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EQUITABLE AND NON-EQUITABLE DEGREE NUMBER OF SOME CLASSES OF GRAPHS

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Abstract

In this article two new sets associated with the vertex set of a finite graph namely equitable degree set and non-equitable degree set are derived. Furthermore, equitable and non-equitable number of some standard classes of graphs identity graph is evaluated.

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1 Introduction and Basic Definitions

Graphs are mathematical structures used to model pairwise relations between objects. To study and compare graphs effectively, several numerical parameters-known as graph invariants-are used [2]. In graph theory, numerical parameters such as the domination number, clique number, and chromatic number play a vital role in understanding the structural and functional characteristics of graphs. Let P denote any property a graph G; P is said be **hereditary** if, whenever a graph G has property P and $G' \subset G$, then G' also has property P [4]. For more definitions, notations and terminologies not covered here,follow [1].

2 Equitable and non-equitable number of a Graph

In this section, two new subsets of the vertex set of the given graph satisfying some special conditions are derived.

Definition 1. Let $G = (V, E)$ be a connected graph. A subset U of V is called **equitable degree set of** G, if $deg(u_i) = deg(u_j)$ and $d(u_i, u_j) = k$, for some $k \in \mathbb{N}$, $\forall\ i, j, u_i, u_j \in U$. The maximum cardinality of equitable degree set of G is called **equitable degree number of** G denoted by $\mathcal{E}(G)$.

Non-equitable degree set of G can be defined similarly.

Definition 2. Let $G = (V, E)$ be a connected graph. A subset W of V is called **non- equitable degree set of** G, if $deg(u_i) \neq deg(u_j)$ and $d(u_i, u_j) \neq k$, for some $k \in \mathbb{N}$, $\forall\ i, j, u_i, u_j \in W$. The maximum cardinality of non-equitable degree set of G is called **non-equitable degree number of** G denoted by $\mathcal{E}^*(G)$.

Consider the example. For this graph in Figure 1, $U = \{v_2, v_3, v_4, v_5\}$ forms an equitable degree set since $deg(v_2) = deg(v_3) = deg(v_4) =$

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$deg(v_5) = 4$ and $d(v_2, v_3) = d(v_2, v_4) = d(v_2, v_5) = d(v_3, v_4) = d(v_3, v_5) = d(v_4, v_5) = 1$. Since there is no other set having cardinality greater than U satisfying the properties of equitable degree set, we can say that $\mathcal{E}(G) = |U| = 4$.

Consider the sets. $W_1 = \{v_1, v_2, v_6\}, W_2 = \{v_1, v_3, v_6\}, W_3 = \{v_1, v_4, v_6\}, W_4 = \{v_1, v_2, v_7\}, W_5 = \{v_1, v_3, v_7\}, W_6 = \{v_1, v_4, v_7\}, W_7 = \{v_2, v_6, v_7\}, W_8 = \{v_3, v_6, v_7\}, W_9 = \{v_4, v_6, v_7\}$.

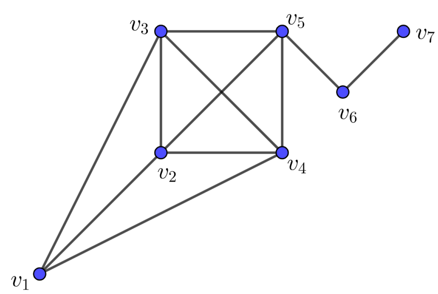

Figure 1: G

Here for the set $W_1, deg(v_1) = 3, deg(v_2) = 4, deg(v_6) = 2$. That is, $deg(v_1) \neq deg(v_2) \neq deg(v_6)$ and $d(v_1, v_2) = 1, d(v_1, v_6) = 3, d(v_2, v_6) = 2$. That is, $d(v_1, v_2) \neq d(v_1, v_6) \neq d(v_2, v_6)$. Hence W_1 forms a non-equitable degree set. Similarly we can verify that the remaining set $W_i, i = 2, 3, 4, 5, 6, 7, 8, 9$ also forms a non-equitable degree set. In this case we can see that $\mathcal{E}^*(G) = 3$.

Let G be a simple graph with order atleast 2. Then G has atleast two vertices of the same degree. Without loss of generality assume that G has two vertices of the same degree. Let v_1 and v_2 be the vertices. Then $U = \{v_1, v_2\}$ forms an equitable degree set of G. This guarantees that for any graph with order $n \geq 2$, equitable degree set exists. But existence of non-equitable degree set cannot be guaranteed. For example, there doesnot exist any non-equitable degree set in G, if G is regular. If the graph G is not regular then the two vertices having different degrees will always forms a non-equitable degree set.

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3 Equitable and Non-equitable number of Some Standard Classes of Graphs

In this section, equitable and non-equitable degree set of some classes of graphs viz. path, complete bipartite graph, complete graph, cycle, r-regular graph derived. Then we evaluate $\mathcal{E}(G)$ and $\mathcal{E}^*(G)$ of these graphs.

Theorem 3. *For $n \geq 3$, $\mathcal{E}(P_n) = \mathcal{E}^*(P_n) = 2$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n in which v_i is adjacent to v_{i-1} and v_{i+1}, where $i = 2, 3, \ldots, n-1$. Let v_1 and v_n be the pendant vertices and $v_2, v_3, \ldots, v_{n-1}$ be the internal vertices. Clearly $deg(v_1) = deg(v_n) = 1$ and $deg(v_i) = 2$, $\forall\, i = 2, 3, \ldots, n-1$.

Consider the sets $V_1 = \{v_1, v_n\}$ and $V_2 = \{v_i, v_j\}$ where $i \neq j$ and $i, j \in \{2, 3, \ldots, n-1\}$. Then V_1 and V_2 forms the equitable degree set for P_n and no other set having more than 2 vertices can satisfy the properties of an equitable degree set for P_n. It is trivial that V_1 has the required property. Consider the set V_2. If possible, let V_2 contains three vertices say, v_i, v_j and v_k, where $i \neq j \neq k$ and $i, j, k \in \{2, 3, \ldots, n-1\}$. Clearly, $deg(v_i) = deg(v_j) = deg(v_k) = 2$. Since, v_i, v_j and v_k are three internal vertices of the path P_n and $i \neq j \neq k$, their indices must satisfy either one of the following: $i < j < k, i < k < j, j < i < k, j < k < i, k < j < i, k < i < j$.

Case 1: $i < j < k$

In this case, $d(u_i, u_j) = i - j$ and $d(u_j, u_k) = k - j$ and $d(u_i, u_k) = k - j$.

Consider,

$$\begin{aligned} |k - i| &= |k - j + j - i| \\ &< |k - j| + |j - i| \\ &= d(u_j, u_k) + d(u_i, u_j). \end{aligned}$$

Hence in this case, the set $\{u_i, u_j, u_k\}$ doesnot form an equitable degree set.

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In a similar manner, **Case (2)-Case(6)** can be proved.
Case 2: $i < k < j$
Case 3: $j < i < k$
Case 4: $j < k < i$
Case 5: $k < j < i$
Case 6: $k < i < j$.
Hence $\mathcal{E}(P_n) = 2$.

Let us consider the sets $H_{1j} = \{v_1, v_j\}$ where $j \in \{2, 3, \dots n-1\}$ and $H_{2j} = \{v_n, v_j\}$ where $j \in \{2, 3, \dots n-1\}$. Clearly H_{1j} and H_{2j} forms non-equitable degree set for P_n and it is the only sets with the maximum cardinality. Hence $\mathcal{E}^*(P_n) = 2$. This completes the proof. □

Theorem 4. *For a complete bipartite graph $K_{m,n}, m, n \geq 2, m \neq n$*

$$\mathcal{E}(K_{m,n}) = \max\{m, n\} \quad \textit{and} \quad \mathcal{E}^*(K_{m,n}) = 2.$$

Proof. Let V be the vertex set of $K_{m,n}$. Then clearly, $|V| = m + n$. Let $V = V_1 \cup V_2$, where $V_1 = \{u_i, i = 1, 2, \dots, m\}$ and $V_2 = \{v_k, k = 1, 2, \dots, n\}$ be the partite sets. Consider the set V_1. Here $deg(u_i) = deg(u_j) = n$ and $d(u_i, u_j) = 2 \ \forall \ i, j \in \{1, 2, \dots, m\}$ with $i \neq j$. So V_1 forms an equitable degree set of $K_{m,n}$. In a similar manner we can show that V_2 also forms an equitable degree set of $K_{m,n}$.

Consider the sets $V_{ij} = \{u_i, v_j\}$ for some $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$. Here $deg(u_i) \neq deg(v_j)$ whenever $m \neq n$. Since the maximum cardinality of the set with this property is 2, we can say that $\mathcal{E}^*(K_{m,n}) = 2$. □

Theorem 5. *For the complete graph $K_n, \ n \geq 2, \ \mathcal{E}(K_n) = n$.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of K_n. Then we know that $deg(v_i) = n-1 \ \forall \ i = 1, 2, \dots, n$ and $d(v_i, v_j) = 1 \ \forall \ i \neq j, i, j = 1, 2, \dots, n$. Hence $U = \{v_1, v_2, \dots, v_n\}$ forms an equitable degree set for G. Since $|U| = n$, and the subset of the vertex set of K_n

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with largest possible cardinality is U, we have $\mathcal{E}(K_n) = n$. This completes the proof. □

Theorem 6. *For the cycle $C_n, n \geq 3$*

$$\mathcal{E}(C_n) = \begin{cases} 3, if \quad n \equiv 0 \mod 3 \\ 2, if \quad n \equiv 1, 2 \mod 3. \end{cases}$$

Theorem 7. *For an r-regular graph G, non-equitable degree set does not exists.*

In this section, the equitable and non-equitable degree set of identity graphs of finite cyclic group is defined and hence determine the equitable degree number and non-equitable degree number of these graphs.

Definition 8. [3] Let G be a group. The identity graph of group G is a graph with the elements of group G as its vertices which satisfies these properties:

a) Two elements x, y in group G are connected by an edge if $xy = e$, with e is the identity element for group G.

b) Each element of G is connected by an edge with the identity element e.

Theorem 9. *Let G_i denotes the identity graph of a group of order n, where n is odd. Then $\mathcal{E}(G_i) = \dfrac{n-1}{2}, \mathcal{E}^*(G_i) = 2$.*

Proof. Let $G = \{g \mid g^n = 1\}$ with n is an odd integer is a cyclic group with multiplication operation and is of the n^{th} order. The elements of G are $\{1, g, g^2, g^3, g^4, \ldots, g^{n-1}\}$. We can see that $gg^{n-1} = 1, g^2g^{n-2} = 1, \ldots, g^{n-1}g = 1$. Thus, for any non-identity element g, we have $g^kg^{n-k} = 1$, for $k \in \mathbb{Z}^+$ and $k < n$. In the identity graph of G, the vertices of the triangle are $1, g^k$ and g^{n-k}.

Consider the following sets $U_1 = \{g, g^3, g^5, \ldots, g^{n-2}\}$ and $U_2 = \{g^2, g^4, g^6, \ldots, g^{n-1}\}$. Then U_1 and U_2 forms equitable degree set of

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G_i since each vertex of U_1 and U_2 is 2 and distance between each vertex in the same is 2. This shows that $\mathcal{E}(G_i) = \dfrac{n-1}{2}$.

For the other result, consider the set $V_i = \{e, g^i\}$ where $i \in \{1, 2, \ldots, n-1\}$. Here $deg(e) = n-1$ and $deg(g^i) = 2$ for any $i = 1, 2, \ldots, n-1$. Each of the sets V_i forms a non-equitable degree set for G_i. Hence $\mathcal{E}^*(G_i) = 2$. This completes the proof. □

Theorem 10. *Let G_i denotes the identity graph of a group of order m, where m is even. Then $\mathcal{E}(G_i) = \dfrac{m-2}{2}, \mathcal{E}^*(G_i) = 2$.*

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