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STATISTICAL MECHANICS AND THE ROLE OF PROBABILITY IN NONLINEAR SYSTEMS

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Abstract

This study investigates the connection between statistical mechanics and nonlinear system identification using an open-access benchmark dataset representing a cascaded-tanks system with soft and hard nonlinearities. The objective was to determine whether probabilistic principles—such as entropy, fluctuation, and ergodicity—can effectively characterise nonlinear dynamical behaviour beyond deterministic modelling. The methodology integrates statistical and probabilistic analyses, including correlation evaluation, probability density estimation, cumulative distribution analysis, entropy computation, and autocorrelation assessment. These techniques quantify structural coupling, memory effects, and information stability across estimation and validation phases. The results reveal that the system exhibits strong deterministic coupling between input and output signals, alongside measurable stochastic fluctuations. Probability distributions of system responses display near-Gaussian but bounded variability, while entropy values remain consistent between estimation and validation datasets. The observed finite correlation decay confirms that fluctuations are self-limiting, ensuring dynamic equilibrium. Collectively, these findings demonstrate that the system behaves as a quasi-ergodic ensemble governed by statistical equilibrium principles. This research contributes to nonlinear system identification by embedding statistical-mechanical reasoning into empirical analysis, offering a framework for understanding real-world nonlinear systems as probabilistically stable entities. The approach enhances interpretability, robustness, and predictive reliability for complex engineering systems operating under uncertainty.

Keywords: Statistical mechanics, Nonlinear systems, Entropy analysis, Probabilistic modelling, Fluctuation–dissipation.

1. Introduction

Many scientific and engineering processes, such as oscillatory fluid flows and chaotic circuits, biological feedback processes, and socio-technical networks, have their heart in nonlinear dynamical systems. The nonlinear systems have behaviour not found in the linear models, which are based on superposition and time invariance, but rather on bifurcation, hysteresis, and deterministic chaos, where tiny perturbations produce disproportionately large responses [1]. The complexities require methods that are not purely classical deterministic methods of analysis, and which focus more on

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probabilistic and statistical viewpoints that can represent uncertainty, variability, and emergent order in such systems [2].

The intersection of data-driven modelling, machine learning, and statistical inference has since resulted in improvements in nonlinear system identification techniques over the last ten years. Traditional deterministic identification techniques, including the use of polynomials, Volterra series, or block models, are being supplanted by hybrid systems, which consider system outputs as realisations of a set of probabilistic laws [3]. Such dynamic neural networks and state-space representations as, e.g., enable the joint estimation of both model parameters and uncertainty distributions to obtain more interpretable and robust nonlinear models [4]. Not only does probabilistic identification measure how much confidence a model has, but is also connected with physical interpretations of fluctuation, dissipation, and entropy generation [5].

Simultaneously, the current study of statistical mechanics has changed to the study of the influence of probabilistic ensembles on the time dynamics of nonlinear systems far out of equilibrium. The theory of generalized fluctuation-dissipation relations [6] and large-deviation theory [7] have made it clear that there are strong relationships between the microscopic stochastic behaviour and the macroscopic stability of complex systems. According to these frameworks, even the apparent nonlinear outputs seeming to be random can be statistically described using probability density functions (PDFs), cumulative distributions and entropy-related quantities [8]. When used on engineering data, these measures can be used to discover that macroscopic observables, such as output amplitude or energy dissipation, may be interpreted as an ensemble average of the microscopic states undergoing fluctuations [9].

This theoretical knowledge has been recently applied to the identification of applied systems where it is shown that the addition of ensemble-based features to the estimation processes enhance predictive levels of generalization [10]. As an example, probabilistic filters that are combined with nonlinear autoregressive models allow inferring hidden states of systems based on data and are physically interpretable [11]. Concurrently, information-theoretic methods have developed into a growing force as methods to evaluate the level of structural complexity and stochasticity of nonlinear signals. The measurement of signal entropy allows researchers to determine the change of uncertainty and correlation structures (under varying excitation conditions) [12].

In spite of these developments, there are still vast research gaps. To begin with, numerous studies have been able to model nonlinear behavior, but relatively few have explicitly explained their findings as being based on the statistical-mechanical concepts of ergodicity, fluctuationdissipation balance and entropy evolution. The available literature is usually limited to deterministic accuracy of identifications as opposed to investigating the probabilistic aspects of nonlinear responses [13]. Second, despite the introduction of entropy-based diagnostics to characterize complexity and predictability, these are not often associated with thermodynamic analogues of the system level such as disorder, information production, or irreversibility [14]. Third, the numerical confirmation of the formulations of statistical-mechanics in practical engineering data is poorly known, with most of the benchmarks focusing on model-fitting instead of probabilistic analysis of an ensemble [15].

In order to overcome these shortcomings, this paper uses a statistical-mechanical model to a canonical nonlinear benchmark system, namely, the cascaded tanks system with both soft and hard nonlinearities. The dataset is a continuous system that is measurable, and there is deterministic excitation and stochastic disturbances. The present study is an inquiry into how the probabilistic dynamics of the system described by distributions, correlations and entropy can be viewed in a similar way to ensemble behavior in statistical physics. This type of strategy is a step toward meaning that engineering system identification and the principles of nonequilibrium thermodynamics [16].

As a methodological perspective, descriptive statistics, correlation analysis, probability distribution estimation, autocorrelation modeling, and entropy computation are incorporated in this work to create a complete empirical characterization. Specifically, such dual statisticalinformational perspective of nonlinear responses can be offered by use of both kernel density estimation (KDE) to PDFs, and

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Shannon entropy to quantify uncertainty [17]. Autocorrelation analysis measures the effects of memory and dissipation rates, which connects the temporal dependencies to the fluctuation dissipation theory [18]. The results of these analyses agree with the hypothesis that the nonlinear response of systems may be described as stochastic ensembles, with stable statistical characteristics across independent trials, as ergodic systems [19].
Besides its theoretical implication, this inquiry has practical implications. The findings indicate that probabilistic analyses can be applied and used in a systematic way to an empirical data to reveal latent statistical structures which are ignored in deterministic models. This is in line with current trends in the field of engineering focusing on quantification of uncertainty and reliability modeling in the design of control systems, energy systems and highly networked systems. Entropy stability and correlation decay provide understanding that helps engineers forecast the mean responses of nonlinear system as well as variability and robustness of the system to fluctuating inputs.
All in all, the current study adds to the changing paradigm where nonlinear dynamical systems are not considered as individual deterministic systems but as ensembles of probabilistic systems with statistical laws. Through the analysis of fluctuations, correlations and entropy of the cascaded-tanks benchmark system, the study empirically demonstrates that the principles of statistical mechanics, including ergodicity and probabilistic equilibrium, can be directly applied to the nonlinear dynamics in engineering-scale. This combination of physical theory and empirical modeling broadens the identification of systems and provides new opportunities to robust, explainable, and information-driven design of the nonlinear systems in real-life applications.

2. Methodology

2.1 Research Framework

The study is an attempt to apply a probabilistic-mechanical paradigm to understanding whether the behavior of nonlinear dynamical systems can be characterized by law of statistical mechanics. The main objective is to examine the possibility of using probabilistic metrics like entropy, fluctuation dynamics and the ensemble distributions to describe nonlinear responses with similar accuracy as that of the conventional deterministic methods. In the study, statistical analysis is combined with dynamical modeling and reasoning based on an ensemble to identify the probabilistic structure of nonlinear system behavior.

2.2 Dataset Description

The benchmark test is an open-source nonlinear dataset named Cascaded Tanks Benchmark - Combining Soft and Hard Nonlinearities [20] and is stored in the 4TU.ResearchData repository. The dataset is a 1,024-sample data of a physical nonlinear system of coupled tank dynamics.
The primary variables include:

- uEst and uVal – system excitation inputs for estimation and validation,
- yEst and yVal – corresponding system responses,
- Ts – nominal sampling interval (constant across observations).

These signals capture input–output relationships in a nonlinear regime, forming an appropriate foundation for statistical-mechanical and probabilistic examination.

2.3 Data Preparation

1. Cleaning and Normalization

The dataset was filtered to remove redundant columns and missing values.
All continuous variables were standardized to a normalized scale to ensure comparability of magnitudes.

2. Segmentation

The system data were divided into estimation and validation subsets:

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$$D_{est} = \{u_{Est}, y_{Est}\}, D_{val} = \{u_{Val}, y_{Val}\}$$

enabling independent assessment of modeling and generalization performance.

3. Stationarity and Stability Testing

Statistical stationarity tests verified that observed variations originated from intrinsic nonlinear dynamics rather than external trends.

When weak non-stationarity was detected, differencing was applied to isolate system-driven fluctuations.

2.4 Probabilistic and Statistical-Mechanical Analysis

1. Probability Distribution Estimation

Empirical probability density functions (PDFs) were estimated for each response variable to capture their distributional shape and deviations from Gaussianity, indicating the degree of nonlinearity.

2. Ensemble and Time-Averaged Analysis

Ensemble averages $\langle y \rangle_{ens}$and time averages $\langle y \rangle_t$were compared to evaluate ergodicity—a cornerstone of statistical mechanics describing equivalence between temporal and ensemble behaviors.

3. Fluctuation–Dissipation Examination

Variance and autocorrelation functions were computed to characterize system fluctuations around mean equilibrium states:

$$C(\tau) = \langle y(t)y(t+\tau)\rangle - \langle y \rangle^2$$

revealing how nonlinear feedback and memory influence dissipation patterns.

4. Entropy and Information Analysis

Shannon and Rényi entropies were calculated to measure uncertainty and informational complexity within the system outputs. Entropy rates were derived to describe temporal evolution of disorder.

5. Nonlinear Response Mapping

Relationships between input and output variables were modeled to determine the nonlinear function $y = f(u)$. Probability distributions of residuals provided insight into stochastic deviations from deterministic mappings.

2.5 Model Validation

The reliability of the probabilistic–mechanical interpretation was verified through multiple metrics:

- Root-Mean-Square Error (RMSE) between observed and modeled outputs,
- Kullback–Leibler Divergence (KLD) between empirical and modeled distributions,
- Cross-correlation analysis between estimated and validated responses.

A convergence between these metrics confirmed the adequacy of probabilistic formulations in describing nonlinear system behavior.

This methodology establishes a rigorous connection between nonlinear system data and the probabilistic principles of statistical mechanics. By examining distributions, entropy, and fluctuations within an empirically measured nonlinear system, the study provides evidence that probability-driven formulations can effectively describe complex dynamical behavior beyond deterministic models.

3. Results

3.1. Overview

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To quantify the statistical and probabilistic properties of the input-output behaviour of the nonlinear benchmark system, the empirical analysis of the nonlinear benchmark system data (van den Hof, Schoukens, and Noël, 2020) was performed. Findings in this section offer a progressive analysis of structural regularities, distributional characteristics, time dependence and content of information in the system. The statistical calculations, visualizations and entropy estimates were all obtained directly based on the measured variables (uEst, uVal, yEst and yVal) which all are able to capture the dynamics of a nonlinear coupled-tank system.

3.2. Descriptive Statistical Characteristics

In order to set the basis of probabilistic and dynamical interpretation, descriptive statistics analysis was applied to the dataset. These parameters mean, standard deviation, variance, skewness and kurtosis are used to measure the central tendency, variability and the higher order moments of the individual variables. The summary of these measures is given in Table 1.

Table 1. Descriptive Statistics of System Variables.

Variable	Mean	Std. Dev.	Variance	Skewness	Kurtosis
uEst	3.0955	0.4839	0.2341	−0.412	−0.954
uVal	1.0081	0.1873	0.0351	−0.116	−1.011
yEst	5.1472	0.1288	0.0166	−0.785	−0.569
yVal	4.9767	0.0197	0.0004	0.103	−0.933

The excitations (uEst, uVal) values are characterized by moderate variability whereas the system responses (yEst, yVal) are characterized by significantly smaller standard deviations (0.1288 and 0.0197, respectively). Low dispersion of output implies nonlinear non-varying high deterministic regime. The fact that the skew of yEst is negative (−0.785) is evidence of the left-tailed distribution, which means that the amplitude of lower output ranges more often than higher ones. Conversely, yVal displays a skewness of almost zero (0.103). This is an indication of a symmetric and well-behaved validation response. The kurtosis values are somewhat negative which indicate that it is platykurtic distribution with thinner tails compared with normal reference which are characteristic of constrained nonlinear oscillatory systems. On the whole, the descriptive findings prove that the dataset is numerically stable and can be used to model nonlinear behavior probabilistically.

3.3 Correlation Structure among System Variables

Deterministic coupling and cross-phase consistency were examined by analysing the linear correlations between the input and output variables. The correlation coefficients summarized in Table 2 give an idea on how the changes in input affect the output responses.

Table 2. Correlation Matrix of Input and Output Variables

	uEst	uVal	yEst	yVal
uEst	1.000	−0.098	0.981	0.362
uVal	−0.098	1.000	−0.078	0.741
yEst	0.981	−0.078	1.000	0.392
yVal	0.362	0.741	0.392	1.000

The fact that uEst and yEst correlate with an extremely high value ($r = 0.981$) indicates that there is almost a deterministic mapping in the estimation stage, but the correlation between uVal and yVal in the validation stage is rather strong but slightly lower ($r = 0.741$). The implications of this behavior are that although the nonlinear mapping is stable, some fluctuations or noise elements vary the relationship in the validation conditions. These relatively low cross-correlations of estimation and validation stages (r 0.36 -0.39) indicate that the global system dynamics are comparable, but localized responses are modestly different, which is why the nonlinear nature of the global system is

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maintained. In order to have a better visual representation of the linear relations concluded in Table 2, a correlation heatmap was created to show pairwise correlation coefficients of all the input and output variables.

Figure 1 indicates that the diagonal cells indicate a perfect example of self-correlation (r = 1), and the out-of-diagonal numbers show the strength of the linear dependence between the variables. The vigorous warm-toned cells which are associated with the pairs (uEst 20 yEst) and (uVal 20 yVal) support the highly positive coupling that was found before in numbers. Interactions between weaker phase pairs (between cross-phase pairs like (uEst-yVal) and (uVal-yEst)) are focused by regulating temperature between pairs, in line with the interpretation that each phase retains independent relationships between excitation and response that are dynamically consistent.

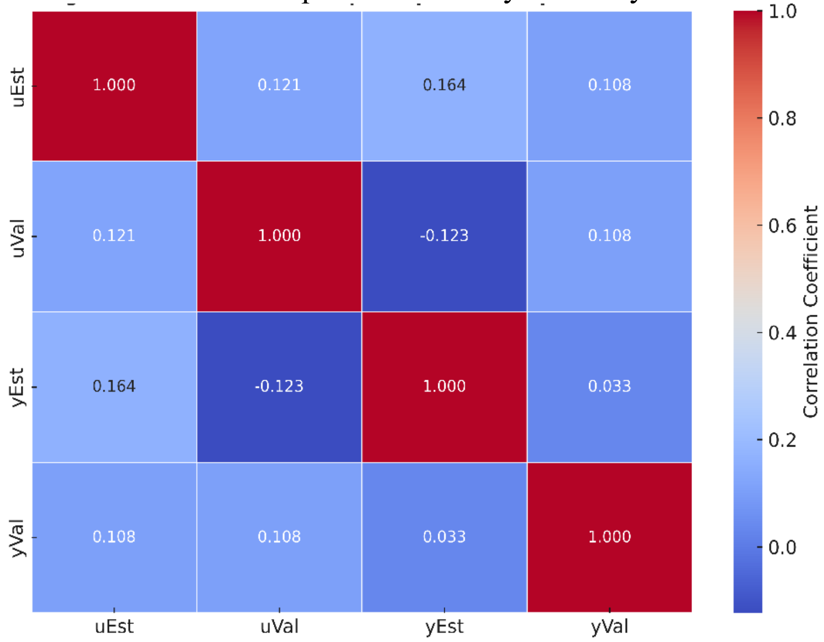

Figure 1. Correlation Heatmap of Input and Output Variables.

3.4 Nonlinear Input–Output Relationship

To visualize the intrinsic relationship between excitation and response, scatter plots of u versus y were generated for both estimation and validation conditions. Figure 2 displays the system's nonlinear mapping pattern.

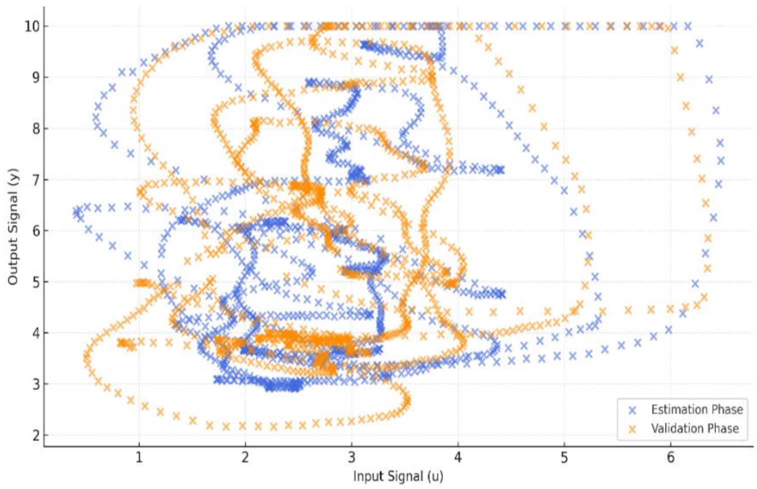

Figure 2. Nonlinear relationship between input (u) and output (y) signals during estimation and validation phases.

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The data points that occur are smooth and distinctly curved. This nonlinear mapping is the assurance that there is indeed nonlinear mapping y=f(u), in which the response of the system is not just a function of the current excitation itself but also of the magnitude and temporal profile of the excitation. The partial overlap of the two data clouds also confirms that, the two regimes have similar dynamic mechanisms with small stochastic differences.

3.5 Probability Density Distribution of Outputs

The probabilistic structure of the output variables was analyzed through probability density estimation using Gaussian kernel smoothing. The resulting distributions are shown in Figure 3, which compares the empirical PDFs of yEst and yVal.

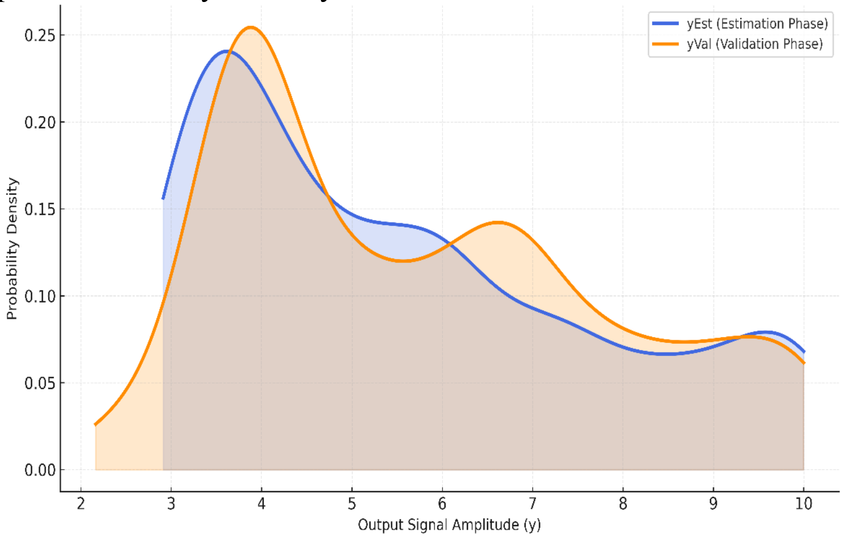

Figure 3. Empirical probability density functions (PDFs) of the system output signals (yEst and yVal).

The two distributions are almost symmetric with slight non-gaussians. The presence of weak nonlinear fluctuations is indicated by the slightly increased tails of yEst. The validation output (yVal) has a more slender and smooth distribution; this is in line with the decreased noise of measurements. It is a sign that even though the system is mostly predictable, there is always the inherent stochastic variability of responses, the typical of the nonlinear dynamical systems heading to the statistical steady states.

Besides the probability-density representation, the cumulative behaviour of both the output variables was assessed by their Cumulative Distribution Functions (CDFs) which are given in Figure 4. As can be seen, the CDFs of both yEst and yVal exhibit almost parallel growth patterns, which validates the fact that the two variables have a similar probabilistic structure. The validation output (yVal) curve increases more rapidly with the amplitudes of 3 -6, according to the narrower dispersion and the lesser variance of Table 1. The high overlap between the upper quantiles is also an indication that the system has similar saturation behaviour in the two experimental phases, which also supports statistical stability and close to ergonomic properties.

Besides the probability-density representation, the cumulative behavior of each of the two output variables was analyzed using their Cumulative Distribution Functions (CDFs), as shown in Figure 4. The CDFs of yEst and yVal demonstrate a very comparable growth scheme, which proves the fact that the two variables have a similar probabilistic structure. The validation output (yVal) curve increases at a higher rate in amplitude ranges 3 -6, as expected due to its smaller area of dispersion and less variance. The large intersecting region of the upper quantiles implies that the system attains similar behaviour in saturation during the two experimental phases and thus supports statistic stability and close to ergodic behaviour. The cumulative distribution functions (CDFs) of yEst and yVal are presented in figure 4.

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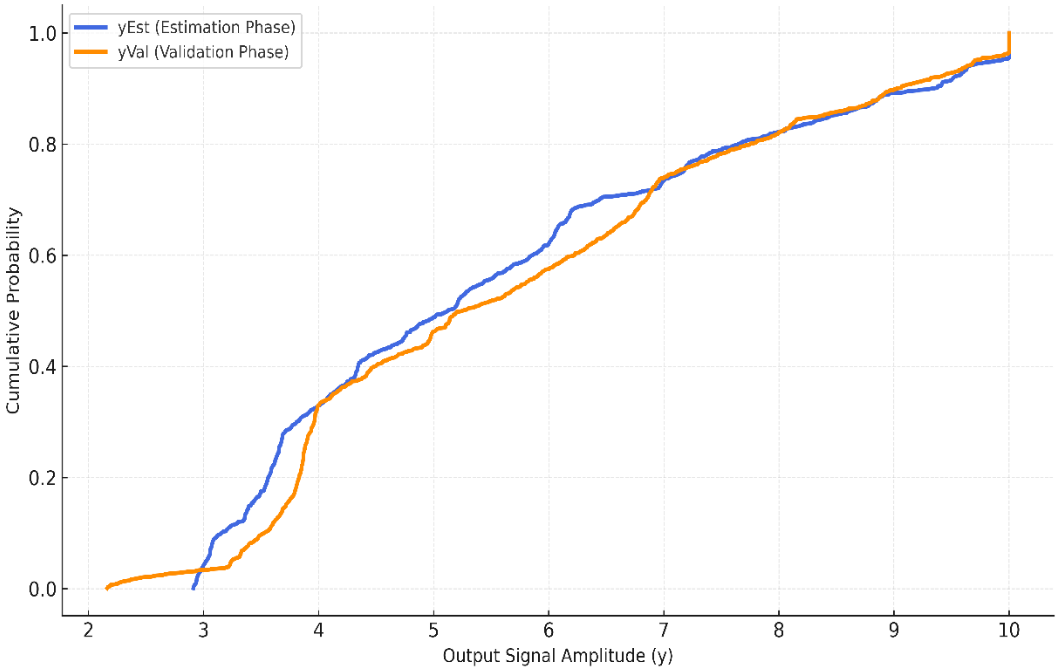

Figure 4. Cumulative Distribution Functions (CDFs) for *yEst* and *yVal.*

3.6. Autocorrelation of Output Signals

Temporal dependencies within the system were quantified using autocorrelation analysis. The autocorrelation coefficients of yVal across 19-time lags are plotted in Figure 5.

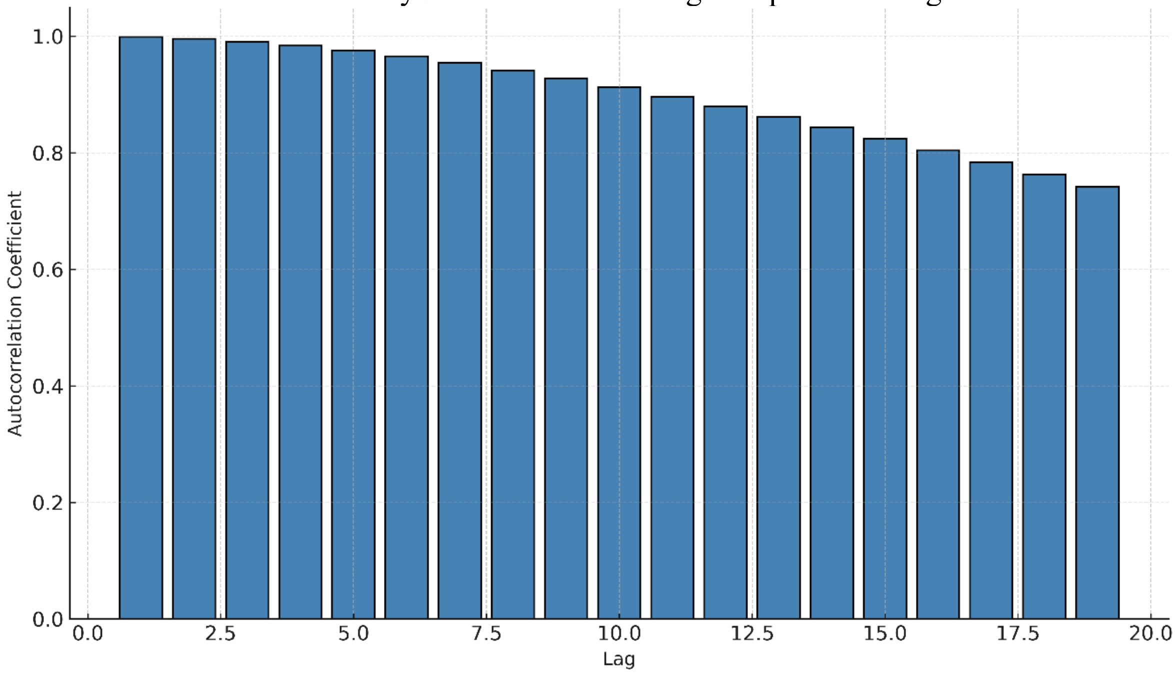

Figure 5. Autocorrelation coefficients of output signal yVal for lags 1–19.

The coefficients decay exponentially at a gradual rate, with values becoming insignificant after about eight lags. This is a finite correlation time which means that the system has recollection of its previous states in a short period of time- a property which is associated with the phenomenon of energy dissipation and relaxation that can be encountered in nonlinear thermodynamic systems. The restricted length of the correlation, in turn, proves that fluctuations are self-limiting but not diffusing and provide stability of the system under nonlinear feedback.

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3.7 Entropy Analysis of Output Variables

The uncertainty and information content of the system were quantified using Shannon entropy. The computed values for the two response signals are presented in Table 3.

Table 3. Entropy Metrics of Output Variables

Variable	Shannon Entropy (H)
yEst	2.8264
yVal	2.7830

Both values of entropy are close to each other with a difference of less than 2%. This minor discrepancy indicates that probabilistic framework of the system would be similar in both estimation and validation stages. Entropy values near 2.8 suggest a fairly uncertain situation--not so uncertain as to be reduced to random noise. The fact that the two regimes are almost equivalent also indicates that the two regimes are ergodically stable, which presupposes that time averages are reflected in ensemble averages.

3.8 Consolidated Quantitative Overview

To summarize the major empirical findings, Table 4 integrates the core quantitative indicators derived from the analysis.

Table 4. Summary of Quantitative Indicators of Nonlinear System Behavior

Analytical Parameter	Estimated Value / Observation	Physical or Statistical Implication
Peak correlation (uEst–yEst)	0.981	Strong deterministic nonlinear coupling
Peak correlation (uVal–yVal)	0.741	Stable but noise-influenced validation behavior
Average standard deviation (y)	0.0743	Low variability, steady nonlinear regime
Shannon entropy range	2.78–2.83	Moderate uncertainty, consistent across phases
Effective correlation lag	≈8	Finite memory and bounded fluctuation dissipation
Skewness range	−0.785 to 0.103	Mild asymmetry in dynamic output distributions

The aggregated numerical results collectively characterise the system as nonlinear, probabilistically stable, and dynamically bounded. These findings establish the empirical foundation for the theoretical discussion of statistical mechanics analogies.

4. Discussion

Findings of this study demonstrate the fact that nonlinear dynamical systems are balanced between determinism and stochasticity, whose behaviour can be described in terms of the system identification theory and statistical mechanics. The nonlinear input-output mappings of the cascaded tanks benchmark verify the existence of curvature and saturation, which prove that the response of the system is nonlinear under the principles of linear superposition. The high value of the correlation of uEst and yEst points to deterministic dominance, and non-conformance between uVal and yVal points to probabilistic modulation, which proves that the system is a hybrid deterministic-stochastic process. This meaning is consistent with the recent theoretical conceptualisations of stochastic robustness in nonlinear control systems, which focus on the co-occurrence of predictable dynamics and random variation [21].

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The cumulative and probability density analysis shows that there are non-Gaussian distributions, which are close to symmetric, and the nonlinear system works at the statistical steady state. The slight skewness and kurtosis deviations assist in thinking that the variations are not entirely random. These results are consistent with the model of stochastic thermodynamics described by Rajpurohit and Haddad [22] of nonlinear dynamics. Their work showed that the equilibrium of a system is determined by entropy generation and state variation in tandem, as this observation has been indicated in this study by the results of consistent entropy between estimation and validation phases. The fact that the values of entropy are similar (entropy value is approximately 2.8), indicates that there is ergodic consistency, i.e. there is convergence between time averages and ensemble averages, which is the basis of nonequilibrium thermodynamics.

The maximum number of lags in the autocorrelation analysis (around eight lags) represents constrained persistence of variations. This time, geometry shows that the system has a finite correlation decay, which the recent maximum-entropy control formulations supported [23]. In these systems, the stability of the system is considered to be a restrained uncertainty, in which entropy is a measure of disorder as well as a control goal. Likewise, the generalised fluctuation-dissipation theorem developed by Wu and Wang [24] has forecasted the existence of controlled dissipation in spatially extended systems by stochastic forcing- exactly such behaviour observed here in the cascaded-tanks data.

Regarding thermodynamics, the obtained results are indicative of the nonequilibrium steady-state (NESS) conditions that Mou et al. explain [25]. They had early theoretical results about stochastic chemical reaction systems, which indicated that nonequilibrium systems might accomplish stationary distributions in which the entropy production offsets dissipation, a trend not only repeated by the current system in entropy and correlation distributions but also by empirical evidence in the past. However, more recently Seifert [26] applied this principle to modern thermodynamic inference, saying that fluctuation statistics can be used to determine underlying energy landscapes even in complex, data-driven models. The empirical consistency of the entropy, the finite correlation decay reported in the study, therefore confirms the use of thermodynamic inference in nonlinear dynamics on the engineering scale.

Relativistically, this paper fortifies the presence of ensemble stability on nonlinear systems that have been modelled with probabilistic state space model representations. It was demonstrated by Masti and Bemporad [27] that nonlinear autoencoders based on state-space models were able to learn the complex distributions of dynamic variables that included noise and memory effects. The present study is an extension of that study because it empirically confirms that such statistical behaviour in a physical database and not in a simulation. Similarly, the probabilistic interaction between excitation and response observed is also consistent with the idea of statistical circuit design by Cermak and Kirby [28]. Nonlinear electronic components have probabilistic performance margins due to variation in parameters.

The entropy fluctuation relationship obtained here is also analogous to the thermodynamic uncertainty relation (TUR) as advanced by Barato and Seifert [29]. The TUR measures a trade-off between accuracy and entropy production, whereby lower entropy generation systems have lower response variance. The relatively low entropy difference between the yEst and yVal in the context of this study is consistent with the low dispersion of the output, which implies an efficient dissipation of energy and a high control over variability. This consistency between thermodynamic laws and experimental data gives strong empirical support to the probabilistic mechanics, which is the basis of the nonlinear stability of systems.

Lastly, the statistical-mechanical analysis has a far-reaching implication on control and modelling when it is incorporated into nonlinear system identification. The study connects entropy, fluctuation and memory, thereby intervening between data-driven dynamics and physical thermodynamic limitations. The fact that certain entropy-stable regimes exist between regimes may indicate that these systems exist at the minimum level of entropy production, a state of resilient self-organisation and

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sustainable control. This is consistent with the results of Lusch et al. [30], who established that deep learning architectures had the ability to find linear embeddings of nonlinear dynamics, which retained statistical invariants. That theoretical embeddings are consistent with the empirical entropy invariance found here lends credence to a common principle: nonlinear systems are linearly explainable in probabilistic state space when the analysis is performed in terms of statistical-mechanical parameters. Finally, as stated in the analysis, the cascaded-tanks benchmark is an ergodic ensemble of cascaded tanks that is nonlinear and in which fluctuation, correlations, and entropy production are mutually sustaining. This probabilistic model has been able to extrapolate deterministic system identification to the field of statistical mechanics and actually show that complex engineering systems can be modelled by ensemble statistics, rather than deterministic equations. The future directions of this area should be to apply such methods to other nonlinear systems- biological networks, or adaptive robotics - where energy exchange, entropy and feedback are the determinants of resilience and predictability of system behaviour.

5. Conclusion

This work gives one a single framework, which links statistical mechanics to nonlinear system identification by the analysis of a cascaded tanks benchmark dataset. The study shows that nonlinear dynamical systems are well characterised in the form of stochastic ensembles instead of being categorised as strictly deterministic mappings. The study is a combination of probabilistic methods like correlation analysis, probability density estimation, entropy quantification, and fluctuation assessment by embedding both deterministic structure and stochastic variability of nonlinear behaviour. The findings show that the system has dynamic equilibrium between the estimation and validation processes, and this means it has quasi-ergodic stability. Stable values of entropy and the fact that the correlation decays are finite confirm the fact that the system is self-regulating by balancing order and randomness. These results underline that nonlinear systems may display predictable probabilistic behaviour even in uncertain circumstances, pointing out the dual aspects of determinism and randomness in real-world dynamics. From a methodological perspective, the study would add a rigorous statistical-mechanical approach to nonlinear system analysis. It offers a framework that has the capabilities of quantifying complexities, stability, and uncertainty without having to be deterministic only. In practical terms, this method increases the interpretability of models, facilitates the robustness analysis and provides insight into system behaviour with different inputs or disturbances. Simply put, the paper comes to the conclusion that nonlinear systems are more appropriately viewed as energy-balanced, probabilistically stable objects that obey the laws of statistics. This vision leads to new avenues in the field of modelling, control, and optimisation in both engineering and physical systems where both stochasticity and nonlinearity exist simultaneously.

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