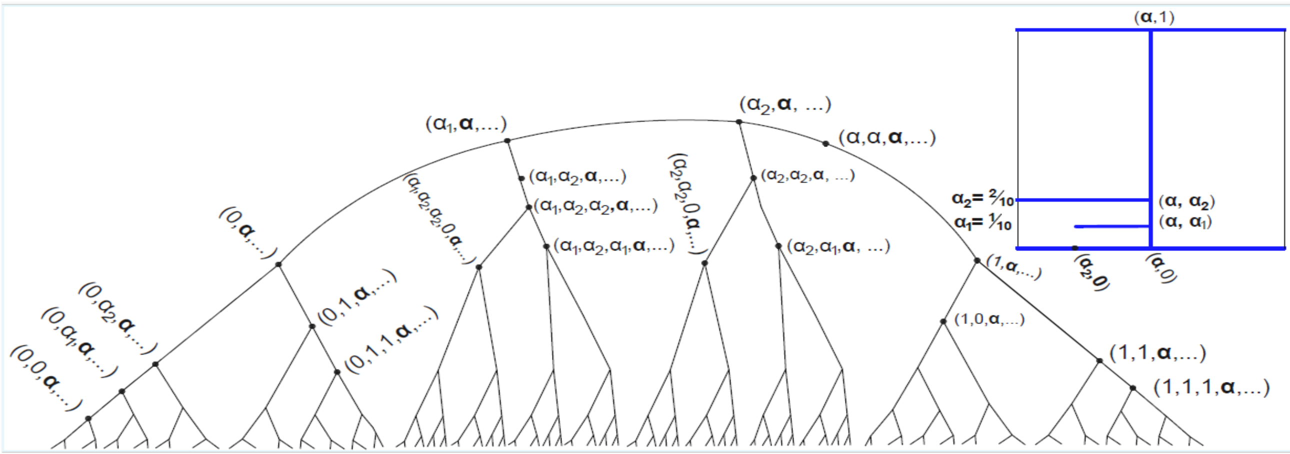

FIGURE 1. The graph f_1 and the representation picture of S

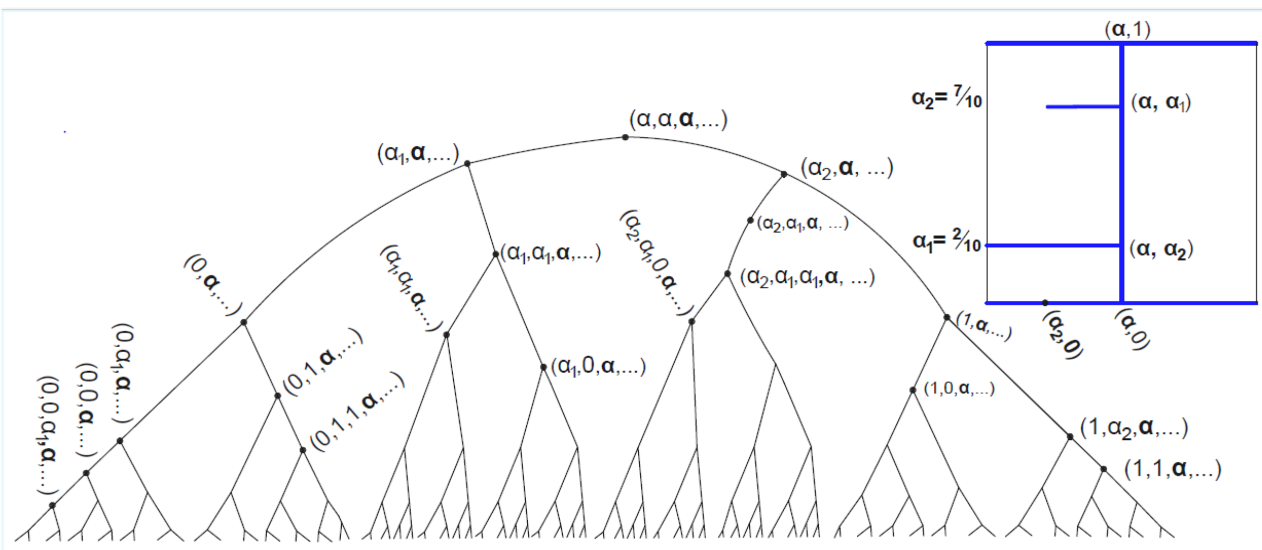

FIGURE 2. The graph f_2 and the representation picture of L

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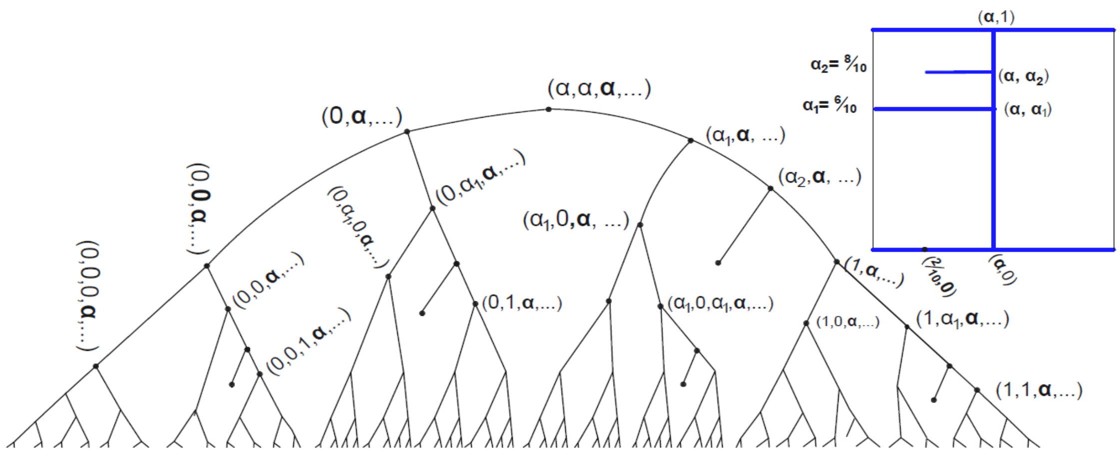

FIGURE 3. The graph f_3 and the representation picture of K

The following examples show that the conditions imposed in the main theorems of the previous chapter are necessary.

Example 3.4. Let $\alpha, \alpha_1, \alpha_2, \alpha_3$ be three distinct elements in (0, 1) with $\alpha_1 > \alpha$, and let $f_3 : I \to 2^I$ be a u.s.c. function defined as follows:

$$f_3(x) = \{0, 1, \alpha_1\}\ for\ x \in [0, \alpha_3], f_3(x) = \{0, 1, \alpha_1, \alpha_2\}\ for\ x \in [\alpha_3, \alpha]$$
$$f_3(\alpha) = I, and\ f_3(x) = \{0, 1\}\ for\ x \in [\alpha, 1]. Let\ K = \lim_{\leftarrow}\{I, f_3\} .$$

As in the first part of the proof of [1, Theorem 2.1], K is a dendrite with all of its ramification points of order 3, but it is not equivalent to G3 due to it has infinitely many free isolated endpoints. The inverse limit picture representation of K is shown in Figure 3, where $\alpha_1 = \frac{6}{10}$, $\alpha_2 = \frac{8}{10}$ $and\ \alpha_3 = \frac{2}{10}$.

Example 3.5. Let $\alpha, \alpha_1, \alpha_2, \alpha_3$ be four distinct elements in (0, 1) with $\alpha_1 < \alpha$. Let $f_4 : I \to 2^I$ be a u.s.c. function defined as follows:

$$f_4(x) = \{0, 1, \alpha_1\}\ for\ x \in [0, \alpha_3], f_4(x) = \{0, 1, \alpha_1, \alpha_2\}\ for\ x \in [\alpha_3, \alpha]$$
$f_4(\alpha) = I, and\ f_4(x) = \{0, 1\}\ for\ x \in [\alpha, 1]$. Let $T = \lim_{\leftarrow}\{I, f_4\}$

The inverse limit T is a dendrite with all of its ramification points are of order 3,

but it is not equivalent to G3 because it has infinitely many free isolated endpoints. The picture representation of T is shown in Figure 4, where $\alpha_1 = \frac{2}{10}$, $\alpha_2 = \frac{8}{10}$ $and\ \alpha_3 = \frac{1}{10}$.

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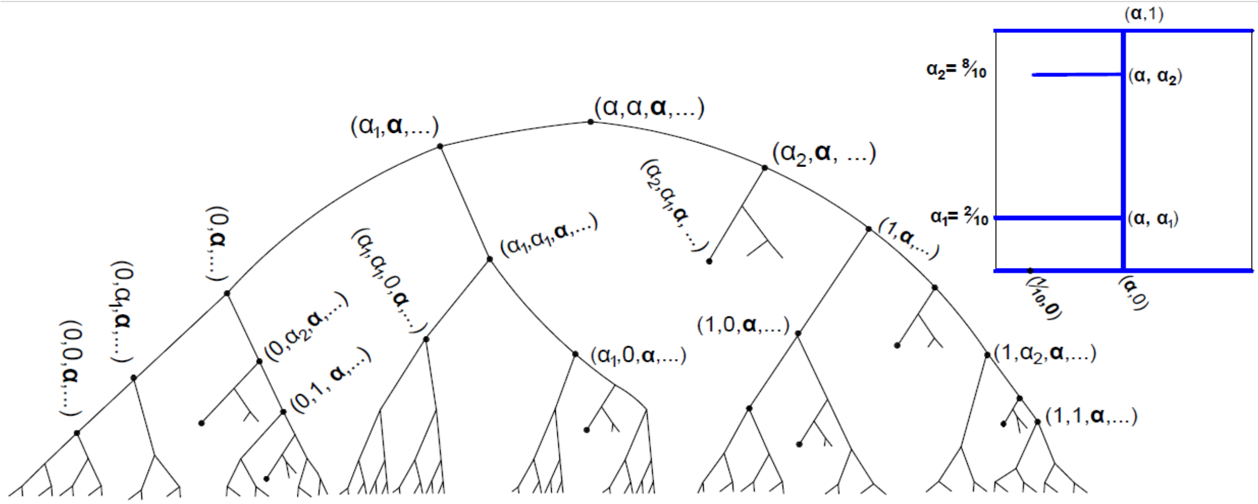

FIGURE 4. The graph of f_4 and the representation picture of T

2. Discussion

We have shown that the Gehman dendrite G_3 can be represented as a generalized inverse limit of the unit interval I with appropriately selected upper semi-continuous bonding functions. Two uncountable families of bonding functions (G_3 in Theorem 3.1.1 with $a\setminus\alpha$) were provided, such that the inverse limit spaces are homeomorphic to G_3. Our construction generalizes the previously known results in Farhan and Mena [1] by presenting a different class of bonding graphs that still yield G_3. One interesting feature of our construction in contrast is that one of the horizontal half-line segments in the bonding graph may have its end point shifted to a different position (the segment being shorter than a half-line then under certain condition) without losing the homeomorphism type G_3[9]. In some sense, this can be seen as uncovering a new kind of freedom in constructing set-valued bonding functions that give rise to a desired continuum as the inverse limit. Our examples also illustrated how fragile the conditions could be for maintaining the G_3 structure. In Examples 3.3 and 3.4, by violating the conditions (e.g. positing an additional parameter besides α in the bonding function or flipping the inequality between $\boldsymbol{a}$ and $\boldsymbol{\alpha}$ without proper modifications), the resultant inverse limit had infinitely many isolated endpoints and hence was not homeomorphic to G_3. These counterexamples highlight that having all ramification points of order 3 alone is not sufficient to characterize a dendrite as G_3, the global endpoint structure (Cantor set rather than presence of isolated endpoints) is equally an indispensable requirement. More generally, this agrees with what is more or less known in continuum theory that a given universal dendrite of given order (here the order 3) often must have some very specific topological property which is no longer true if additional endpoints get glued in a different way. Our construction thus provides a constructive demonstration of how the Cantor-set endpoint condition can be prescribed (or destroyed) in inverse limits.

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It is natural to ask whether similar families of bonding functions can be devised for obtaining the Gehman dendrite G_3 of higher order $n > 3$, or other classic continua, as generalized inverse limits. The approach in this paper, together with prior results, In conclusion, this work expands the known results of Farhan and Mena [1] in generalized inverse limits with u.s.c. bonding maps that produce well-known continua. By obtaining G_3 from new families of bonding functions, we not only replicate a classic continuum via inverse limits but also deepen the understanding of how subtle changes in bonding graph topology reflect on the inverse limit's topological properties.

5. Conclusion

We provided a formal inverse-limit realization of the Gehman dendrite G_3 using u.s.c. bonding maps on the unit interval. The construction is close to, but not identical with, the family of Farhan and Mena [1], which shows that G_3 is stable under a small class of graph modifications. Future work may look at higher-order Gehman dendrites or at realizing similar dendrites via bonding maps with prescribed dynamical properties.

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