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A-CUT NEUTROSOPHIC Z-NUMBERS WITH SHAPLEY NORMALIZED WEIGHTED BONFERRONI MEAN AND ITS ALGEBRAIC PROPERTIES

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Abstract

Previous research has demonstrated that the Bonferroni Mean (BM) and its extensions possess valuable properties for capturing interrelationships between criteria in fuzzy multi-criteria decision-making (MCDM). However, most existing BM-based models are restricted to handling pairwise interactions, limiting their ability to represent complex dependencies among multiple criteria. To overcome the limitation mentioned, this research introduces a new Shapley-based Normalized Weighted Bonferroni Mean aggregation operator within the framework of α-cut Neutrosophic Z-numbers (α-NZNs). The proposed method integrates the α-cut technique to transform continuous neutrosophic Z-number information into interval representations at a given confidence level, thereby simplifying computations while retaining uncertainty characteristics. The Shapley measure is then employed to capture the overall interaction among all criteria, not only in pairs, by considering their cooperative importance. The developed α-cut NZN Shapley Normalized Weighted Bonferroni Mean (α-NZN-SNWBM) exhibits key mathematical properties including reducibility, monotonicity, commutativity, boundedness, and idempotency, reflecting its robustness in modelling complex relationships. Furthermore, a structured decision-making procedure is proposed, incorporating the α-NZN-SNWBM as the core aggregation step. A detailed numerical case study is presented to demonstrate the applicability of the approach, and sensitivity analysis confirms the stability and consistency of the ranking results across varying parameter values. The findings indicate that the proposed approach effectively captures higher-order interdependencies among criteria in uncertain and imprecise decision-making environments, offering a more comprehensive tool for real-world MCDM problems.

Keywords: α-cut Neutrosophic Z-numbers set, Bonferroni mean, Aggregation operator, Decision-making

Introduction

Recently, the focus in multi-criteria decision-making which known as (MCDM) has grown considerably, as its theories and methods have proven effective in addressing complex real-world problems. Among the most widely adopted frameworks in MCDM is fuzzy set (FS) theory, conducted by Zadeh [1], which has been extensively applied due to its ability to model vagueness, uncertainty, and subjectivity inherent in decision-making processes. FS-based methods have achieved remarkable success [2–4]; however, they describe only the level of membership and cannot explicitly represent non-membership, which limits their ability to capture certain aspects of uncertainty.

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To overcome this limitation, Atanassov [5] extended FS to intuitionistic fuzzy sets (IFS) by representing a non-membership function. Like FS, IFS has been applied extensively in MCDM, but both frameworks focus primarily on fuzziness and do not explicitly account for indeterminacy. This gap was addressed by Smarandache [6], who first conducted the neutrosophic set (NS), representing each component with three independent membership degrees: truth, falsity, and indeterminacy. The independence of these components makes NS more expressive in representing uncertain, inconsistent, and incomplete information.

Since its introduction, NS theory has been adapted into more practical forms, including the (INS) [8] and (SVNS) [7], which expresses truth, indeterminacy, and falsity memberships as intervals within [0,1]. Unlike IFS or interval-valued intuitionistic fuzzy sets (IVIFS), in which truth and falsity degrees considered dependent and indeterminacy is derived indirectly, INS models indeterminacy explicitly and independently. This flexibility allows INS to more effectively handle uncertainty and incomplete information, making it suitable for a wide range of real-world applications. Notably, INS was utilized in different areas, including similarity measures [9–11], entropy measures [11–14], correlation coefficients [15,16], and subset hood measures [17,18]. It has also been extended into different forms, such as the (IVNSS) [19], (INLS) [20], and the (INHFS) [21].

In MCDM, aggregation is a crucial step for combining information from multiple criteria, experts, or alternatives. Numerous aggregation operators have been developed, ranging from classical weighted and ordered means [22–26] to more advanced models such as Hamacher [27,28], power [29,30], Einstein [31,32], Choquet [33], Bonferroni [34], and Heronian [35,36] means. Among these, the Bonferroni mean (BM) stands out for its ability to capture interrelationships between input arguments, though it primarily considers pairwise interactions. Several extensions including (GBM) [37], (PBM) [38], and (NWBM) [39], have been proposed, but these also focus on pairwise dependencies.

To capture more complex interactions among all criteria, the Shapley fuzzy measure offers a powerful alternative. Unlike traditional probability measures constrained by additivity, Shapley measures can model superadditive or subadditive relationships, providing greater flexibility and accommodating incomplete weight information [40–43]. While BM and Shapley measures have been studied extensively in isolation, no existing work has integrated them within the α-cut neutrosophic z-numbers set framework.

This paper addresses this gap by proposing the α-cut neutrosophic z-numbers Shapley normalized weighted Bonferroni mean (α-NZN-SNWBM). This approach combines the strengths of NWBM and Shapley measures to account for both pairwise and overall interactions among criteria in the α-NZN environment. The proposed operator is analyzed in terms of desirable mathematical properties such as commutativity, monotonicity, idempotency, and boundedness, and its effectiveness is demonstrated through a numerical MCDM example.

2. Preliminaries

This part highlights the main definitions which will enhance the α-NZN-SNWBM aggregation operator.

2.1 The α-cut Neutrosophic with Z-numbers Set (α-NZNS)

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Definition 1. **[44]** Let X be a universe set with generic elements in B influenced by x. A neutrosophic set B in X is indicated by truth membership function $T_B(x)$, indeterminacy membership function $I_B(x)$ and falsity membership function $F_B(x)$. A neutrosophic set (NS) will be denoted as represented in Eq. (1).

$$B = \{x, \langle T_B(x), I_B(x), F_B(x)\rangle \backslash x \in X\} \quad (1)$$

In which the functions $T_B(x), I_B(x)$ and $F_B(x)$ are considered both real standard or non-standard subsets of $[^-0,1^+]$, that is $T_B(x) \to [^-0,1^+]$, $I_B(x) \to [^-0,1^+]$ and $F_B(x) \to [^-0,1^+]$. Thus, there is no restriction on the sum of $T_B(x), I_B(x)$ and $F_B(x)$.

$$0^- \leq \sup T_B(x) + \sup I_B(x) + \sup F_B(x) \leq 3^+ \quad (2)$$

Definition 2. Let $\check{S} = \{x, \langle (T_V^\alpha(x), T_R^\alpha(x)), (I_V^\alpha(x), \mathrm{I}_\mathrm{R}^\alpha(\mathrm{x})), (F_V^\alpha(x), F_R^\alpha(x))\rangle \backslash x \in X\}$, where $T_V^\alpha(x)$: Truth value at α-cut level, with $T_R^\alpha(x)$ as its associated reliability at α-level, $I_V^\alpha(x)$: Indeterminacy value at α-cut level, with $I_R^\alpha(x)$ as its associated reliability at α-level, $F_V^\alpha(x)$: Falsity value at α-cut level, with $F_R^\alpha(x)$ as its associated reliability at α-level. Each value component lies within [0,1] for a given $\alpha \in [0,1]$

$$0 \leq T_V^\alpha(x) + I_V^\alpha(x) + F_V^\alpha(x) \leq 3 \text{ and } 0 \leq T_R^\alpha(x) + I_R^\alpha(x) + F_R^\alpha(x) \leq 3 \quad (3)$$

The operational laws of α-NZN are defined as follows

Definition 3. Let $\check{A} = \langle (T_{V_1}^\alpha, T_{R_1}^\alpha), (I_{V_1}^\alpha, I_{R_1}^\alpha), (F_{V_1}^\alpha, F_{R_1}^\alpha)\rangle$ and $\check{B} = \langle (T_{V_2}^\alpha, T_{R_2}^\alpha), (I_{V_2}^\alpha, I_{R_2}^\alpha), (F_{V_2}^\alpha, F_{R_2}^\alpha)\rangle$, be two α-NZN Then,

i) Addition

$$\check{A} \oplus \check{B} = \langle (T_{V_1}^\alpha + T_{V_2}^\alpha - T_{V_1}^\alpha T_{V_2}^\alpha, T_{R_1}^\alpha + T_{R_2}^\alpha - T_{R_1}^\alpha T_{R_2}^\alpha), (I_{V_1}^\alpha I_{V_2}^\alpha, I_{R_1}^\alpha I_{R_2}^\alpha), (F_{V_1}^\alpha F_{V_2}^\alpha, F_{R_1}^\alpha F_{R_2}^\alpha)\rangle \quad (4)$$

ii) Multiplication

$$\check{A} \otimes \check{B} = \langle (T_{V_1}^\alpha T_{V_2}^\alpha, T_{R_1}^\alpha T_{R_2}^\alpha), (I_{V_1}^\alpha + I_{V_2}^\alpha - I_{V_1}^\alpha I_{V_2}^\alpha, I_{R_1}^\alpha + I_{R_2}^\alpha - I_{R_1}^\alpha I_{R_2}^\alpha), (F_{V_1}^\alpha + F_{V_2}^\alpha - F_{V_1}^\alpha F_{V_2}^\alpha, F_{R_1}^\alpha + F_{R_2}^\alpha - F_{R_1}^\alpha F_{R_2}^\alpha)\rangle \quad (5)$$

iii) Scalar Multiplication

$$\lambda\check{A} = \langle \left(1 - \left(1 - T_{V_1}^\alpha\right)^\lambda, 1 - \left(1 - T_{R_1}^\alpha\right)^\lambda\right), \left((I_{V_1}^\alpha)^\lambda, (I_{R_1}^\alpha)^\lambda\right), \left((F_{V_1}^\alpha)^\lambda, (F_{R_1}^\alpha)^\lambda\right)\rangle \quad (6)$$

where $\lambda > 0$

iv) Power

$$(\check{A})^\lambda = \langle \left((T_{V_1}^\alpha)^\lambda, (T_{R_1}^\alpha)^\lambda\right), \left(1 - \left(1 - I_{V_1}^\alpha\right)^\lambda, 1 - \left(1 - I_{R_1}^\alpha\right)^\lambda\right), \left(1 - \left(1 - F_{V_1}^\alpha\right)^\lambda, 1 - \left(1 - F_{R_1}^\alpha\right)^\lambda\right)\rangle \quad (7)$$

where $\lambda > 0$

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To compare the two presented α-NZN, the function score of α-NZN utilized in the study. It was conducted by Sahin [45] which represented as the following.

Definition 4 [45]. Let $\breve{A}$ in previous definition be α-NZN, then the score function $Y_{(\breve{A})}$ is defined follows:

$$Y_{(\breve{A})}=\frac{\left(2+T^{\alpha}_{V_i}+T^{\alpha}_{R_i}-2\,I^{\alpha}_{V_i}-2\,I^{\alpha}_{R_i}-F^{\alpha}_{V_i}-F^{\alpha}_{R_i}\right)}{4}, \text{ for } Y_{(\breve{A}_i)}\in[0,1] \tag{8}$$

Respectively, if $\mathrm{Y}_{(\breve{\mathrm{A}})} > \mathrm{Y}_{(\breve{\mathrm{B}})}$, where $Y_{(\breve{A})} \in [0, 1]$ and supposed $\breve{A}$ and $\breve{B}$ are two α-NZN then $\breve{A}$ is bigger than $\breve{B}$, represented by $\breve{A} > \breve{B}$.

2.2 Shapley Fuzzy Measure

The Shapley fuzzy measure that considered as an instrument that can measure the correlations among input arguments and its coalition for this context, the fuzzy measure's concepts.
Fuzzy measure that considers as a significant method in capturing the connection between the components was firstly introduced by Sugeno [46]. This approach has been considered in different areas such as game theory and decision-making. The terms of λ -fuzzy measure are represented as the following

Definition 5. [46] Let $N = \{1,\ldots, n\}$ be a set of criteria and $P(N)$ be a set of function of N. Then a fuzzy measure on N is a set function μ: $P(N) \rightarrow [0,1]$ which satisfies the following axioms:
i. Boundary condition: $\mu(\emptyset) = 0$ and $\mu(N) = 1$.
ii. Monotonicity: if S, $T \in P(\mathrm{N})$ and $S \subseteq T$ then $\mu(S) \subseteq \mu(T)$.

Aiming to have simplicity, [46] introduced a significant type of fuzzy measure that is λ -fuzzy measure $P(N)$.

$$\mu(S \cup T) = \mu(S) + \mu(T) - \lambda\mu(S) \,.\, \mu(T). \tag{9}$$

where $S \cap T = \emptyset$ for all S, $T \in P(N)$ and $\lambda > -1$.

1) If $\lambda = 0$, then $\mu(S \cup T) = \mu(S) + \mu(T)$. This issue is known as an additive measure that does not indicate any interaction with S and T.
2) If $\lambda > 0$ then then $(S \cup T) > \mu(S) + \mu(T)$. This issue shows interaction positively synergic occurs with S and T and the set S, T have a multiplicative effect.
3) If $\lambda < 0$, then $\mu(S \cup T) < \mu(S) + \mu(T)$. The set S, T has a substitutive impact and it has a negatively synergic connection with S and T.
Feather, fuzzy measure will be represented as below:

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$$\mu_\lambda(S) = \begin{cases} \frac{1}{\lambda}\left(\prod_{C_i=S}^{n}\left(1+\lambda\mu_\lambda(\{C_i\})\right)-1\right) & ,\text{if } \lambda \neq 0 \\ \sum_{C_i=S}^{n}\mu_\lambda(\{C_i\}) & ,\text{if } \lambda = 0 \end{cases} \tag{10}$$

and the λ value will be obtained by finding the solution of the equation:

$$\prod_{C_i=S}^{n}\left(1+\lambda\mu_\lambda(\{C_i\})\right) = \lambda+1 \tag{11}$$

According to the fuzzy measures and fuzzy integral, Shapley (1953) proposed Shapley fuzzy measure. The λ-fuzzy measure has a significant impact on the measure of Shapley fuzzy. However, traditional weighting methods, the fuzzy measure can reflect both the importance of each individual criterion and the interactions among criteria. According to the fuzzy measure definition which stated in Definition 5, the Shapley fuzzy measure is represented as the following

$$\varphi_i(\mu, N) = \sum_{\substack{i,j=1 \\ i\neq j}}^{n}\frac{(|N|-|V|-1)!\,|V|!}{|N|!}\left(\mu(V\cup i)-\mu(V)\right),\ \forall i \in N \tag{12}$$

where μ possess a fuzzy measure on N. $|N|$ and $|V|$ highlights the cardinality of the set of N and V. $\varphi_i(\mu, N)$ is obviously referred to the weight on set N as $\varphi_i(\mu, N) > 0$ and $\sum_{\substack{i,j=1 \\ i\neq j}}^{n}\varphi_i(\mu, N) = 1$

2.3 Bonferroni mean (BM)

Bonferroni (1950) was the first to introduce the Bonferroni mean, to handle the connections between the individual input arguments. The definition of BM aggregation operator is given according the following

Definition 6. [47] Let $g, k \geq 0$, and A_i (i =1,.., n) be a set of positive numbers, then BM operator is defined as follows:

$$BM^{g,k}(A_i) = \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1 \\ i\neq j}}^{n} A_i^g A_j^k\right)^{\frac{1}{g+k}} \tag{13}$$

The BM possess a desirable characteristic including monotonicity, commutativity, boundedness and idempotency. [47] Extended BM operator to weighted BM (WBM) defined as below:

Definition 7. [47] Let $g, k \geq 0$ and A_i be a set of positive numbers . $w_i = (w_1, w_2, \ldots, w_n)$ is `the weight vector of A_i where w_i shows the significant degree of A_i, satisfying $w_i > 0$ and $\sum_{\substack{i,j=1 \\ i\neq j}}^{n} w_i = 1$. Then WBM is calculated as the following:

$$WBM^{g,k}(A_i) = \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1 \\ i\neq j}}^{n} (w_i A_i)^g (w_j A_j)^k\right)^{\frac{1}{g+k}} \tag{14}$$

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The WBM operator possesses satisfied the desirable characteristics of commutativity, boundedness and monotonicity but not satisfied the idempotency feature.
A clear extended version of BM was introduced by Zhou and He (2012) known as the normalized weighted Bonferroni mean (NWBM). The definition of NWBM is stated as the following

Definition 8. [48] Let g, $k \geq 0$, and A_i be a set of positive numbers, $w_i = (w_1, w_2, \dots, w_n)$ is the weight vector of A_i where w_i shows the significant degree of A_i, satisfying $w_i > 0$ and $\sum_{\substack{i,j=1 \\ i\neq j}}^{n} w_i = 1$. Then

$$NWBM^{g,k}(A_i) = \left(\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \frac{w_i w_j}{1-w_i} A_i^g A_j^k \right)^{\frac{1}{g+k}} \quad (15)$$

The NWBM operator carries important characteristics for any aggregation operators including reducibility, boundedness, idempotency, commutativity and monotonicity which have the ability to show the connections among the aggregated arguments.

3. Bonferroni Mean Operators with α-cut Neutrosophic Z-numbers Shapley Normalized Weighted

This part introduced α-cut neutrosophic with z-numbers Shapley normalized weighted Bonferroni mean (α-NZN-SNWBM) aggregation operator and mentions some feaatures that satisfy the introduced operator. The α-NZN-SNWBM is conducted with the combination of Shapley fuzzy measure and Bonferroni mean concepts in the α-cut neutrosophic with z-numbers. The represented aggregation operator is considered as the interactions among input arguments as well as their combined effects.
Preceding the action of presenting the α-NZN-SNWBM deeply, BM operator into α-cut neutrosophic with z-numbers environment should be applied which called the α-NZN-BM aggregation operator.

3.1 The α-NZN-BM and α-NZN-SNWBM Operator

Definition 9. Let $\check{S}_i (i=1,..,n)$ be a set of α-NZN. Then,

$$\alpha\text{-}NZN\text{-}BM^{g,k}(\check{S}_i) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} (\check{S}_i^g \otimes \check{S}_j^k) \right)^{\frac{1}{g+k}} \quad (16)$$

Then α-NZN-$BM^{g,k}$ is called the α-cut Neutrosophic Z-numbers Bonferroni mean (α-NZN-BM). Subsequently, this study will clearly propose the α-NZN-SNWBM operator.

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Definition 10. Let $\check{S}_i$(i =1,..,n) be a set of α-NZN. Then,

$$\alpha\text{-}NZN\text{-}SNWBM^{g,k}\left(\check{S}_i\right)=\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}\left(\check{S}_i^g\otimes\check{S}_j^k\right)\right)^{\frac{1}{g+k}} \quad (17)$$

where $\varphi_i(\mu,N)$ is the Shapley fuzzy measure taking into consideration the associated fuzzy measure μ on N satisfying $\varphi_i(\mu,N)\in[0,1]$ and $\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varphi_i(\mu,N)=1$, then *α-NZN-SNWBM* ^{g,k} is called the α-cut Neutrosophic Z-numbers Shapley Normalized Weighted Bonferroni Mean (α-NZN -SNWBM).

Theorem 1. Let g, $k\geq 0$ and $\check{S}_i$ (i =1,..,n) be a set of α-NZN, resulting the aggregated value by the Eq. (17) and value of α-NZN, can be represent by

$$\alpha\text{-}NZN\text{-}SNWBM^{g,k}\left(\check{S}_i\right)=$$

$$\left\langle\left(\left(1-\prod_{J=1}^{n}\left(1-(T_{V_i}^{\alpha})^g(T_{V_j}^{\alpha})^k\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{g+k}},\left(1-\prod_{J=1}^{n}\left(1-(T_{R_i}^{\alpha})^g(T_{R_j}^{\alpha})^k\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{g+k}}\right),\right.$$

$$\left(\left(\prod_{J=1}^{n}\left((I_{V_i}^{\alpha})^g(I_{V_j}^{\alpha})^k\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{g+k}},\left(\prod_{J=1}^{n}\left((I_{R_i}^{\alpha})^g(I_{R_j}^{\alpha})^k\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{g+k}}\right),$$

$$\left.\left(\left(\prod_{J=1}^{n}\left((F_{V_i}^{\alpha})^g(F_{V_j}^{\alpha})^k\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{g+k}},\left(\prod_{J=1}^{n}\left((F_{R_i}^{\alpha})^g(F_{R_j}^{\alpha})^k\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{g+k}}\right)\right\rangle. \quad (18)$$

where i, j = 1,…,n

*Proof***.** By the power law of α-NZN in Eq. (7), we have

$(\check{S}_i)^g=\langle((T_V^{\alpha})^g,(T_R^{\alpha})^k),(1-(1-I_V^{\alpha})^g,1-(1-I_R^{\alpha})^g),(1-(1-F_V^{\alpha})^g,1-(1-F_R^{\alpha})^g\rangle.$

Product rule of α-NZN in Eq. (5), we have

$(\check{S}_i)^g\otimes(\check{S}_j)^k$=$\langle((T_{V_i}^{\alpha})^g\left(T_{V_j}^{\alpha}\right)^k,(T_{R_i}^{\alpha})^g\left(T_{R_j}^{\alpha}\right)^k),((I_{V_i}^{\alpha})^g+\left(I_{V_j}^{\alpha}\right)^k-(I_{V_i}^{\alpha})^g\left(I_{V_j}^{\alpha}\right)^k,(I_{R_i}^{\alpha})^g+\left(I_{R_j}^{\alpha}\right)^k$ $-(I_{R_i}^{\alpha})^g\left(I_{R_j}^{\alpha}\right)^k),((F_{V_i}^{\alpha})^g+\left(F_{V_j}^{\alpha}\right)^k-(F_{V_i}^{\alpha})^g\left(F_{V_j}^{\alpha}\right)^k,(F_{R_i}^{\alpha})^g+\left(F_{R_j}^{\alpha}\right)^k-(F_{R_i}^{\alpha})^g\left(F_{R_j}^{\alpha}\right)^k)\rangle.$

Consequently, using a scalar multiplication rule of α-NZN in Eq. (6), the result is:

$\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}\left(\check{S}_i^g\otimes\check{S}_j^k\right)=$

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$$\langle\left(1-\left(1-\left(T_{V_i}^{\alpha}\right)^{g}\left(T_{V_j}^{\alpha}\right)^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}},1-\left(1-\left(T_{R_i}^{\alpha}\right)^{g}\left(T_{R_j}^{\alpha}\right)^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right),$$

$$\left(\left(\left(I_{V_i}^{\alpha}\right)^{g}+\left(I_{V_j}^{\alpha}\right)^{k}-\left(I_{V_i}^{\alpha}\right)^{g}\left(I_{V_j}^{\alpha}\right)^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}},\left(\left(I_{R_i}^{\alpha}\right)^{g}+\left(I_{R_j}^{\alpha}\right)^{k}-\left(I_{R_i}^{\alpha}\right)^{g}\left(I_{R_j}^{\alpha}\right)^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right),$$

$$\left(\left(\left(F_{V_i}^{\alpha}\right)^{g}+\left(F_{V_j}^{\alpha}\right)^{k}-\left(F_{V_i}^{\alpha}\right)^{g}\left(F_{V_j}^{\alpha}\right)^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}},\left(\left(F_{R_i}^{\alpha}\right)^{g}+\left(F_{R_j}^{\alpha}\right)^{k}-\right.\right.$$

$$\left.\left.\left(F_{R_i}^{\alpha}\right)^{g}\left(F_{R_j}^{\alpha}\right)^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)\rangle.$$

By addition rule of α-NZN in Eq. (4), we have

$$\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\varphi_i\varphi_j}{1-\varphi_i}\left(\check{S}_i^{g}\otimes\check{S}_j^{k}\right)=$$

$$\langle\begin{pmatrix}1-\prod_{k=1}^{n}\left(1-\left(T_{V_i}^{\alpha}\right)^{g}\left(T_{V_j}^{\alpha}\right)^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}},\\ 1-\prod_{k=1}^{n}\left(1-\left(T_{R_i}^{\alpha}\right)^{g}\left(T_{R_j}^{\alpha}\right)^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\end{pmatrix},\begin{pmatrix}\prod_{k=1}^{n}\left(\left(I_{V_i}^{\alpha}\right)^{g}\left(I_{V_j}^{\alpha}\right)^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}},\\ \prod_{k=1}^{n}\left(\left(I_{R_i}^{\alpha}\right)^{g}\left(I_{R_j}^{\alpha}\right)^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\end{pmatrix},$$

$$\begin{pmatrix}\prod_{k=1}^{n}\left(\left(F_{V_i}^{\alpha}\right)^{g}\left(F_{V_j}^{\alpha}\right)^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}},\\ \prod_{k=1}^{n}\left(\left(F_{R_i}^{\alpha}\right)^{g}\left(F_{R_j}^{\alpha}\right)^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\end{pmatrix}\rangle.$$

Consequently, using the power rule of α-NZN in (4), the result is

$$\alpha\text{-}NZN\text{-}SNWBM^{g,k}\left(\check{S}_i\right)=\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}\left(\left(\check{S}_i\right)^{g}\otimes\left(\check{S}_j\right)^{k}\right)\right)$$

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$$=\left\langle \begin{pmatrix} \left(1-\prod_{J=1}^{n}\left(1-\ (T^{\alpha}_{V_i})^{g}(T^{\alpha}_{V_j})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{g+k}}, \\ \left(1-\prod_{J=1}^{n}\left(1-\ (T^{\alpha}_{R_i})^{g}(T^{\alpha}_{R_j})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{g+k}} \end{pmatrix}, \begin{pmatrix} \left(\prod_{J=1}^{n}\left((I^{\alpha}_{V_i})^{g}(I^{\alpha}_{V_j})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{g+k}}, \\ \left(\prod_{J=1}^{n}\left((I^{\alpha}_{R_i})^{g}(I^{\alpha}_{R_j})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{g+k}} \end{pmatrix}, \begin{pmatrix} \left(\prod_{J=1}^{n}\left((F^{\alpha}_{V_i})^{g}(F^{\alpha}_{V_j})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{g+k}}, \\ \left(\prod_{J=1}^{n}\left((F^{\alpha}_{R_i})^{g}(F^{\alpha}_{R_j})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{g+k}} \end{pmatrix} \right\rangle.$$

which completes the proof.

The theorems of reducibility, idempotency, commutativity, monotonicity and boundedness of α-NZN-SNWBM were introduced, accompanied with mathematical proofs.
In what follows, we will prove that *α-NZN-SNWBM* clustering operator has the following desirable properties.

Theorem 2 (Reducibility**)** If $\varphi_i(\mu,N)=(\frac{1}{q},\frac{1}{q},\dots,\frac{1}{q})^T$, then

$$\alpha\text{-}NZN\text{-}SNWBM^{g,k}\left(\check{S}_i\right)=\alpha\text{-}NZN\text{-}BM^{g,k}\left(\check{S}_i\right) \tag{19}$$

Proof. Since $\varphi_i(\mu,N)=(\frac{1}{q})$, then according to Eq. (17), we get

$$\alpha\text{-}NZN\text{-}SNWBM^{g,k}\left(\check{S}_i\right)=\left(\sum_{\substack{i,j=1\\ i\neq j}}^{m}\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}(\check{S}_i^{g}\otimes\check{S}_j^{k})\right)^{\frac{1}{g+k}}$$

$$=\left(\sum_{\substack{i,j=1\\ i\neq j}}^{m}\frac{(\frac{1}{q})(\frac{1}{q})}{1-(\frac{1}{q})}(\check{S}_i^{g}\otimes\check{S}_j^{k})\right)^{\frac{1}{g+k}}\Rightarrow\left(\sum_{\substack{i,j=1\\ i\neq j}}^{m}\frac{1}{q(q-1)}(\check{S}_i^{g}\otimes\check{S}_j^{k})\right)^{\frac{1}{g+k}}=\alpha\text{-}NZN\text{-}BM^{g,k}\left(\check{S}_1,\check{S}_2,\dots\check{S}_m\right)$$

Then it proves Theorem 2.

The availability of the reduction property means that the operator or the aggregation method behaves in a manner consistent with reality; when the input information is identical, then the final evaluation will not be affected.

Theorem 3 (Idempotency**)** Let $\check{S}_i$(i = 1,…,m) be a set of α-NZN and are all equal $\check{S}_i=\check{S}$, for all i. Then,

$$\alpha\text{-}NZN\text{-}SNWBM^{g,k}\left(\check{S}_i\right)=\check{S}. \tag{20}$$

Proof. Since $\check{S}_i=\check{S}$ for all i, we get

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α-NZN-$SNWBM^{g,k}\left(\check{S}_i\right) = \left(\sum_{\substack{i,j=1 \\ i\neq j}}^{m} \frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)} (\check{S}_i^{g} \otimes \check{S}_j^{k})\right)^{\frac{1}{g+k}}$

$= \left(\sum_{\substack{i,j=1 \\ i\neq j}}^{m} \frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)} (\check{S}_i^{g} \otimes \check{S}_j^{k})\right)^{\frac{1}{g+k}} \Rightarrow \left(\sum_{\substack{i,j=1 \\ i\neq j}}^{m} \frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)} (\check{S}^{g+k})\right)^{\frac{1}{g+k}}$

$= \check{S}\left(\sum_{\substack{i,j=1 \\ i\neq j}}^{m} \frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)} (\check{S}^{g+k})\right)^{\frac{1}{g+k}} \Rightarrow \check{S}.$

Then it proves Theorem 3.

Idempotence is another property of α-NZN-$SNWBM^{g,k}\left(\check{S}_i\right)$, which explains the repetition of addition and multiplication operations without changing the initial results.

Theorem 4 (Commutativity) Let $\tilde{\check{\boldsymbol{S}}}_{\boldsymbol{i}}$ be any permutation of $\check{\boldsymbol{S}}_{\boldsymbol{i}}$, then

$$\alpha\text{-}NZN\text{-}SNWBM^{g,k}\left(\tilde{\check{S}}_i\right) = \alpha\text{-}NZN\text{-}SNWBM^{g,k}\left(\check{S}_i\right) \tag{21}$$

Proof. Since $\tilde{\check{S}}_i$ be any permutation of $\check{S}_i$, then

$\left(\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)} (\tilde{\check{S}}_i^{g} \otimes \tilde{\check{S}}_j^{k})\right)^{\frac{1}{g+k}} = \left(\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)} (\tilde{\check{S}}_i^{g} \otimes \tilde{\check{S}}_j^{k})\right)^{\frac{1}{g+k}}$.Thus,

α-NZN-$SNWBM^{g,k}\left(\tilde{\check{S}}_i\right) = \alpha$-$NZN$-$SNWBM^{g,k}\left(\check{S}_i\right)$

Then it proves Theorem 4.

Theorem Commutativity is algebra theorem that allows us to swap numbers and get the same answer. This theorem is applicable to the proposed α-NZN-SNWBM $^{g,k}\left(\check{S}_i\right)$.

Theorem 5 (Monotonicity**)** Let $\check{S}_i$ and $\check{S}^*_i$ be two sets of α-NZNs where(i =1,..,n).

If $\check{S}_i \le \check{S}^*_i$, then α-NZN-$SNWBM^{g,k}\left(\check{S}_i\right) \le \alpha$-$NZN$-$SNWBM^{g,k}\left(\check{S}^*_i\right)$ (22)

Proof.

1) For the truth membership part of α-NZN

Since $T_{V_i}^{\alpha} \le T_{V_i}^{\alpha\,*}$ and $T_{V_j}^{\alpha} \le T_{V_j}^{\alpha\,*}$ for all i and j with $g,\ k > 0$, then for the lower bounded of truth membership function, we have

$(T_{V_i}^{\alpha})^{g} \le (T_{V_i}^{\alpha\,*})^{g}$, $(T_{V_j}^{\alpha})^{k} \le (T_{V_j}^{\alpha\,*})^{k} \Rightarrow$1-$(T_{V_i}^{\alpha})^{g}(T_{V_j}^{\alpha})^{k} \ge$ 1-$(T_{V_i}^{\alpha\,*})^{g}(T_{V_j}^{\alpha\,*})^{k}$

$\Rightarrow \prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(1-(T_{V_i}^{\alpha})^{g}(T_{V_j}^{\alpha})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}} \ge \prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(1-(T_{V_i}^{\alpha\,*})^{g}(T_{V_j}^{\alpha\,*})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}$

$\Rightarrow$1-$\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(1-(T_{V_i}^{\alpha})^{g}(T_{V_j}^{\alpha})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}} \le$ 1-$\prod_{\substack{i,j=1 \\ i\neq j}}^{n}\left(1-(T_{V_i}^{\alpha\,*})^{g}(T_{V_j}^{\alpha\,*})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}$

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$$\Rightarrow\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(T_{V_i}^{\alpha})^{g}(T_{V_j}^{\alpha})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{g+k}}\leq\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(T_{V_i}^{\alpha *})^{g}(T_{V_j}^{\alpha *})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{g+k}}$$

2) For the indeterminacy member ship part of α-NZN

Since $I_{V_i}^{\alpha}\leq I_{V_i}^{\alpha *}$ and $I_{V_j}^{\alpha}\leq I_{V_J}^{\alpha *}$ for all *i* and *j* with *g*, $k>0$, then we have

$$\Rightarrow I_{V_i}^{\alpha}\leq I_{V_i}^{\alpha *},\ I_{V_j}^{\alpha}\leq I_{V_J}^{\alpha *}\ \Rightarrow\ (I_{V_i}^{\alpha})^{g}(I_{V_j}^{\alpha})^{k}\leq(I_{V_i}^{\alpha *})^{g}(I_{V_j}^{\alpha *})^{k}$$

$$\Rightarrow\prod_{J=1}^{n}\left((I_{V_i}^{\alpha})^{g}(I_{V_j}^{\alpha})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\leq\prod_{J=1}^{n}\left((I_{V_i}^{\alpha *})^{g}(I_{V_j}^{\alpha *})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}$$

$$\Rightarrow\left(\prod_{J=1}^{n}\left((I_{V_i}^{\alpha})^{g}(I_{V_j}^{\alpha})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{g+k}}\leq\left(\prod_{J=1}^{n}\left((I_{V_i}^{\alpha *})^{g}(I_{V_j}^{\alpha *})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{g+k}}$$

3) For the falsity member ship part of α-NZN

Since $F_{V_i}^{\alpha}\leq F_{V_i}^{\alpha *}$ and $F_{V_j}^{\alpha}\leq F_{V_J}^{\alpha *}$ for all *i* and *j* with *g*, $k>0$, then we have

$$\Rightarrow F_{V_i}^{\alpha}\leq F_{V_i}^{\alpha *},\ F_{V_j}^{\alpha}\leq F_{V_J}^{\alpha *}\ \Rightarrow(F_{V_i}^{\alpha})^{g}(F_{V_j}^{\alpha})^{k}\leq(F_{V_i}^{\alpha *})^{g}(F_{V_j}^{\alpha *})^{k}$$

$$\Rightarrow\prod_{J=1}^{n}\left((F_{V_i}^{\alpha})^{g}(F_{V_j}^{\alpha})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\leq\prod_{J=1}^{n}\left((F_{V_i}^{\alpha *})^{g}(F_{V_j}^{\alpha *})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}$$

$$\Rightarrow\left(\prod_{J=1}^{n}\left((F_{V_i}^{\alpha})^{g}(F_{V_j}^{\alpha})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{g+k}}\leq\left(\prod_{J=1}^{n}\left((F_{V_i}^{\alpha *})^{g}(F_{V_j}^{\alpha *})^{k}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{g+k}}$$

Monotonicity considered as a property to validate α-NZN-SNWBM ^{g,k} can clearly strictly increasing or strictly decreasing.

Theorem 6 (Boundedness**)** Let $\check{\boldsymbol{S}}_{\boldsymbol{i}}$($i$ =1,..,n) be a set of α-NZN and

$\check{S}_{min}$=⟨$\min\left(T_{V_i}^{\alpha},T_{R_i}^{\alpha}\right),\max\left(I_{V_i}^{\alpha},I_{R_i}^{\alpha}\right),\max\left(F_{V_i}^{\alpha},F_{R_i}^{\alpha}\right)$⟩

$\check{S}_{\max}$=⟨$\max\left(T_{V_i}^{\alpha},T_{R_i}^{\alpha}\right),\min\left(I_{V_i}^{\alpha},I_{R_i}^{\alpha}\right),\min\left(F_{V_i}^{\alpha},F_{R_i}^{\alpha}\right)$ ⟩, then

$\check{S}_{\min}\leq$ *α-NZN-SNWBM* $^{g,k}\left(\check{S}_{i}\right)\leq\check{S}_{\max}$

Proof. Since $\check{S}_{i}\geq\check{S}_{\max}$, we have

α-NZN-SNWBM $^{g,k}\left(\check{S}_{i}\right)\geq$ *α-NZN-SNWBM* $^{g,k}\left(\check{S}_{\min},\check{S}_{\min},\ldots..,\check{S}_{\min}\right)=\check{S}_{\min}$.

Like, we can get

α-NZN-SNWBM $^{g,k}\left(\check{S}_{i}\right)\leq$ *α-NZN-SNWBM* $^{g,k}\left(\check{S}_{\max},\check{S}_{\max},\ldots..,\check{S}_{\max}\right)=\check{S}_{\max}$. Then,

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$$\check{S}_{\min} \leq \alpha\text{-}NZN\text{-}SNWBM^{g,k}\left(\check{S}_i\right) \leq \check{S}_{\max} \qquad (23)$$

As we have seen, this theorem indicates when the α-NZN-SNWBM remains stable on the closed and bounded interval, so the α-NZN-SNWBM is a bounded function.

3.2 Cases the α-NZN-SNWBM Operator

The researchers will explain other special case shows that α-NZN-SNWBM aggregation operator.

1- If $k = 0$, then

$$\alpha\text{-}NZN\text{-}SNWBM^{g,0}\left(\check{S}_i\right) = \left(\sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}\left(\check{S}_i^{\,g} \otimes \check{S}_j^{\,0}\right)\right)^{\frac{1}{g+k}} = \left(\sum_{\substack{i,j=1\\ i\neq j}}^{n} \varphi_i(\mu,N)\ \check{S}_i^{\,g}\right)^{\frac{1}{g}} \qquad (24)$$

$$=\left\langle \begin{pmatrix} \left(1-\prod_{J=1}^{n}\left(1-(T^{\alpha}_{V_i})^{g}\right)^{\varphi_i(\mu,N)}\right)^{\frac{1}{g}}, \\ \left(1-\prod_{J=1}^{n}\left(1-(T^{\alpha}_{R_i})^{g}\right)^{\varphi_i(\mu,N)}\right)^{\frac{1}{g}} \end{pmatrix}, \begin{pmatrix} \left(\prod_{J=1}^{n}\left((I^{\alpha}_{V_i})^{g}\right)^{\varphi_i(\mu,N)}\right)^{\frac{1}{g}}, \\ \left(\prod_{J=1}^{n}\left((I^{\alpha}_{R_i})^{g}\right)^{\varphi_i(\mu,N)}\right)^{\frac{1}{g}} \end{pmatrix}, \begin{pmatrix} \left(\prod_{J=1}^{n}\left((F^{\alpha}_{V_i})^{g}\right)^{\varphi_i(\mu,N)}\right)^{\frac{1}{g}}, \\ \left(\prod_{J=1}^{n}\left((F^{\alpha}_{R_i})^{g}\right)^{\varphi_i(\mu,N)}\right)^{\frac{1}{g}} \end{pmatrix} \right\rangle$$

.

2- If $g = 1$ and $k = 0$, then

$$\alpha\text{-}NZN\text{-}SNWBM^{1,0}\left(\check{S}_i\right)=\left\langle\left(1-\prod_{J=1}^{n}\left(1-(T^{\alpha}_{V_i})\right)^{\varphi_i(\mu,N)},\ 1-\prod_{J=1}^{n}\left(1-(T^{\alpha}_{R_i})\right)^{\varphi_i(\mu,N)}\right),\right.$$
$$\left.\left(\prod_{J=1}^{n}\left(I^{\alpha}_{V_i}\right)^{\varphi_i(\mu,N)},\ \prod_{J=1}^{n}\left(I^{\alpha}_{R_i}\right)^{\varphi_i(\mu,N)}\right), \left(\prod_{J=1}^{n}\left(F^{\alpha}_{V_i}\right)^{\varphi_i(\mu,N)},\ \prod_{J=1}^{n}\left(F^{\alpha}_{R_i}\right)^{\varphi_i(\mu,N)}\right)\right\rangle.$$
$$\Rightarrow \alpha\text{-}NZN\text{-}WA(S^{\alpha}_{Z_1}, S^{\alpha}_{Z_2},\ldots, S^{\alpha}_{Z_n}) \qquad (25)$$

which is called the α-cut neutrosophic z-numbers weighted average (α-NZN-WA) aggregation operator.

3- If $g = 0$, $k = 0$, then α-NZN-SNWBM $0,0\left(\check{S}_i\right) = \sum_{\substack{i,j=1\\ i\neq j}}^{n}(S^{\alpha}_{Z_n})^{\varphi_i(\mu,N)}$

$$=\left\langle\left(\prod_{J=1}^{n}\left(T^{\alpha}_{V_i}\right)^{\varphi_i(\mu,N)},\prod_{J=1}^{n}\left((T^{\alpha}_{R_i})\right)^{\varphi_i(\mu,N)}\right), \left(1-\prod_{J=1}^{n}\left(1-(I^{\alpha}_{V_i})\right)^{\varphi_i(\mu,N)},1-\prod_{J=1}^{n}\left(1-(I^{\alpha}_{R_i})\right)^{\varphi_i(\mu,N)}\right), \left(1-\prod_{J=1}^{n}\left(1-(F^{\alpha}_{V_i})\right)^{\varphi_i(\mu,N)},1-\prod_{J=1}^{n}\left(1-(F^{\alpha}_{R_i})\right)^{\varphi_i(\mu,N)}\right)\right\rangle. \qquad (26)$$

$\Rightarrow$ α-NZN-WG $(S^{\alpha}_{Z_1}, S^{\alpha}_{Z_2},\ldots, S^{\alpha}_{Z_n})$. That is called the α-cut neutrosophic z-numbers weighted geometric (α-NZN-WG) aggregation operator.

4- If $g = k$, then

$$\alpha\text{-}NZN\text{-}SNWBM^{g,g}\left(\check{S}_i\right) = \left(\sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}\left(\check{S}_i^{g} \otimes \check{S}_j^{g}\right)\right)^{\frac{1}{g+g}} = \left(\sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}\left(\check{S}_i\check{S}_j\right)^{g}\right)^{\frac{1}{2g}}$$

i.e.

α-NZN-SNWBM$^{g,g}\left(\check{S}_i\right)=$

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$$\langle\left(\left(1-\prod_{J=1}^{n}\left(1-\ (T_{V_i}^{\alpha}T_{V_j}^{\alpha})^{g}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{2g}},\left(1-\prod_{J=1}^{n}\left(1-\ (T_{R_i}^{\alpha}T_{R_j}^{\alpha})^{g}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{2g}}\right),$$

$$\left(\left(\prod_{J=1}^{n}\left((I_{V_i}^{\alpha}I_{V_j}^{\alpha})^{g}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{2g}},\left(\prod_{J=1}^{n}\left(\ (I_{R_i}^{\alpha}I_{R_j}^{\alpha})^{g}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{2g}}\right),$$

$$\left(\left(\prod_{J=1}^{n}\left((F_{V_i}^{\alpha}F_{V_j}^{\alpha})^{g}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{2g}},\left(\prod_{J=1}^{n}\left((F_{R_i}^{\alpha}F_{R_j}^{\alpha})^{g}\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{2g}}\right)\rangle. \tag{27}$$

5- If $g = k = 1$, then

$$\alpha\text{-}NZN\text{-}SNWBM^{1,1}(\check{S}_i)=\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}\left(\ \check{S}_i^{g}\otimes\check{S}_j^{g}\right)\right)^{\frac{1}{2}}=\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}(\check{S}_i\check{S}_j)\right)^{\frac{1}{2}}$$

i.e,

$$\alpha\text{-}NZN\text{-}SNWBM^{1,1}(\check{S}_i)=$$

$$\langle\left(\left(1-\prod_{J=1}^{n}\left(1-\ (T_{V_i}^{\alpha}T_{V_j}^{\alpha})\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{2}},\left(1-\prod_{J=1}^{n}\left(1-\ (T_{R_i}^{\alpha}T_{R_j}^{\alpha})\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{2}}\right),$$

$$\left(\left(\prod_{J=1}^{n}\left((I_{V_i}^{\alpha}I_{V_j}^{\alpha})\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{2}},\left(\prod_{J=1}^{n}\left((I_{R_i}^{\alpha}I_{R_j}^{\alpha})\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{2}}\right),$$

$$\left(\left(\prod_{J=1}^{n}\left((F_{V_i}^{\alpha}F_{V_j}^{\alpha})\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{2}},\left(\prod_{J=1}^{n}\left((F_{R_i}^{\alpha}F_{R_j}^{\alpha})\right)^{\frac{\varphi_i(\mu,N)\varphi_j(\mu,N)}{1-\varphi_i(\mu,N)}}\right)^{\frac{1}{2}}\right)\rangle.$$

which we call the α-cut neutrosophic z-numbers weighted interrelated average (α-NZN-WIA) operator. The definition of α-NZN-SNWBM and its algebraic properties have been investigated. We could see the application of the proposed α-NZN-SNWBM to decision-making problem in the subsequent section.

4. MCDM Approach Based on α-NZN-SNWBM Aggregation Operator

An MCDM approach based on the proposed α-NZN-SNWBM operator is developed in this section.
Let $\check{A}_i = \{\check{A}_1, \check{A}_2 .., \check{A}_m\}$ and $C_j=\{c_1, c_2, \dots. c_n\}$ be a set of alternatives and criteria respectively. An assessment is carried out by the pairwise comparison on *i*-th alternative with reflecting to *j*-th criterion using the linguistic variables of α-cut neutrosophic z-numbers. The successive phases of the presented decision-making method utilizing the α-NZN-SNWBM are shown as the following:

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Step 1. Create a decision matrix *D,* after surveying experts, $D = \langle T_{Z_{ij}}, I_{Z_{ij}}, F_{Z_{ij}} \rangle_{(m\times n)}$ in the MCDM.

Step2. Set the value of α-cut to redues the number of alternative, then $D^{\alpha} = \langle T^{\alpha}_{Z_{ij}}, I^{\alpha}_{Z_{ij}}, F^{\alpha}_{Z_{ij}} \rangle_{(m\times n)}$

Step 3. Shapley fuzzy measure.

The Shapley fuzzy measure can be obtained by Equations (11-12) and the fuzzy measure of each criterion is given by experts.

Step 4. Comprehensive evaluation.

The comprehensive evaluation value of each alternative A_i {i = 1,...,m) can be calculated using the α-NZN-SNWBM operator given in Eq. (18) and the obtained Shapley fuzzy measure in Step 3.

Step 5. Score function.

The Eq. (8) is used to compute the score function for all alternative $\check{A}_i$.

Step 6. Rank.

The alternatives are ranked according to the derived score function.

The general proposed method in figure 1.

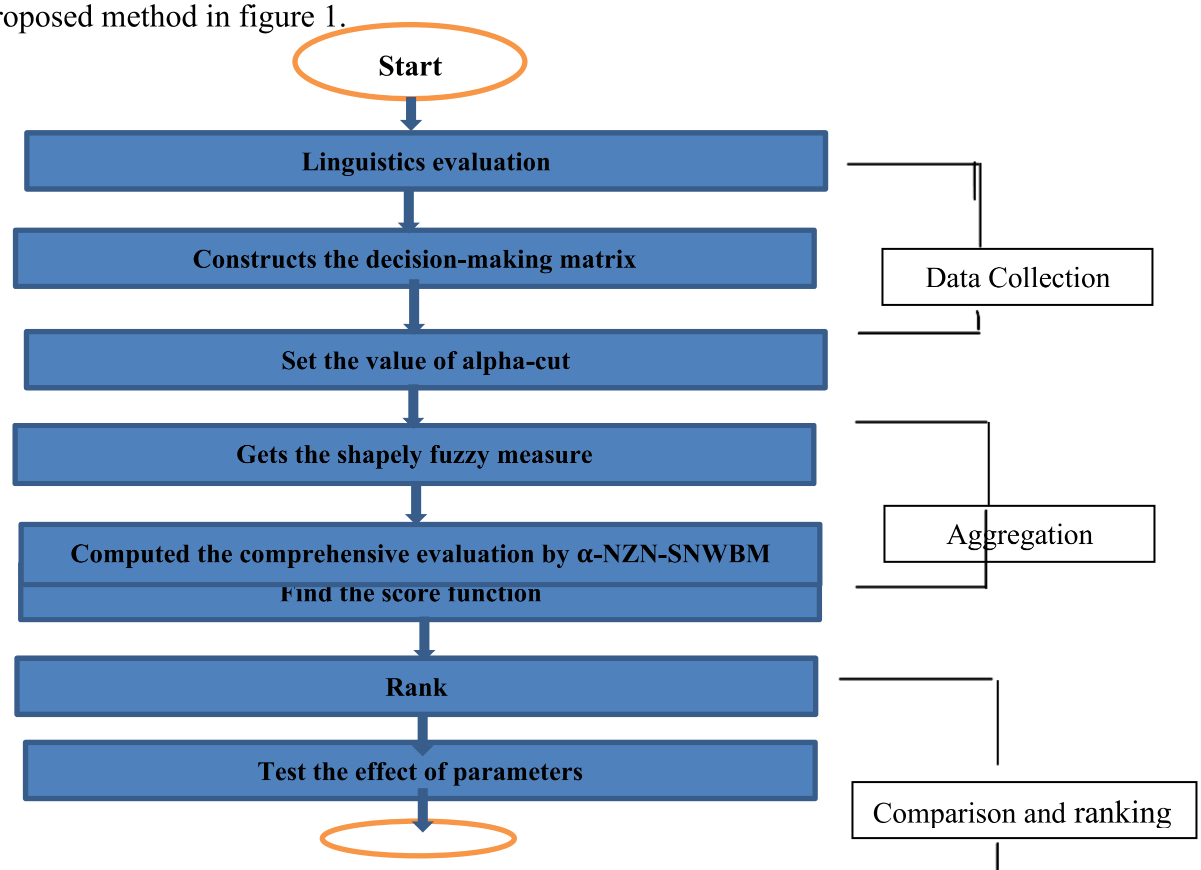

Figure 1. Framework of the proposed method

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The following section provides an example of a company selecting a mobile device supplier for is employed to illustrate the applicability of the α-NZN-SNWBM operator.

Suppose the company is tasked with choosing an appropriate supplier from several mobile device companies. The expert panel identifies four suppliers/alternatives, $A=\{A_1, A_2, A_3, A_4\}$, all offering mobile devices manufactured in 2023. The selection process is based on three evaluation criteria: C_1 (the cost of device), C_2 (the quality of the device), and C_3 (the warranty of device). The four alternatives are found by experts according to the stated criteria and under α-cut neutrosophic z-numbers environment as represented in the decision matrix *D*.

4.1 The procedure of the proposed method

The six-steps approach and computations proceeded the achievement of the finalized ranking of the alternatives is given as the following:

Step 1. Create a decision matrix after surveying experts $D = \langle T_{z_i}, I_{z_i}, F_{z_i}\rangle_{(m\times n)}$ in the MCDM [49].

$$D = \begin{array}{c} A_i \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \left[\begin{array}{ccc} C_1 & C_2 & C_3 \\ (T^{\alpha}_{V_i}, T^{\alpha}_{R_i}), (I^{\alpha}_{V_i}, I^{\alpha}_{R_i}), (F^{\alpha}_{V_i}, F^{\alpha}_{R_i}) & (T^{\alpha}_{V_i}, T^{\alpha}_{R_i}), (I^{\alpha}_{V_i}, I^{\alpha}_{R_i}), (F^{\alpha}_{V_i}, F^{\alpha}_{R_i}) & (T^{\alpha}_{V_i}, T^{\alpha}_{R_i}), (I^{\alpha}_{V_i}, I^{\alpha}_{R_i}), (F^{\alpha}_{V_i}, F^{\alpha}_{R_i}) \\ \langle(0.8,0.8),(0.1,0.8),(0.1,0.7)\rangle & \langle(0.7,0.7),(0.1,0.7),(0.1,0.9)\rangle & \langle(0.8,0.6),(0.2,0.7),(0.3,0.7)\rangle \\ \langle(0.9,0.7),(0.1,0.4),(0.2,0.2)\rangle & \langle(0.7,0.7),(0.1,0.1),(0.2,0.3)\rangle & \langle(0.9,0.8),(0.4,0.2),(0.2,0.4)\rangle \\ \langle(0.7,0.6),(0.2,0.1),(0.1,0.2)\rangle & \langle(0.8,0.8),(0.2,0.3),(0.3,0.1)\rangle & \langle(0.8,0.7),(0.3,0.4),(0.4,0.3)\rangle \\ \langle(0.7,0.8),(0.1,0.3),(0.2,0.4)\rangle & \langle(0.8,0.6),(0.1,0.2),(0.2,0.2)\rangle & \langle(0.9,0.7),(0.2,0.4),(0.2,0.2)\rangle \end{array}\right]$$

Step 2.Set the value of α-cut to redues the number of alternative, if $\alpha = 0.5$, then
$D^{\alpha=0.5} = \langle T^{\alpha}_{Z_{iJ}}, I^{\alpha}_{Z_{iJ}}, F^{\alpha}_{Z_{iJ}}\rangle_{(m\times n)}$
$D^{\alpha=0.5} =$

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \left[\begin{array}{ccc} C_1 & C_2 & C_3 \\ \text{Excluded} & \text{Excluded} & \text{Excluded} \\ \langle(0.9,0.7),(0.1,0.4),(0.2,0.2)\rangle & \langle(0.7,0.7),(0.1,0.1),(0.2,0.3)\rangle & \langle(0.9,0.8),(0.4,0.2),(0.2,0.4)\rangle \\ \langle(0.7,0.6),(0.2,0.1),(0.1,0.2)\rangle & \langle(0.8,0.8),(0.2,0.3),(0.3,0.1)\rangle & \langle(0.8,0.7),(0.3,0.4),(0.4,0.3)\rangle \\ \langle(0.8,0.8),(0.1,0.3),(0.1,0.4)\rangle & \langle(0.7,0.7),(0.1,0.3),(0.1,0.2)\rangle & \langle(0.8,0.6),(0.2,0.3),(0.3,0.2)\rangle \end{array}\right]$$

Step 3. Obtain the Shapley fuzzy measure of criteria.
For this step, the fuzzy measure of all criterion is gained by the consensus experts' views as the following:
$\mu(c_1) = 0.3$, $\mu(c_2) = 0.35$, $\mu(c_3) = 0.45$

The λ value solving by Eq. (11), the result is as follows
$\lambda + 1 = (1+0.4\,\lambda)\,(1+0.25\,\lambda)\,(1+0.45\,\lambda)$,
$0.045\,\lambda^3 + 0.3925\,\lambda^2 + 0.1\,\lambda = 0 \Rightarrow \lambda \approx 0, -0.26, -8.15$

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Since $\lambda \geq$ -1 is only acceptable, we chose to take λ = -0.26.

The computing fuzzy measure of criteria is obtained by Eq. (10).

$\mu(c_1, c_2) = \frac{1}{\lambda}((1 + \lambda\mu(c_1))(1 + \lambda\mu(c_2)) - 1)$

$\mu(c_1, c_2) = \frac{1}{-0.26}((1 - 0.26(0.4))(1 - 0.26(0.25)) - 1)$

$\mu(c_1, c_2) = 0.62$;

By similar calculation, we have

$\mu(c_1, c_3) = 0.71$, $\mu(c_2, c_3) = 0.76$, $\mu(c_1, c_2, c_3) = 1$

The Shapely fuzzy measure can be obtained by Eq. (12)

$\varphi_i(\mu, N) = \sum_{\substack{V \subseteq N/i \\ i \neq j}}^{n} \frac{(|N|-|V|-1)!|V|!}{|N|!}\left(\mu(V \cup i) - \mu(V)\right), \forall i \in N$

$\varphi_1(\mu, N) = \frac{(3-0-1)!\,0!}{3!}\left(\mu(c_1) - \mu(\emptyset)\right) + \frac{(3-1-1)!1!}{3!}\left(\left(\mu(c_1, c_2) - \mu(c_2)\right) + \left(\mu(c_1, c_3) - \mu(c_3)\right)\right) + \frac{(3-2-1)!2!}{3!}\left(\mu(c_1, c_2, c_3) - \mu(c_2, c_3)\right)$

$\varphi_1(\mu, N) = \frac{2!\,0!}{3!}(0.3 - 0) + \frac{1!\,1!}{3!}((0.62 - 0.35)(0.71 - 0.45)) + \frac{0!\,2!}{3!}(1 - 0.76)$

$\varphi_1(\mu, N) = 0.27$

In similar manner, we get

$\varphi_2(\mu, N) = 0.32$, $\varphi_3(\mu, N) = 0.41$

Step 4. Comprehensive evaluation for each alternative is calculated by Eq. (18).

The aggregated assessment value of all alternatives $\breve{A}_i \{i = 1,.., n\}$ is obtained as follows:

when $g = k = 1$, we get

$\breve{A}_1 = \langle\ Excluded\ \rangle$

$\breve{A}_2 = \langle(0.8396, 0.7374), (0.1687, 0.1948), (0.2, 0.297)\rangle$

$\breve{A}_3 = \langle\ (0.7715, 0.7047), (0.2331, 0.2425), (0.2425, 0.2066)\rangle$

$\breve{A}_4 = \langle\ (0.7674, 0.6912), 0.1299, 0.3), (0.1514, 0.1886)\rangle$.

Step 5. Calculate the score function values.

Eq. (8) is used to calculate the score function of each alternative $\breve{A}_i$, respectively.

$\breve{A}_1$ is *Excluded*, $Y_{(S^{\alpha}_{Z_2})}$ =0.5882, $Y_{(S^{\alpha}_{Z_3})}$=0.5198, $Y_{(S^{\alpha}_{Z_4})}$=0.5647.

Step 6. Rank the alternatives

The score function values provide the basis for ranking the alternatives as $\breve{A}_2 > \breve{A}_4 > \breve{A}_3$ and $\breve{A}_1$ is excluded. Therefore, the best alternative is $\breve{A}_2$.

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4.2 Effect of parameters

To examine the effect of the parameters *g* and *k* on the results of alternatives under diverse values of *g, k* can be used. The consequence ranking output and the obtained score functions are represented in the following table:

Table 1. The ordering of alternatives for varying *g, k* values

g, k	Scour function	Rank of alternative
$g=1, k=1$	$Y_{(S^{\alpha}_{Z_2})}=0.5882$,$Y_{(S^{\alpha}_{Z_3})}=0.5198$,$Y_{(S^{\alpha}_{Z_4})}=0.5647$	$\breve{A}_2 > \breve{A}_4 > \breve{A}_3$
$g=0, k=0$	$Y_{(S^{\alpha}_{Z_2})}=0.5282$,$Y_{(S^{\alpha}_{Z_3})}=0.4588$,$Y_{(S^{\alpha}_{Z_4})}=0.5379$	$\breve{A}_4 > \breve{A}_2 > \breve{A}_3$
$g=1, k=2$	$Y_{(S^{\alpha}_{Z_2})}=0.5903$,$Y_{(S^{\alpha}_{Z_3})}=0.5226$,$Y_{(S^{\alpha}_{Z_4})}=0.5663$	$\breve{A}_2 > \breve{A}_4 > \breve{A}_3$
$g=2, k=0$	$Y_{(S^{\alpha}_{Z_2})}=0.5282$,$Y_{(S^{\alpha}_{Z_3})}=0.4588$,$Y_{(S^{\alpha}_{Z_4})}=0.5379$	$\breve{A}_4 > \breve{A}_2 > \breve{A}_3$
$g=1, k=3$	$Y_{(S^{\alpha}_{Z_2})}=0.5639$,$Y_{(S^{\alpha}_{Z_3})}=0.5216$,$Y_{(S^{\alpha}_{Z_4})}=0.5621$	$\breve{A}_2 > \breve{A}_4 > \breve{A}_3$
$g=5, k=3$	$Y_{(S^{\alpha}_{Z_2})}=0.5908$,$Y_{(S^{\alpha}_{Z_3})}=0.5171$,$Y_{(S^{\alpha}_{Z_4})}=0.5679$	$\breve{A}_2 > \breve{A}_4 > \breve{A}_3$
$g=3, k=5$	$Y_{(S^{\alpha}_{Z_2})}=0.5915$,$Y_{(S^{\alpha}_{Z_3})}=0.5236$,$Y_{(S^{\alpha}_{Z_4})}=0.5676$	$\breve{A}_2 > \breve{A}_4 > \breve{A}_3$
$g=10, k=2$	$Y_{(S^{\alpha}_{Z_2})}=0.5647$,$Y_{(S^{\alpha}_{Z_3})}=0.4965$,$Y_{(S^{\alpha}_{Z_4})}=0.5472$	$\breve{A}_2 > \breve{A}_4 > \breve{A}_3$
$g=10, k=1$	$Y_{(S^{\alpha}_{Z_2})}=0.5951$,$Y_{(S^{\alpha}_{Z_3})}=0.5205$,$Y_{(S^{\alpha}_{Z_4})}=0.5735$	$\breve{A}_2 > \breve{A}_4 > \breve{A}_3$
$P=0.02, q=1$	$Y_{(S^{\alpha}_{Z_2})}=0.5994$,$Y_{(S^{\alpha}_{Z_3})}=0.5322$,$Y_{(S^{\alpha}_{Z_4})}=0.5721$	$\breve{A}_2 > \breve{A}_4 > \breve{A}_3$

According to Table 1, it is obvious that all the ranking orders are mainly similar as accepted through $g = 0$, $k = 0$ and $g = 2$, $k = 0$. Rank number four will be often $\breve{A}_3$ regard less of the changes in parameter. The parameters *g, k* in α-NZN-SNWBM aggregation operator may differ the results of decision particularly for the first, second place. Generally, $g = k = 1$ may be assigned in α-NZN-SNWBM aggregation operator since it considers both simple to calculate and practical, while keeping in mind the relationships between the criteria.

5. Conclusion

(NWBM) operator has proven to be an effective aggregation tool for managing interactions between input arguments. However, most existing researches on NWBM have been confined to fuzzy set environments, making them less suitable for decision-making problems that involve indeterminacy and complex uncertainty. Moreover, the conventional NWBM primarily handles interactions between pairs of arguments, without considering the overall relationships among all criteria.

To address these limitations, this study introduces a novel aggregation operator that integrates, Shapley fuzzy measure by the NWBM following the (α-NZN) framework. The resulting α-NZN Shapley Normalized Weighted Bonferroni Mean (α-NZN-SNWBM) effectively manages both the overall interaction among input arguments and α-cut levels and z-numbers the presence of truth, indeterminacy,

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and falsity degrees, thereby offering a more comprehensive approach to complex decision-making scenarios.

The proposed α-NZN-SNWBM offers several advantages:

• First, Enhanced Interaction modelling by combining the Shapley fuzzy measure with NWBM, the operator captures the importance of each criterion as well as the influence of their combined interactions.

• Secondly, Robust Uncertainty representation the α-NZN framework effectively models uncertainty, vagueness, and indeterminacy, which are common in real-world multi-criteria decision-making problems.

• thirdly Desirable Mathematical Properties the proposed operator satisfies essential features including idempotency, reducibility, monotonicity, commutativity, and boundedness, ensuring reliable and consistent results.

• Finally, Flexible Decision Parameters The ranking results can be fine-tuned by adjusting parameters g and k, enabling decision-makers to align the selection process with specific requirements.

The α-NZN-SNWBM can be extended to various fuzzy and neutrosophic environments, including IFS, IVIFS, and SVNS, and can be utilized in solving multi-criteria and multi-expert decision-making issues. Future research may also explore its integration with advanced optimization techniques such as genetic algorithms, as well as the development of other aggregation operators (e.g., geometric Bonferroni means) under the α-NZN setting.

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