International Journal of Applied Mathematics

Volume 38 No. 11s 2025,

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

CAPUTO-TYPE MARICHEV-SAIGO-MAEDA FRACTIONAL DERIVATIVE AND GENERAL CLASS POLYNOMIAL ASSOCIATED WITH S-FUNCTION

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Abstract

In this paper, we study and discuss Caputo-type Marichev-Saigo-Maeda fractional derivative of the S-function defined by Saxena and Daiya [15]. A uniqueness theorem is defined for the S-function in term of generalized Wright hypergeometric function. The results are also derived with the help of general class polynomial. Some special cases are also discussed in a closed and compact form

Mathematics Subject Classification: - 33C60, 26A33 33E12, 33B15

Key words and Phrases: - S- function, Caputo-type Marichev-Saigo-Maeda fractional derivative, generalized Wright hypergeometric function.

1. INTRODUCTION

The intuitive idea of fractional order calculus is as old as integer order calculus. Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. fractional calculus has gained increased attention across various fields of science and engineering, including physics, chemistry, biology, fluid dynamics, astrophysics, electrical engineering, image processing and others.

In this paper we develop some new relation between Caputo-type Marichev-Saigo-Maeda fractional derivative and S-function with the help of general class polynomial in term of generalized wright hypergeometric function.

CAPUTO-TYPE MARICHEV-SAIGO-MAEDA FRACTIONAL DERIVATIVE

Let $\upsilon, \mu, \rho \in \mathbb{C}$ $with\,\Re(\upsilon) > 0,\ x \in \Re^{+}$, then the left and right Caputo-type fractional differential operators associated with the Gauss hypergeometric function are defined Reo et al. [9] as follows

$$\left({}^{C}D_{0+}^{\upsilon,\mu,\rho} f\right)(x) = \left(I_{0+}^{-\upsilon+[\Re(\upsilon)]+1,\,-\mu-[\Re(\upsilon)]-1,\,\upsilon+\rho-[\Re(\upsilon)]-1} f^{([\Re(\upsilon)]+1)}\right)(x)$$

(1.1)

and

$$\left({}^{C}D_{-}^{\upsilon,\mu,\rho} f\right)(x) = (-1)^{[\Re(\upsilon)]+1}\left(I_{0+}^{-\upsilon+[\Re(\upsilon)]+1,-\mu-[\Re(\upsilon)]-1,\upsilon+\rho} f^{([\Re(\upsilon)]+1)}\right)(x) \tag{1.2}$$

then

Let $\upsilon,\upsilon',\mu,\mu',\eta \in \mathbb{C}$ $with\, \Re(\eta) > 0,\ x \in \Re^{+}$, then the left and right Caputo-type Marichev-Saigo-Maeda fractional differential operators associated with the Gauss hypergeometric function are defined Kataria and Vellaisamy [5] as follows (See also[1])

$$\left({}^{C}D_{0+}^{\upsilon,\upsilon',\mu,\mu',\eta} f\right)(x) = \left(I_{0+}^{-\upsilon',-\upsilon,-\mu'+[\Re(\upsilon)]+1,-\mu,-\eta+[\Re(\upsilon)]+1,} f^{([\Re(\upsilon)]+1)}\right)(x)$$

(1.3)

and

$$\left({}^{C}D_{-}^{\upsilon,\upsilon',\mu,\mu',\eta} f\right)(x) = (-1)^{[\Re(\upsilon)]+1}\left(I_{-}^{-\upsilon'.-\upsilon,-\mu'-\mu+[\Re(\upsilon)]+1,-\eta+[\Re(\upsilon)]+1} f^{([\Re(\upsilon)]+1)}\right)(x)$$

(1.4)

2. RESULTS REQUIRED IN THE SEQUEL

Let $\upsilon,\upsilon',\mu,\mu',\eta \in \mathbb{C}$, $\Re\text{e}\,(\lambda) - m > \max\left\{0,\ \Re\text{e}\left(\eta-\upsilon-\upsilon'-\mu'\right), \Re\text{e}\left(\mu-\upsilon\right)\right\},\ m = \left[\Re(\eta)\right]+1.$ then (See [1])

$$\left({}^{C}D_{0+}^{\upsilon,\upsilon',\mu,\mu',\eta}\, t^{\lambda-1}\right)(x) = \frac{\Gamma(\lambda)\,\Gamma\left(\lambda-\eta+\upsilon+\upsilon'+\mu'-m\right)\Gamma\left(\lambda-\mu+\upsilon-m\right)}{\Gamma\left(\lambda-\mu-m\right)\Gamma\left(\lambda-\eta+\upsilon+\upsilon'\right)\Gamma\left(\lambda-\eta+\upsilon+\mu'-m\right)}\, x^{\lambda+\upsilon+\upsilon'-\eta-1}$$

(2.1)

and

Let $\upsilon,\upsilon',\mu,\mu',\eta \in \mathbb{C}$, $\Re\text{e}\,(\lambda) + m > \max\left\{\Re\left(-\mu'\right), \Re\text{e}\left(\upsilon'+\mu-\eta\right), \Re\text{e}\left(\upsilon+\upsilon'-\eta\right)+m\right\},\ m = \left[\Re(\eta)\right]+1.$ then

$$\left({}^{C}D_{-}^{\upsilon,\upsilon',\mu,\mu',\eta}\, t^{-\lambda}\right)(x) = \frac{\Gamma\left(\lambda+\mu'-m\right)\Gamma\left(\lambda-\upsilon'-\mu+\eta+m\right)\Gamma\left(\lambda-\upsilon-\upsilon'+\eta\right)}{\Gamma(\lambda)\Gamma\left(\lambda-\upsilon'+\mu'+m\right)\Gamma\left(\lambda-\upsilon-\upsilon'-\mu+\eta+m\right)}\, x^{\upsilon+\upsilon'-\eta-\lambda}$$

(2.2)

Note 1.1: For further details of the left and right Caputo-type Marichev-Saigo-Maeda fractional differential operators associated with the Gauss hypergeometric function one can refer to the papers Maged and Younis [1], Saigo [10], Saigo and Maedo [11] and Samko et al [12].

3. S-FUNCTION

The S-function introduced by Saxena and Daiya [15] is defined as follows

$$\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;x\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n\,(\gamma)_{n\tau,k}}{(b_1)_n\ldots(b_q)_n\,\Gamma_k\,(n\alpha+\beta)}\frac{x^n}{n!}$$

(3.1)

$k\in R$; $\alpha,\beta,\gamma,\tau\in C$; $\mathrm{Re}(\alpha)>0$, a_i $(i=1,2,\ldots,p)$, b_j $(j=1,2,\ldots,q)$, $\mathrm{Re}(\alpha)>k\,\mathrm{Re}(\tau)$ *and* $p<q+1$.

The Pochhammer symbol $(\lambda)_\mu,(\lambda,\mu\in C)$ with $(1)_n=n!$ for $n\in N$ defined in terms of gamma function as

$$(\lambda)_\mu=\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)}=\begin{cases}1 & (\mu=0;\ x\in C\setminus\{0\})\\ \lambda(\lambda+1)\ldots(\lambda+\mu-1) & (\mu=n\in N;\ x\in C)\end{cases}$$

(3.2)

The k-Pochhammer symbol was introduced in [3] in the form:

$$(x)_{n,k}=x(x+k)(x+2k)\ldots(x+(n-1)k)\ \textit{and}\ (x)_{(n+r)q,k}=(x)_{rq,k}\,(x+qrk)_{nq,k}\,, \tag{3.3}$$

where $x\in C, k\in R$ *and* $n\in N$.

Proposition 1 Let $\gamma\in C$ *and* $k,s\in R$, then the following identity holds:

$$\Gamma_s(\gamma)=\left(\frac{s}{k}\right)^{\frac{\gamma}{s}-1}\Gamma_k\left(\frac{k\gamma}{s}\right)\quad \textit{and}\quad \Gamma_k(\gamma)=k^{\frac{\gamma}{k}-1}\,\Gamma\left(\frac{\gamma}{k}\right), \tag{3.4}$$

Proposition 2 Let $\gamma\in C$ *and* $k,s\in R$, then the following identity holds:

$$(\gamma)_{nq,s}=\left(\frac{s}{k}\right)^{nq}\left(\frac{k\gamma}{s}\right)_{nq,k}\quad \textit{and}\quad (\gamma)_{nq,s}=k^{nq}\left(\frac{\gamma}{k}\right)_{nq}$$

(3.5)

Note 1.1: For further details of k-Pochammer symbol, k-special functions and fractional Fourier transforms one can refer to the papers by Romero et al [7,8].

Special cases

(i) when $p=q=0$ in equation (3.1) it reduces to generalized k-Mittag-Leffler function, defined by Saxena, Daiya, Singh [14].

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$\underset{(0,0)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}(-;-;x)== \sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)}\frac{x^n}{n!}=E_{k,\alpha,\beta}^{\gamma,\tau}(x) \quad \text{where} \quad \text{Re}\left(\frac{\alpha}{k}-\tau\right)>p-q.$$

(3.6)

(ii) Similarly, for τ = 1, equation (3.1) yields where $\text{Re}(\alpha)>kp.$

(3.7)

(iii) When k = 1, equation (3.7) yields K-function, defined by Sharma [18] where $\text{Re}(\alpha)>p-q$

$$\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,1,1)}{S}}\left(a_1,...,a_p;b_1,...,b_q;x\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n(\gamma)_n}{(b_1)_n...(b_q)_n\Gamma(n\alpha+\beta)}\frac{x^n}{n!}=\underset{(p,q)}{\overset{(\alpha,\beta,\gamma)}{K}}\left(a_1,...,a_p;b_1,...,b_q;x\right)$$

(3.8)

(iv) If we set $\gamma=1,$ in equation (3.8), it reduces to the generalized M-Series defined by Sharma and Jain [19]

$$\underset{(p,q)}{\overset{(\alpha,\beta,1,1,1)}{S}}\left(a_1,...,a_p;b_1,...,b_q;x\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n x^n}{(b_1)_n...(b_q)_n\Gamma(n\alpha+\beta)}=\underset{(p,q)}{\overset{(\alpha,\beta)}{M}}\left(a_1,...,a_p;b_1,...,b_q;x\right)$$

(3.9)

where $\text{Re}(\alpha)>p-q-1$

(v) when $p=q=0, \tau=k=1$ in equation (1) it reduces to generalized Mittag-Leffler function,

$$\underset{(0,0)}{\overset{(\alpha,\beta,\gamma,1,1)}{S}}(-;-;x)== \sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(n\alpha+\beta)}\frac{x^n}{n!}=E_{\alpha,\beta}^{\gamma}(x) \tag{3.10}$$

(vi) If we set γ =1 in (3.10) it reduces to the generalized Mittag-Leffler function

$$\text{E}_{1,1}\left[(\alpha,\beta);\text{z}\right]=\text{E}^{1}_{\alpha,\beta}[\text{z}]=\sum_{\text{n}=0}^{\infty}\frac{\text{z}^n}{\Gamma(n\alpha+\beta)}=\text{E}_{\alpha,\beta}[\text{z}] \tag{3.11}$$

Where $\alpha,\beta\in C;\ R(\alpha)>0,\ \text{Re}(\beta)>0$

If we further set $\beta=1$ in education (3.11) in the above expression, we obtain the Mittag-Leffler functions defined and studied by Mittag-Leffler [6].

$$\text{E}_{1,1}\left[(\alpha,1);\text{z}\right]=\text{E}^{1}_{\alpha,1}[\text{z}]=\sum_{\text{n}=0}^{\infty}\frac{\text{z}^n}{\Gamma(\text{n}\alpha+1)}=\text{E}_{\alpha}[\text{z}] \qquad \alpha\in C;\ \text{Re}(\alpha)>0. \tag{3.12}$$

This function plays a very important role in the theory of fractional calculus. A detailed account of Mittag-Leffler function and their applications can be found in the survey paper by Haubold, Mathai and Saxena [4] and Saxena et al [13,16,17].

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

The generalized Wright hypergeometric function ${}_p\Psi_q(z)$ is defined by Wright [22,23,24] in the following form:

$$ {}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix};Z\right]=\sum_{n=0}^{\infty}\frac{\left[\prod_{i=1}^{p}\Gamma(a_i+A_in)\right]}{\left[\prod_{j=1}^{q}\Gamma(b_j+B_jn)\right]}\frac{z^n}{n!} \tag{3.13}$$

where $a_i\,,b_j\in C$ and $A_i\,,B_j\in R\;(i=1,\ldots.,p\,;\;j=1,\ldots,q)$ and the defining series (2.12) converges for

$$\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i>-1.$$

Also $S_V^U[x]$ occurring in the sequel represents the general class of polynomials [20, p.1 equ. (6),21]

$$S_V^U[x]=\sum_{R=0}^{[V/U]}(-V)_{UR}\,A_{(V,R)}\frac{x^R}{R!} \tag{3.14}$$

where U is an arbitrary positive integer, V = 0, 1, 2, . . . and the coefficients A (V, R) $A_{V,R}(V,R\geq 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{V,R}$ yields a number of known polynomials as its special cases. (See also [13]).

In this session we use Gamma and Beta function

$$\int_0^{\infty}e^{-pz}\,z^{e-1}\,dz=\frac{\Gamma(e)}{pe},\mathrm{Re}(e)>0,\;\mathrm{Re}(p)>0. \tag{3.15}$$

And $$\int_0^{\infty}z^{\alpha-1}\,(1-z)^{\beta-1}\,dz=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},\mathrm{Re}(\alpha)>0,\;\mathrm{Re}(\beta)>0.$$

(3.16)

The following results are also needed in the analysis that follows:

Now, left and right Caputo-type Marichev-Saigo-Maeda fractional derivatives of the S-function is given as

4. Main Result

Received: August 07, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Theorem 1: Let $k\in R$; $\alpha,\beta,\gamma,\tau\in C$; $\mathrm{Re}(\alpha)>0$, a_i $(i=1,2,...,p)$, b_j $(j=1,2,...,q)$, $\mathrm{Re}(\alpha)>k$, $\mathrm{Re}(\tau)$ *and* $p<q+1$, further $\zeta,\zeta',\upsilon,\upsilon'\eta\in \mathrm{C}$ with $m=\left[\Re(\eta)\right]+1$, $\Re(\eta)>0$, $x\in\Re$ and

Let $\Re\mathrm{e}\,(\lambda)-m>\max\left\{0,\Re\mathrm{e}\left(-\upsilon'\right),\Re\mathrm{e}\left(\eta-\zeta-\zeta'-\upsilon'\right),\Re\mathrm{e}\left(\upsilon-\zeta\right)+m\right\}$. Let ${}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}$ be the right sided operator of Caputo type Marichev Saigo Maeda fractional derivative. Then

$$
{}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{\lambda-1}\,\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}\left(a_1,...,a_p;b_1,...,b_q;a\,t^p\right)\right)
$$

$$
=\frac{k^{1-\frac{\beta}{k}}\,x^{\lambda+\zeta+\zeta'-\eta-1}}{\Gamma\left(\frac{\gamma}{k}\right)}\;{}_{p+4}\Psi_{q+4}\left[\begin{array}{l} a_1...a_p;\left(\frac{\gamma}{k},\tau\right)\;;\;(\lambda,p) \\ b_1...b_q;\left(\frac{\beta}{k},\frac{\alpha}{k}\right);\;(\lambda-\upsilon-m,p) \\ \left(\lambda-\eta+\zeta+\zeta'+\upsilon'-m,p\right);\left(\lambda-\upsilon+\zeta-m,p\right) \\ \left(\lambda-\eta+\zeta+\zeta,p\right);\left(\lambda-\eta+\zeta+\upsilon'-m,p\right)\end{array}\;\middle|\;k^{\tau-\frac{\alpha}{k}}\,a\,x^p\right]
$$

(4.1.1)

Proof :- Using equation (1.3) , (2.1) and (3.1), we have

$$
{}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{\lambda-1}\,\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}\left(a_1,...,a_p;b_1,...,b_q;a\,t^p\right)\right)
$$

$$
={}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{\lambda-1}\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n\,(\gamma)_{n\tau,k}}{(b_1)_n...(b_q)_n\,\Gamma_k\,(n\alpha+\beta)}\frac{\left(at^p\right)^n}{n!}\right)
$$

$$
=\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n\,(\gamma)_{n\tau,k}\,(a)^n}{(b_1)_n...(b_q)_n\,\Gamma_k\,(n\alpha+\beta)\;n!}\;{}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{pn+\lambda-1}\right)
$$

$$
=\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n\,(\gamma)_{n\tau,k}\,(a)^n}{(b_1)_n...(b_q)_n\,\Gamma_k\,(n\alpha+\beta)\;n!}\;\frac{\Gamma\left(pn+\lambda\right)\Gamma\left(pn+\lambda-\eta+\zeta+\zeta'+\upsilon'-m\right)}{\Gamma\left(pn+\lambda-\upsilon-m\right)\Gamma\left(pn+\lambda-\eta+\zeta+\zeta'\right)}
$$

$$
\times\frac{\Gamma\left(pn+\lambda-\upsilon+\zeta-m\right)}{\Gamma\left(pn+\lambda-\eta+\zeta+\upsilon'-m\right)}x^{pn+\lambda+\zeta+\zeta'-\eta-1}
$$

Using equation (3.4) and (3.5)

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n\,(k)^{n\tau}\left(\frac{\gamma}{k}\right)_{n\tau}(a)^n}{(b_1)_n\ldots(b_q)_n\,(k)^{\frac{n\alpha+\beta}{k}-1}\,\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)n!}\;\frac{\Gamma(pn+\lambda)}{\Gamma(pn+\lambda-\upsilon-m)}$$

$$\times\frac{\Gamma\left(pn+\lambda-\eta+\zeta+\zeta'+\upsilon'-m\right)\Gamma\left(pn+\lambda-\upsilon+\zeta-m\right)}{\Gamma\left(pn+\lambda-\eta+\zeta+\zeta'\right)\Gamma\left(pn+\lambda-\eta+\zeta+\upsilon'-m\right)}\,x^{pn+\lambda+\zeta+\zeta'-\eta-1}$$

$$=\frac{k^{1-\frac{\beta}{k}}\,x^{\lambda+\zeta+\zeta'-\eta-1}}{\Gamma\left(\frac{\gamma}{k}\right)}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n\,\Gamma\left(\frac{\gamma}{k}+n\tau\right)}{(b_1)_n\ldots(b_q)_n\,\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)}\;\frac{\Gamma(pn+\lambda)}{\Gamma(pn+\lambda-\upsilon-m)}$$

$$\times\frac{\Gamma\left(pn+\lambda-\eta+\zeta+\zeta'+\upsilon'-m\right)\Gamma\left(pn+\lambda-\upsilon+\zeta-m\right)}{\Gamma\left(pn+\lambda-\eta+\zeta+\zeta'\right)\Gamma\left(pn+\lambda-\eta+\zeta+\upsilon'-m\right)}\,\frac{\left(ax^p\right)^n}{n!}$$

$$=\frac{k^{1-\frac{\beta}{k}}\,x^{\lambda+\zeta+\zeta'-\eta-1}}{\Gamma\left(\frac{\gamma}{k}\right)}\;{}_{p+4}\Psi_{q+4}\left[\begin{matrix}a_1\ldots a_p;\left(\frac{\gamma}{k},\tau\right);\ (\lambda,p)\\ b_1\ldots b_q;\left(\frac{\beta}{k},\frac{\alpha}{k}\right);\ (\lambda-\upsilon-m,p)\\ \left(\lambda-\eta+\zeta+\zeta'+\upsilon'-m,p\right);\left(\lambda-\upsilon+\zeta-m,p\right)\\ \left(\lambda-\eta+\zeta+\zeta,p\right);\left(\lambda-\eta+\zeta+\upsilon'-m,p\right)\end{matrix}\;\middle|\;k^{\tau-\frac{\alpha}{k}}\,a\,x^p\right]$$

This completes the proof of Theorem 1.

Corollary 1.1

when $p=q=0$ in theorem (1) it reduces to generalized k-Mittag-Leffler function, defined by Saxena, Daiya, Singh [14].

$$ {}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{\lambda-1}\,\underset{(0,0)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}\left(-;-;at^p\right)\right)={}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{\lambda-1}\,E_{k,\alpha,\beta}^{\gamma,\tau}\left(at^p\right)\right)$$

$$=\frac{k^{1-\frac{\beta}{k}}x^{\lambda+\zeta+\zeta'-\eta-1}}{\Gamma\left(\frac{\gamma}{k}\right)}\,{}_4\Psi_4\left[\begin{matrix}\left(\frac{\gamma}{k},\tau\right)\;;\;(\lambda,p);(\lambda-\eta+\zeta+\zeta'+\upsilon'-m,p);(\lambda-\upsilon+\zeta-m,p)\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right);\;(\lambda-\upsilon-m,p);(\lambda-\eta+\zeta+\zeta,p);(\lambda-\eta+\zeta+\upsilon'-m,p)\end{matrix}\middle|\,k^{\tau-\frac{\alpha}{k}}\,ax^{p}\right]$$

Corollary 1.2

when $\tau=1$ in Corollary 1.1, it reduces to generalized k-Mittag-Leffler function, defined by Saxena, Daiya, Singh [14].

$${}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{\lambda-1}\underset{(0,0)}{S}^{(\alpha,\beta,\gamma,1,k)}(-;-;at^{p})\right)={}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{\lambda-1}\,E_{k,\alpha,\beta}^{\gamma}(at^{p})\right)$$

$$=\frac{k^{2-\frac{\beta}{k}-\frac{\alpha}{k}}x^{\lambda+\zeta+\zeta'-\eta-1}}{\Gamma\left(\frac{\gamma}{k}\right)}\,{}_4\Psi_4\left[\begin{matrix}\left(\frac{\gamma}{k},1\right)\;;\;(\lambda,p);(\lambda-\eta+\zeta+\zeta'+\upsilon'-m,p);(\lambda-\upsilon+\zeta-m,p)\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right);\;(\lambda-\upsilon-m,p);(\lambda-\eta+\zeta+\zeta,p);(\lambda-\eta+\zeta+\upsilon'-m,p)\end{matrix}\middle|\,ax^{p}\right]$$

Corollary 1.3

when $k=1$ in Corollary 1.2, it reduces to generalized k-Mittag-Leffler function,

$${}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon',\eta}\left(t^{\lambda-1}\underset{(0,0)}{S}^{(\alpha,\beta,\gamma,1,1)}(-;-;at^{p})\right)={}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{\lambda-1}\,E_{\alpha,\beta}^{\gamma}(at^{p})\right)$$

$$=\frac{x^{\lambda+\zeta+\zeta'-\eta-1}}{\Gamma(\gamma)}\,{}_4\Psi_4\left[\begin{matrix}(\gamma,1)\;;\;(\lambda,p);(\lambda-\eta+\zeta+\zeta'+\upsilon'-m,p);(\lambda-\upsilon+\zeta-m,p)\\ (\beta,\alpha);\;(\lambda-\upsilon-m,p);(\lambda-\eta+\zeta+\zeta,p);(\lambda-\eta+\zeta+\upsilon'-m,p)\end{matrix}\middle|\,ax^{p}\right]$$

Corollary 1.4

when $\tau=k=1$ in theorem (1) it reduces to generalized k-Mittag-Leffler function, defined by Sharma [14].

$${}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{\lambda-1}\underset{(p,q)}{S}^{(\alpha,\beta,\gamma,1,1)}(a_1,\dots,a_p;b_1,\dots,b_q;at^{p})\right)$$

$$={}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{\lambda-1}\underset{(p,q)}{K}^{(\alpha,\beta,\gamma)}(a_1,\dots,a_p;b_1,\dots,b_q;at^{p})\right)$$

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Volume 38 No. 11s 2025,

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$= \frac{x^{\lambda+\zeta+\zeta'-\eta-1}}{\Gamma(\gamma)} {}_{p+4}\Psi_{q+4}\left[\begin{matrix} a_1...a_p;(\gamma,1)\ ;\ (\lambda,p) \\ b_1...b_q;(\beta,\alpha);\ (\lambda-\upsilon-m,p) \\ (\lambda-\eta+\zeta+\zeta'+\upsilon'-m,p);(\lambda-\upsilon+\zeta-m,p) \\ (\lambda-\eta+\zeta+\zeta,p);(\lambda-\eta+\zeta+\upsilon'-m,p) \end{matrix} \middle| \, a x^p \right]$$

Theorem 2: Let $k \in R$; $\alpha,\beta,\gamma,\tau \in C$; $\mathrm{Re}(\alpha)>0$, a_i $(i=1,2,...,p)$, b_j $(j=1,2,...,q)$, $\mathrm{Re}(\alpha)>k$, $\mathrm{Re}(\tau)$ *and* $p<q+1$, further $\zeta,\zeta',\upsilon,\upsilon'\eta \in \mathrm{C}$ with $m=[\Re(\eta)]+1$, $\Re(\eta)>0$, $x\in\Re$ and Let $\Re\mathrm{e}(\lambda)+m>\max\{\Re\mathrm{e}(-\upsilon'),\Re\mathrm{e}(\upsilon+\zeta'-\eta),\Re\mathrm{e}(\zeta+\zeta'-\eta)+m\}$. Let ${}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}$ be the left sided operator of Caputo type Marichev Saigo Maeda fractional derivative. Then

$${}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{-\lambda}\ \underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}\left(a_1,...,a_p;b_1,...,b_q;at^{-p}\right)\right)$$

$$= \frac{k^{1-\frac{\beta}{k}} x^{\zeta+\zeta'-\eta-\lambda}}{\Gamma\left(\frac{\gamma}{k}\right)} {}_{p+4}\Psi_{q+4}\left[\begin{matrix} a_1...a_p;\left(\frac{\gamma}{k},\tau\right)\ ;\ (\lambda+\upsilon+m,p) \\ b_1...b_q;\left(\frac{\beta}{k},\frac{\alpha}{k}\right);\ (\lambda,p) \\ (\lambda-\zeta'-\upsilon+\eta+m,p);(\lambda-\zeta-\zeta'+\eta,p) \\ (\lambda-\zeta'+\upsilon'+m,p);(\lambda-\zeta-\zeta'-\upsilon+\eta+m,p) \end{matrix} \middle| \, ax^p \right] \quad (4.2.1)$$

Proof :- Using equation (1.4) , (2.2) and (3.1), we have

$${}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{-\lambda}\ \underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}\left(a_1,...,a_p;b_1,...,b_q;at^{-p}\right)\right)$$

$$= {}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{-\lambda}\ \sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n(\gamma)_{n\tau,k}}{(b_1)_n...(b_q)_n\Gamma_k(n\alpha+\beta)}\frac{\left(at^{-p}\right)^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n(\gamma)_{n\tau,k}\,(a)^n}{(b_1)_n...(b_q)_n\Gamma_k(n\alpha+\beta)\,n!}\ {}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{-(pn+\lambda)}\right)$$

$$= \sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n(\gamma)_{n\tau,k}\,(a)^n}{(b_1)_n...(b_q)_n\Gamma_k(n\alpha+\beta)\,n!}\ \frac{\Gamma(pn+\lambda+\upsilon'+m)\Gamma(pn+\lambda-\zeta'-\upsilon+\eta+m)}{\Gamma(pn+\lambda)\Gamma(pn+\lambda-\zeta'+\upsilon'+m)}$$

$$\times\frac{\Gamma(pn+\lambda-\zeta-\zeta'+\eta)}{\Gamma(pn+\lambda-\zeta-\zeta'-\upsilon+\eta+m)}x^{\zeta+\zeta'-\eta-pn-\lambda}$$

Using equation (2.4) and (2.5)

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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n\,(k)^{n\tau}\left(\frac{\gamma}{k}\right)_{n\tau}(a)^n}{(b_1)_n\ldots(b_q)_n\,(k)^{\frac{n\alpha+\beta}{k}-1}\,\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)n!}\;\frac{\Gamma\left(pn+\lambda+\upsilon'+m\right)\Gamma\left(pn+\lambda-\zeta'-\upsilon+\eta+m\right)}{\Gamma\left(pn+\lambda\right)\Gamma\left(pn+\lambda-\zeta'+\upsilon'+m\right)}$$

$$\times\frac{\Gamma\left(pn+\lambda-\zeta-\zeta'+\eta\right)}{\Gamma\left(pn+\lambda-\zeta-\zeta'-\upsilon+\eta+m\right)}x^{\zeta+\zeta'-\eta-pn-\lambda}$$

$$=\frac{k^{1-\frac{\beta}{k}}\,x^{\zeta+\zeta'-\eta-\lambda}}{\Gamma\left(\frac{\gamma}{k}\right)}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n\,\Gamma\left(\frac{\gamma}{k}+n\tau\right)}{(b_1)_n\ldots(b_q)_n\,\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)}\;\frac{\Gamma\left(pn+\lambda+\upsilon'+m\right)\Gamma\left(pn+\lambda-\zeta'-\upsilon+\eta+m\right)}{\Gamma\left(pn+\lambda\right)\Gamma\left(pn+\lambda-\zeta'+\upsilon'+m\right)}$$

$$\times\frac{\Gamma\left(pn+\lambda-\zeta-\zeta'+\eta\right)}{\Gamma\left(pn+\lambda-\zeta-\zeta'-\upsilon+\eta+m\right)}\frac{\left(ax^{-p}\right)^n}{n!}$$

$$=\frac{k^{1-\frac{\beta}{k}}\,x^{\zeta+\zeta'-\eta-\lambda}}{\Gamma\left(\frac{\gamma}{k}\right)}\,{}_{p+4}\Psi_{q+4}\left[\begin{matrix}a_1\ldots a_p;\left(\frac{\gamma}{k},\tau\right)\;;\;\left(\lambda+\upsilon+m,p\right)\\ b_1\ldots b_q;\left(\frac{\beta}{k},\frac{\alpha}{k}\right);\;\left(\lambda,p\right)\\ \left(\lambda-\zeta'-\upsilon+\eta+m,p\right);\left(\lambda-\zeta-\zeta'+\eta,p\right)\\ \left(\lambda-\zeta'+\upsilon'+m,p\right);\left(\lambda-\zeta-\zeta'-\upsilon+\eta+m,p\right)\end{matrix}\;\middle|\;k^{\tau-\frac{\alpha}{k}}\,a\,x^{-p}\right]$$

Corollary 2.1

when $p=q=0$ in Theorem (2) it reduces to generalized k-Mittag-Leffler function, defined by Saxena, Daiya, Singh [14].

$${}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{-\lambda}\,\underset{(0,0)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}\left(-;-;at^{-p}\right)\right)={}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{-\lambda}\,E_{k,\alpha,\beta}^{\gamma,\tau}(at^{-p})\right)$$

$$=\frac{k^{1-\frac{\beta}{k}}\,x^{\zeta+\zeta'-\eta-\lambda}}{\Gamma\left(\frac{\gamma}{k}\right)}\,{}_{4}\Psi_{4}\left[\begin{matrix}\left(\frac{\gamma}{k},\tau\right)\;;\;\left(\lambda+\upsilon+m,p\right)\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right);\;\left(\lambda,p\right)\\ \left(\lambda-\zeta'-\upsilon+\eta+m,p\right);\left(\lambda-\zeta-\zeta'+\eta,p\right)\\ \left(\lambda-\zeta'+\upsilon'+m,p\right);\left(\lambda-\zeta-\zeta'-\upsilon+\eta+m,p\right)\end{matrix}\;\middle|\;k^{\tau-\frac{\alpha}{k}}\,a\,x^{-p}\right]$$

Corollary 2.2

when $\tau = 1$ in Corollary 1.1, it reduces to generalized k-Mittag-Leffler function, defined by Saxena, Daiya, Singh [14].

$$ {}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{-\lambda}\overset{(\alpha,\beta,\gamma,1,k)}{\underset{(0,0)}{S}}\left(-;-;at^{-p}\right)\right) = {}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{-\lambda}\ E_{k,\alpha,\beta}^{\gamma}(at^{-p})\right) $$

$$ = \frac{k^{2-\frac{\beta}{k}-\frac{\alpha}{k}}x^{\zeta+\zeta'-\eta-\lambda}}{\Gamma\left(\frac{\gamma}{k}\right)}\ {}_4\Psi_4\left[\begin{matrix}\left(\frac{\gamma}{k},1\right)\ ;\ (\lambda+\upsilon+m,p)\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right);\ (\lambda,p)\\ (\lambda-\zeta'-\upsilon+\eta+m,p);(\lambda-\zeta-\zeta'+\eta,p)\\ (\lambda-\zeta'+\upsilon'+m,p);(\lambda-\zeta-\zeta'-\upsilon+\eta+m,p)\end{matrix}\ \middle|\ ax^{-p}\right] $$

Corollary 2.3

when $k = 1$ in Corollary 2.2, it reduces to generalized k-Mittag-Leffler function,

$$ {}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon',\eta}\left(t^{-\lambda}\overset{(\alpha,\beta,\gamma,1,1)}{\underset{(0,0)}{S}}\left(-;-;at^{-p}\right)\right) = {}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{-\lambda}\ E_{\alpha,\beta}^{\gamma}(at^{-p})\right) $$

$$ = \frac{x^{\zeta+\zeta'-\eta-\lambda}}{\Gamma(\gamma)}\ {}_4\Psi_4\left[\begin{matrix}(\gamma,1)\ ;\ (\lambda+\upsilon+m,p)\\ (\beta,\alpha);\ (\lambda,p)\\ (\lambda-\zeta'-\upsilon+\eta+m,p);(\lambda-\zeta-\zeta'+\eta,p)\\ (\lambda-\zeta'+\upsilon'+m,p);(\lambda-\zeta-\zeta'-\upsilon+\eta+m,p)\end{matrix}\ \middle|\ ax^{-p}\right] $$

Corollary 2.4

when $\tau = k = 1$ in theorem (1) it reduces to generalized k-Mittag-Leffler function, defined by Sharma [14].

$$ {}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{-\lambda}\overset{(\alpha,\beta,\gamma,1,1)}{\underset{(p,q)}{S}}\left(a_1,\dots,a_p;b_1,\dots,b_q;at^{-p}\right)\right) $$

$$ = {}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{-\lambda}\overset{(\alpha,\beta,\gamma)}{\underset{(p,q)}{K}}\left(a_1,\dots,a_p;b_1,\dots,b_q;at^{-p}\right)\right) $$

$$= \frac{x^{\zeta+\zeta'-\eta-\lambda}}{\Gamma(\gamma)} {}_{p+4}\Psi_{q+4}\left[\begin{matrix} a_1...a_p;(\gamma,1)\ ;\ (\lambda+\upsilon+m,p) \\ b_1...b_q;(\beta,\alpha);\ (\lambda,p) \\ (\lambda-\zeta'-\upsilon+\eta+m,p);(\lambda-\zeta-\zeta'+\eta,p) \\ (\lambda-\zeta'+\upsilon'+m,p);(\lambda-\zeta-\zeta'-\upsilon+\eta+m,p) \end{matrix} \middle| \ ax^{-p}\right]$$

CAPUTO-TYPE MARICHEV-SAIGO-MAEDA FRACTIONAL DERIVATIVES OF THE S-FUNCTION WITH THE GENERAL CLASS OF POLYNOMIALS $S_V^U[x]$

Theorem 3: Let $k\in R$; $\alpha,\beta,\gamma,\tau\in C$; $\mathrm{Re}(\alpha)>0$, a_i $(i=1,2,...,p)$, b_j $(j=1,2,...,q)$, $\mathrm{Re}(\alpha)>k$, $\mathrm{Re}(\tau)$ *and* $p<q+1$, further $\zeta,\zeta',\upsilon,\upsilon'\eta\in\mathrm{C}$ with $m=[\Re(\eta)]+1$, $\Re(\eta)>0$, $x\in\Re$ and $\Re\mathrm{e}(\lambda)-m>\max\{0,\Re\mathrm{e}(-\upsilon'),\Re\mathrm{e}(\eta-\zeta-\zeta'-\upsilon'),\Re\mathrm{e}(\upsilon-\zeta)+m\}$, Let ${}^{c}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}$ be the left sided operator of Caputo type Marichev Saigo Maeda fractional derivative with the general class of polynomials $S_V^U[x]$ exists and the following relation holds:

$${}^{c}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{\lambda-1}\, {}_{(p,q)}^{(\alpha,\beta,\gamma,\tau,k)}S\left(a_1,...,a_p;b_1,...,b_q;at^p\right) S_V^U\left[\delta t^{\psi}\right]\right)$$

$$=\frac{k^{1-\frac{\beta}{k}}x^{\lambda+\zeta+\zeta'-\eta-1}}{\Gamma\left(\frac{\gamma}{k}\right)} S_V^U\left[\delta x^{\psi}\right]$$

$$\times {}_{p+4}\Psi_{q+4}\left[\begin{matrix} a_1...a_p;\left(\frac{\gamma}{k},\tau\right)\ ;\ (\lambda+\psi R,p) \\ b_1...b_q;\left(\frac{\beta}{k},\frac{\alpha}{k}\right);\ (\lambda+\psi R-\upsilon-m,p) \\ (\lambda+\psi R-\eta+\zeta+\zeta'+\upsilon'-m,p);(\lambda+\psi R-\upsilon+\zeta-m,p) \\ (\lambda+\psi R-\eta+\zeta+\zeta,p);(\lambda+\psi R-\eta+\zeta+\upsilon'-m,p) \end{matrix} \middle| k^{\tau-\frac{\alpha}{k}}\ a x^{p}\right] \quad (4.3.1)$$

Proof :- Using equation (1.3) , (2.1), (3.1), and (3.14) we have

$${}^{c}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{\lambda-1}\, {}_{(p,q)}^{(\alpha,\beta,\gamma,\tau,k)}S\left(a_1,...,a_p;b_1,...,b_q;at^p\right) S_V^U\left[\delta t^{\psi}\right]\right)$$

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$= {}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{\lambda-1}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n(\gamma)_{n\tau,k}}{(b_1)_n\ldots(b_q)_n\Gamma_k(n\alpha+\beta)}\frac{\left(at^p\right)^n}{n!}\sum_{R=0}^{[V/U]}(-V)_{UR}A_{(V,R)}\frac{[\delta t^{\psi}]^R}{R!}\right)$$

$$= \sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n(\gamma)_{n\tau,k}\left(a\right)^n}{(b_1)_n\ldots(b_q)_n\Gamma_k(n\alpha+\beta)\,n!}\sum_{R=0}^{[V/U]}(-V)_{UR}A_{(V,R)}\frac{[\delta]^R}{R!}\,{}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{pn+\lambda+\psi R-1}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n(\gamma)_{n\tau,k}\left(a\right)^n}{(b_1)_n\ldots(b_q)_n\Gamma_k(n\alpha+\beta)\,n!}\sum_{R=0}^{[V/U]}(-V)_{UR}A_{(V,R)}\frac{[\delta]^R}{R!}\frac{\Gamma\left(pn+\lambda+\psi R\right)}{\Gamma\left(pn+\lambda+\psi R-\upsilon-m\right)}$$
$$\times\frac{\Gamma\left(pn+\lambda+\psi R-\eta+\zeta+\zeta'+\upsilon'-m\right)\Gamma\left(pn+\lambda+\psi R-\upsilon+\zeta-m\right)}{\Gamma\left(pn+\lambda+\psi R-\eta+\zeta+\upsilon'-m\right)\Gamma\left(pn+\lambda+\psi R-\eta+\zeta+\zeta'\right)}x^{pn+\lambda+\psi R+\zeta+\zeta'-\eta-1}$$

Using equation (2.4) and (2.5)

$$=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n(k)^{n\tau}\left(\frac{\gamma}{k}\right)_{n\tau}\left(a\right)^n}{(b_1)_n\ldots(b_q)_n(k)^{\frac{n\alpha+\beta}{k}-1}\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)n!}\sum_{R=0}^{[V/U]}(-V)_{UR}A_{(V,R)}\frac{[\delta]^R}{R!}\frac{\Gamma\left(pn+\lambda+\psi R\right)}{\Gamma\left(pn+\lambda+\psi R-\upsilon-m\right)}$$
$$\times\frac{\Gamma\left(pn+\lambda+\psi R-\eta+\zeta+\zeta'+\upsilon'-m\right)\Gamma\left(pn+\lambda+\psi R-\upsilon+\zeta-m\right)}{\Gamma\left(pn+\lambda+\psi R-\eta+\zeta+\zeta'\right)\Gamma\left(pn+\lambda+\psi R-\eta+\zeta+\upsilon'-m\right)}x^{pn+\lambda+\psi R+\zeta+\zeta'-\eta-1}$$

On interpreting the above result by mean of (2.4) and (2.5) we obtain the desired result (4.3.1).

Corollary 3.1

when $p=q=0$ in Theorem (3) it reduces to generalized k-Mittag-Leffler function, defined by Saxena, Daiya, Singh [14].

$${}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{\lambda-1}\,\underset{(0,0)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}\left(-;-;at^p\right)S_V^U\left[\delta t^{\psi}\right]\right)$$

$$= {}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{\lambda-1}E_{k,\alpha,\beta}^{\gamma,\tau}(at^p)\;S_V^U\left[\delta t^{\psi}\right]\right)=\frac{k^{1-\frac{\beta}{k}}x^{\lambda+\zeta+\zeta'-\eta-1}}{\Gamma\left(\frac{\gamma}{k}\right)}S_V^U\left[\delta x^{\psi}\right]$$

$$\times{}_4\Psi_4\left[\begin{matrix}\left(\frac{\gamma}{k},\tau\right);\;(\lambda+\psi R,p),\left(\lambda+\psi R-\eta+\zeta+\zeta'+\upsilon'-m,p\right);\left(\lambda+\psi R-\upsilon+\zeta-m,p\right)\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right);\;(\lambda+\psi R-\upsilon-m,p);(\lambda+\psi R-\eta+\zeta+\zeta,p);\left(\lambda+\psi R-\eta+\zeta+\upsilon'-m,p\right)\end{matrix}\middle|\;k^{\tau-\frac{\alpha}{k}}ax^p\right]$$

Corollary 3.2

when $\tau=1$ in Corollary 3.1, it reduces to generalized k-Mittag-Leffler function, defined by Saxena, Daiya, Singh [14].

$$ {}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon',\eta}\left(t^{\lambda-1}\underset{(0,0)}{\overset{(\alpha,\beta,\gamma,1,k)}{S}}\left(-;-;at^{p}\right)S_{V}^{U}\left[\delta t^{\psi}\right]\right)={}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon',\eta}\left(t^{\lambda-1}E_{k,\alpha,\beta}^{\gamma}(at^{p})\,S_{V}^{U}\left[\delta t^{\psi}\right]\right) $$

$$ =\frac{k^{2-\frac{\beta}{k}-\frac{\alpha}{k}}x^{\lambda+\zeta+\zeta'-\eta-1}}{\Gamma\left(\frac{\gamma}{k}\right)}S_{V}^{U}\left[\delta x^{\psi}\right] $$

$$ \times{}_{4}\Psi_{4}\left[\begin{matrix}\left(\frac{\gamma}{k},1\right);\ (\lambda+\psi R,p),\left(\lambda+\psi R-\eta+\zeta+\zeta'+\upsilon'-m,p\right);\left(\lambda+\psi R-\upsilon+\zeta-m,p\right)\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right);\ (\lambda+\psi R-\upsilon-m,p);(\lambda+\psi R-\eta+\zeta+\zeta,p);\left(\lambda+\psi R-\eta+\zeta+\upsilon'-m,p\right)\end{matrix}\ \middle|\ ax^{p}\right] $$

Corollary 3.3

when $k=1$ in Corollary 3.2, it reduces to generalized Mittag-Leffler function,

$$ {}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon',\eta}\left(t^{\lambda-1}\underset{(0,0)}{\overset{(\alpha,\beta,\gamma,1,1)}{S}}\left(-;-;at^{p}\right)S_{V}^{U}\left[\delta t^{\psi}\right]\right) $$

$$ ={}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon',\eta}\left(t^{\lambda-1}E_{\alpha,\beta}^{\gamma}(at^{p})S_{V}^{U}\left[\delta t^{\psi}\right]\right)=\frac{x^{\lambda+\zeta+\zeta'-\eta-1}}{\Gamma(\gamma)}S_{V}^{U}\left[\delta x^{\psi}\right] $$

$$ \times{}_{4}\Psi_{4}\left[\begin{matrix}(\gamma,1);\ (\lambda+\psi R,p),\left(\lambda+\psi R-\eta+\zeta+\zeta'+\upsilon'-m,p\right);\left(\lambda+\psi R-\upsilon+\zeta-m,p\right)\\ (\beta,\alpha);\ (\lambda+\psi R-\upsilon-m,p);(\lambda+\psi R-\eta+\zeta+\zeta,p);\left(\lambda+\psi R-\eta+\zeta+\upsilon'-m,p\right)\end{matrix}\ \middle|\ ax^{p}\right] $$

Corollary 3.4

when $\tau=k=1$ in Theorem (3) it reduces to generalized k-Mittag-Leffler function, defined by Sharma [14].

$$ {}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon',\eta}\left(t^{\lambda-1}\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,1,1)}{S}}\left(a_{1},\ldots,a_{p};b_{1},\ldots,b_{q};at^{p}\right)S_{V}^{U}\left[\delta t^{\psi}\right]\right) $$

$$ ={}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon',\eta}\left(t^{\lambda-1}\underset{(p,q)}{\overset{(\alpha,\beta,\gamma)}{K}}\left(a_{1},\ldots,a_{p};b_{1},\ldots,b_{q};at^{p}\right)\right)S_{V}^{U}\left[\delta t^{\psi}\right] $$

$$= \frac{x^{\lambda+\zeta+\zeta'-\eta-1}}{\Gamma(\gamma)} S_V^U\left[\delta x^{\psi}\right]$$

$$\times {}_{p+4}\Psi_{q+4}\left[\begin{matrix} a_1 \ldots a_p; (\gamma,1) \; ; \; (\lambda+\psi R, p) \\ b_1 \ldots b_q; (\beta,\alpha); \; (\lambda+\psi R-\upsilon-m, p) \\ (\lambda+\psi R-\eta+\zeta+\zeta'+\upsilon'-m, p); (\lambda+\psi R-\upsilon+\zeta-m, p) \\ (\lambda+\psi R-\eta+\zeta+\zeta, p); (\lambda+\psi R-\eta+\zeta+\upsilon'-m, p) \end{matrix} \middle| \, a x^{p} \right]$$

Theorem 4: Let $k \in R$; $\alpha, \beta, \gamma, \tau \in C$; $\operatorname{Re}(\alpha) > 0$, a_i $(i = 1, 2, \ldots, p)$, b_j $(j = 1, 2, \ldots, q)$, $\operatorname{Re}(\alpha) > k$, $\operatorname{Re}(\tau)$ *and* $p < q+1$, further $\zeta, \zeta', \upsilon, \upsilon' \eta \in C$ with $m = [\Re(\eta)] + 1$, $\Re(\eta) > 0$, $x \in \Re$ andLet $\Re e(\lambda) + m > \max\{\Re e(-\upsilon'), \Re e(\upsilon+\zeta'-\eta), \Re e(\zeta+\zeta'-\eta)+m\}$. Let ${}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}$ be the left sided operator of Caputo type Marichev Saigo Maeda fractional derivative with the general class of polynomials $S_V^U[x]$ exists and the following relation holds:

$${}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{-\lambda} \, S_{(p,q)}^{(\alpha,\beta,\gamma,\tau,k)}\left(a_1,\ldots,a_p; b_1,\ldots,b_q; a t^{-p}\right) S_V^U\left[\delta t^{-\psi}\right]\right)$$

$$= \frac{k^{1-\frac{\beta}{k}} x^{\zeta+\zeta'-\eta-\lambda}}{\Gamma\left(\frac{\gamma}{k}\right)} S_V^U\left[\delta x^{-\psi}\right]$$

$${}_{p+4}\Psi_{q+4}\left[\begin{matrix} a_1 \ldots a_p; \left(\frac{\gamma}{k}, \tau\right) \; ; \; (\psi R+\lambda+\upsilon+m, p) \\ b_1 \ldots b_q; \left(\frac{\beta}{k}, \frac{\alpha}{k}\right); \; (\psi R+\lambda, p) \\ (\psi R+\lambda-\zeta'-\upsilon+\eta+m, p); (\psi R+\lambda-\zeta-\zeta'+\eta, p) \\ (\psi R+\lambda-\zeta'+\upsilon'+m, p); (\psi R+\lambda-\zeta-\zeta'-\upsilon+\eta+m, p) \end{matrix} \middle| \, k^{\tau-\frac{\alpha}{k}} a x^{-p} \right] \quad (4.4.1)$$

Proof :- Using equation (1.4), (2.2), (3.1) and (3.14), we have

$${}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{-\lambda} \, S_{(p,q)}^{(\alpha,\beta,\gamma,\tau,k)}\left(a_1,\ldots,a_p; b_1,\ldots,b_q; a t^{-p}\right) S_V^U\left[\delta t^{-\psi}\right]\right)$$

$$= {}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{-\lambda} \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n (\gamma)_{n\tau,k}}{(b_1)_n \ldots (b_q)_n \Gamma_k(n\alpha+\beta)} \frac{\left(a t^{-p}\right)^n}{n!} \sum_{R=0}^{[V/U]} (-V)_{UR} A_{(V,R)} \frac{[\delta t^{-\psi}]^R}{R!}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n (\gamma)_{n\tau,k} (a)^n}{(b_1)_n \ldots (b_q)_n \Gamma_k(n\alpha+\beta) \, n!} \sum_{R=0}^{[V/U]} (-V)_{UR} A_{(V,R)} \frac{[\delta]^R}{R!} \, {}^{C}D_{0+}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{-(pn+\psi R+\lambda)}\right)$$

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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n (\gamma)_{n\tau,k} (a)^n}{(b_1)_n \ldots (b_q)_n \Gamma_k (n\alpha+\beta)\, n!} \sum_{R=0}^{[V/U]} (-V)_{UR} A_{(V,R)} \frac{[\delta]^R}{R!} \frac{\Gamma\left(pn+\psi R+\lambda+\upsilon'+m\right)}{\Gamma\left(pn+\psi R+\lambda\right)}$$
$$\times \frac{\Gamma\left(pn+\psi R+\lambda-\zeta'-\upsilon+\eta+m\right)\Gamma\left(pn+\psi R+\lambda-\zeta-\zeta'+\eta\right)}{\Gamma\left(pn+\psi R+\lambda-\zeta'+\upsilon'+m\right)\Gamma\left(pn+\psi R+\lambda-\zeta-\zeta'-\upsilon+\eta+m\right)} x^{\zeta+\zeta'-\eta-pn-\lambda-\psi R}$$

On interpreting the above result by mean of (2.4) and (2.5) we obtain the desired result (4.3.1).

Corollary 4.1

when $p=q=0$ in theorem (1) it reduces to generalized k-Mittag-Leffler function, defined by Saxena, Daiya, Singh [14].

$${}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta} \left(t^{-\lambda} \underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}} \left(-;-;a t^{-p}\right) S_V^U \left[\delta t^{-\psi}\right] \right)$$
$$= {}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta} \left(t^{-\lambda} E_{k,\alpha,\beta}^{\gamma,\tau} (a t^{-p})\ S_V^U \left[\delta t^{-\psi}\right] \right) = \frac{k^{1-\frac{\beta}{k}} x^{\zeta+\zeta'-\eta-\lambda}}{\Gamma\left(\frac{\gamma}{k}\right)} S_V^U \left[\delta x^{-\psi}\right]$$
$${}_4\Psi_4 \left[\begin{matrix} \left(\frac{\gamma}{k},\tau\right);\ (\psi R+\lambda+\upsilon+m, p) \\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right);\ (\psi R+\lambda, p) \\ \left(\psi R+\lambda-\zeta'-\upsilon+\eta+m, p\right); \left(\psi R+\lambda-\zeta-\zeta'+\eta, p\right) \\ \left(\psi R+\lambda-\zeta'+\upsilon'+m, p\right); \left(\psi R+\lambda-\zeta-\zeta'-\upsilon+\eta+m, p\right) \end{matrix} \middle|\ k^{\tau-\frac{\alpha}{k}} a x^{-p} \right] \quad (32)$$

Corollary 4.2

when $\tau=1$ in Corollary 4.1, it reduces to generalized k-Mittag-Leffler function, defined by Saxena, Daiya, Singh [14].

$${}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon',\eta} \left(t^{-\lambda} \underset{(0,0)}{\overset{(\alpha,\beta,\gamma,1,k)}{S}} \left(-;-;a t^{-p}\right) S_V^U \left[\delta t^{-\psi}\right] \right)$$
$$= {}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta} \left(t^{-\lambda} E_{k,\alpha,\beta}^{\gamma} (a t^{-p})\ S_V^U \left[\delta t^{-\psi}\right] \right) = \frac{k^{2-\frac{\beta}{k}-\frac{\alpha}{k}} x^{\zeta+\zeta'-\eta-\lambda}}{\Gamma\left(\frac{\gamma}{k}\right)} S_V^U \left[\delta x^{-\psi}\right]$$
$${}_4\Psi_4 \left[\begin{matrix} \left(\frac{\gamma}{k},1\right);\ (\psi R+\lambda+\upsilon+m, p) \\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right);\ (\psi R+\lambda, p) \\ \left(\psi R+\lambda-\zeta'-\upsilon+\eta+m, p\right); \left(\psi R+\lambda-\zeta-\zeta'+\eta, p\right) \\ \left(\psi R+\lambda-\zeta'+\upsilon'+m, p\right); \left(\psi R+\lambda-\zeta-\zeta'-\upsilon+\eta+m, p\right) \end{matrix} \middle|\ a x^{-p} \right]$$

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Corollary 4.3

when $k=1$ in Corollary 4.2, it reduces to generalized k-Mittag-Leffler function,

$$ {}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon',\eta}\left(t^{-\lambda}\, \underset{(0,0)}{\overset{(\alpha,\beta,\gamma,1,1)}{S}}\left(-;-;at^{-p}\right)S_{V}^{U}\left[\delta t^{-\psi}\right]\right) $$

$$ = {}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{-\lambda}\; E_{\alpha,\beta}^{\gamma}(at^{-p})\; S_{V}^{U}\left[\delta t^{-\psi}\right]\right) = \frac{x^{\zeta+\zeta'-\eta-\lambda}}{\Gamma(\gamma)} S_{V}^{U}\left[\delta x^{-\psi}\right] $$

$$ {}_4\Psi_4\left[\begin{matrix}(\gamma,1)\;\;;\;\;(\psi R+\lambda+\upsilon+m,p)\\ (\beta,\alpha)\;;\;(\psi R+\lambda,p)\\ (\psi R+\lambda-\zeta'-\upsilon+\eta+m,p);\,(\psi R+\lambda-\zeta-\zeta'+\eta,p)\\ (\psi R+\lambda-\zeta'+\upsilon'+m,p);\,(\psi R+\lambda-\zeta-\zeta'-\upsilon+\eta+m,p)\end{matrix}\;\Big|\; ax^{-p}\right] $$

Corollary 4.4

when $\tau=k=1$ in theorem (1) it reduces to generalized k-Mittag-Leffler function, defined by Sharma [14].

$$ {}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{-\lambda}\, \underset{(p,q)}{\overset{(\alpha,\beta,\gamma,1,1)}{S}}\left(a_1,\dots,a_p;b_1,\dots,b_q;at^{-p}\right)S_{V}^{U}\left[\delta t^{-\psi}\right]\right) $$

$$ = {}^{C}D_{-}^{\zeta,\zeta',\upsilon,\upsilon'\eta}\left(t^{-\lambda}\, \underset{(p,q)}{\overset{(\alpha,\beta,\gamma)}{K}}\left(a_1,\dots,a_p;b_1,\dots,b_q;at^{-p}\right)\; S_{V}^{U}\left[\delta t^{-\psi}\right]\right) $$

$$ = \frac{x^{\zeta+\zeta'-\eta-\lambda}}{\Gamma(\gamma)} S_{V}^{U}\left[\delta x^{-\psi}\right] $$

$$ {}_4\Psi_4\left[\begin{matrix}a_1\dots a_p,(\gamma,1)\;\;;\;\;(\psi R+\lambda+\upsilon+m,p)\\ b_1\dots b_q\;,(\beta,\alpha)\;;\;(\psi R+\lambda,p)\\ (\psi R+\lambda-\zeta'-\upsilon+\eta+m,p);\,(\psi R+\lambda-\zeta-\zeta'+\eta,p)\\ (\psi R+\lambda-\zeta'+\upsilon'+m,p);\,(\psi R+\lambda-\zeta-\zeta'-\upsilon+\eta+m,p)\end{matrix}\;\Big|\; ax^{-p}\right] $$

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Received: August 07, 2025

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