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Dihedral Groups of Transnormal Embeddings of $\mathbf{S}^1$ in Euclidean Spaces

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Abstract

Through this work, dihedral groups are exploited to study the transnormality of $\mathbf{S}^1$ in Euclidean spaces. We show that generating polytopes of such embeddings follow the transitive action deduced by the fundamental group of the normal plane at the image of $\mathbf{S}^1$. We take advantage of the orientability of $\mathbf{S}^1$ and the spliting of the generating frame into two sets of the same cardinality, each lies in one of two parallel planes. We show that if k is odd, $k \geq 3$, then $2k$-transnormal embeddings of $\mathbf{S}^1$ in $\mathbf{R}^n$ exist with regular right prisms generating polytopes.

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Key Words and Phrases: Transnormal curves, generating frames, generating polytopes, transitive action, dihedral groups.

1 Introduction

In this work, we take M as a smooth (C^∞) m-manifold that is connected and boundaryless with $f : M \longrightarrow \mathbf{R}^n$ being a smooth embedding of M into $\mathbf{R}^n$. We assume that the image of M is $V = f(M)$. Hence it is natural to assign to each point $p \in V$ a unique m-plane T_pV that is tangent to V at p and a unique compliment $(n-m)$-plane N_pV that is normal to T_pV at p. Equivalently, , there exist mappings T and N with $T(p) = T_pV$ and $N(p) = N_pV$. As a generalization of the concept of constant width, the m-manifold V is said to be *transnormal* in the Euclidean space $\mathbf{R}^n$ iff $\forall p, q \in V$, if $q \in N(p)$, then $N(q) = N(p)$. The idea of transnormality was introduced and discussed by S.Robertson in [12, 13]. We define an equivalence relation on V by writing $p \sim q$ to mean $q \in N(p)$. The factor space $V/\sim$ can be identified with the space of normal planes to V, say $W = N(V)$, topologised as a subspace of $Gr_{n-m,n}$, where $Gr_{n-m,n}$ is the *Grassmannian* of affine $(n-m)$-planes in $\mathbf{R}^n$. Also $N : V \to W$ is a covering map [12]. In 1981, B.Wegner showed that for any transnormal embedding V in $\mathbf{R}^n$, the order of N as a covering map is always finite [15]. For V being transnormal in $\mathbf{R}^n$, if the order of the covering map N is k, then we say that V is a k-*transnormal* manifold. It is clear that compact transnormal manifolds have an even order of transnormality. For a survey article see [14].

For $p_1 \in V$, we define the *generating frame* of V at p_1 to be the set

$$\phi(p_1) = V \cap N(p_1).$$

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2 Isometries and Transitivity

The transitive action on generating frames was explained by S.Robertson in [14] and, more recently, used by K.Al-Banawi to deduce new results regarding generating polytopes of tori in [9]. Since a transnormal manifold is connected, paths between points in the generating frame are lift uniquely to each other.

Proposition 1. *[9] The generating frames at any two distinct points of a transnormal manifold are isometric.*

It is in this way we write the generating frame of $\mathbf{S}^1$ at any point as $\phi(\mathbf{S}^1)$. If the image of $\mathbf{S}^1$ is k-transnormal, then $\#(\phi(\mathbf{S}^1)) = k$.

Proposition 2. *[9] There exits a transitive group of isometries on the generating frame of a transnormal manifold.*

Thus, the generating frame $\phi(\mathbf{S}^1)$ should lie on a Euclidean $(n-2)$-sphere that is located in the $(n-1)$-normal plane $N(p)$, where p lies on the image of $\mathbf{S}^1$ [16]. When $\mathbf{S}^1$ is k-transnormally embedded in $\mathbf{R}^2$, and since $n - m = 1$, the subset of the real line under the transitive action is a set with exactly two elements. Hence $k = 2$ [14]. The idea of transitivity was explained by K.Al-Banawi and S.Carter in [1] while describing the generating frames of some transnormal curves in higher spaces. Also their study was extended to construct transnormal partial tubes about such transnormal curves [2]. By a generating polytope of V (at p) we mean the convex hull of the generating frame of V (at p) [3]. In [5, 6], K.Al-Banawi managed to build different generating polytopes and calculated radii of transnormal tori in $\mathbf{R}^4$ being examples of spherical partial tubes. In [7], K.Al-Banawi estimated the order of some compact transnormal manifolds in Euclidean spaces. While K.Al-Banawi used Morse theory [11] in [4] to study transnormal immersions and embeddings of $\mathbf{S}^1$, recently, K.Al-Banawi used normal holonomy in a process of building parallel transnormal curves in $\mathbf{R}^4$ of different even orders [8]. In [9], K.Al-Banawi uesd the

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idea of the transitive action as s tool towards showing that dihedral groups describe the generating polytyopes of transnormal tori in $\mathbf{R}^4$.In [9] , K.Al-Banawi showed that the generating polytope of a transnormal torus in $\mathbf{R}^4$ is a rectangle.

3 Dihedral Groups of Generating Polytopes

Let $f : \mathbf{S}^1 \longrightarrow \mathbf{R}^n$ be an embedding of $\mathbf{S}^1$ in $\mathbf{R}^n$, $n \geq 3$. Assume that the image $f(\mathbf{S}^1)$ is k-transnormal in $\mathbf{R}^n$. If H is the space of normal planes of $f(\mathbf{S}^1)$, then we write $\pi(H)$ as the fundamental group of H. Let Γ be the group of isometries of the generating frame $\phi(\mathbf{S}^1)$. Then, as in [9], the homomorphism $h : \pi(H) \longrightarrow \Gamma$ leads to a transitive group of isometries of $\phi(\mathbf{S}^1)$. If $p \in f(\mathbf{S}^1)$, $\mathcal{G}= h(\pi(H))$, then $\mathcal{G}$ acts transitively on $\phi(p)$, the generating frame of $f(\mathbf{S}^1)$ at p. Due to the transitive action on $\phi(p)$, $\phi(p)$ is a subset of the graph of a Euclidean circle $\mathbf{C}$ that lies in a 2-dimensional subplane of the normal plane at p, say $\tilde{N}(p)$. Hence the generating polytope of $f(\mathbf{S}^1)$ is a semi-regular or a regular $2m$-gon, $m \geq 1$.

Proposition 3. *Let $f : \mathbf{S}^1 \longrightarrow \mathbf{R}^n$ be a k-transnormal embedding of $\mathbf{S}^1$ in $\mathbf{R}^n$, $n \geq 3$. Let the group of action $\mathcal{G}$ be as above. If the generating polytope of $f(\mathbf{S}^1)$ is a semi-regular $2m$-gon, $m \geq 1$, then $\mathcal{G}= D_m$.*

Proof. If the generating polytope of $f(\mathbf{S}^1)$ is a semi-regular $2m$-gon, then Γ, the group of isometries of the generating frame $\phi(\mathbf{S}^1)$, is the dihedral group D_m with generators R and F where R is a rotation of the semi-regular $2m$-gon in $\tilde{N}(p)$, the 2-dimensional subplane of the normal plane at p, through an angle $\frac{2\pi}{m}$ about the centre of $\mathbf{C}$ and F is a reflection about an arbitrary axis chosen from the m axes of symmetry of the semi-regular $2m$-gon. Equivalently,

$$D_m =< R, F : R^m = F^2 = (RF)^2 = e > .$$

Since no proper subgroup of D_m acts transitively on $\phi(\mathbf{S}^1)$, we have

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$\mathcal{G}= D_m$. □

Proposition 4. *Let $f : \mathbf{S}^1 \longrightarrow \mathbf{R}^n$ be a k-transnormal embedding of $\mathbf{S}^1$ in $\mathbf{R}^n$, $n \geq 3$. Let the group of action $\mathcal{G}$ be as above. If the generating polytope of $f(\mathbf{S}^1)$ is a regular $2m$-gon, $m \geq 1$, then* $\mathcal{G}= Z_{2m}$, D_m or D_{2m}.

Proof. If the generating polytope of $f(\mathbf{S}^1)$ is a regular $2m$-gon, then Γ, the group of isometries of the generating frame $\phi(\mathbf{S}^1)$, is the dihedral group D_{2m} with generators R and F where R is a rotation of the regular $2m$-gon in $\tilde{N}(p)$, the 2-dimensional subplane of the normal plane at p, through an angle $\frac{\pi}{m}$ about the centre of $\mathbf{C}$ and F is a reflection about an arbitrary axis chosen from the $2m$ axes of symmetry of the regular $2m$-gon. Equivalently,

$$D_{2m} =< R, F : R^{2m} = F^2 = (RF)^2 = e > .$$

In this case, there are two proper subgroups of D_{2m} acting transitively on $\phi(\mathbf{S}^1)$, namely the cyclic group $Z_{2m} =< R : R^{2m} = e >$ and the group D_m generated by R^2 and F, that is,

$$D_m =< R^2, F : (R^2)^m = F^2 = (R^2F)^2 = e > .$$

Hence the choices of $\mathcal{G}$ are Z_{2m}, D_m and D_{2m}. □

Since $f(\mathbf{S}^1)$ is orientable, for each $p \in f(\mathbf{S}^1)$, one can choose a suitable orientation for $N(p)$ relative to some orientation of the Euclidean space $\mathbf{R}^n$. Upon moving from the point p to the point $q \in \phi(p)$, $N(q)$ acquires two orientations, one induced by the choosen orientation of $\mathbf{R}^n$ and another orientation relative to the orientation of $N(p)$ since $N(q) = N(p)$. The point q is called *positive* (with respect to p) if these two orientations agree, otherwise q is called *negative*. Since $f(\mathbf{S}^1)$ is compact, for each $p \in f(\mathbf{S}^1)$, there is a plane in $\mathbf{R}^n$ of dimension $n-1$ that is parallel to $N(p)$ with no points of intersection with $f(\mathbf{S}^1)$ at all. By moving two planes of the same dimension onto one another, one can conclude that a normal plane

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$N(p)$ intersects $f(\mathbf{S}^1)$ at an even number of points. Consequently, the generating frame $\phi(p)$ is exactly the union of two disjoint sets of the same cardinality, one contains the positive points, the other contains the negative points.

The next theorem is regarding transnormal $f(\mathbf{S}^1)$ having their generating frame as a subset of a 2-dimensional subplane of the normal plane with $\pi_1(W)$ being Abelian.

Theorem 1. *Let $f : \mathbf{S}^1 \longrightarrow \mathbf{R}^n$ be a k-transnormal embedding of $\mathbf{S}^1$ in $\mathbf{R}^n$, $n \geq 3$. For $p \in f(\mathbf{S}^1)$, let $\phi(p)$ be the generating frame of $f(\mathbf{S}^1)$ at p, H the space of normal planes of $f(\mathbf{S}^1)$, and $\pi(H)$ the fundamental group of H. If $\phi(p)$ lies on a 2-dimensional subplane of the normal plane of $f(\mathbf{S}^1)$ at p and $\pi(H)$ is Abelian, then $k = 2$ and the generating polytope is a line segment.*

Proof. If Γ is the group of isometries of the generating frame $\phi(\mathbf{S}^1)$, then the image of the homomorphism $h : \pi(H) \longrightarrow \Gamma$, namely $\mathcal{G}= h(\pi(H))$, acts transitively on $\phi(p)$. Since $\pi(H)$ is Abelian, $\mathcal{G}= h(\pi(H))$ is also Abelian. Since $f(\mathbf{S}^1)$ is compact and $\phi(p)$ lies on a 2-dimensional subplane of the normal plane of $f(\mathbf{S}^1)$ at p, the generating polytope of $f(\mathbf{S}^1)$ is a semi-regular or a regular $2m$-gon, $m \geq 1$.
If the generating polytope of $f(\mathbf{S}^1)$ is a semi-regular $2m$-gon, then $\mathcal{G}= D_m$. But D_m is Abelian only if $m = 1$ or $m = 2$. So $k = 2$ or $k = 4$. Since it must be possible to separate the vertices of the $2m$-gon, by a plane, into 2 isometric transitive sets, each with dimension 1. This can only happen if $k = 2$, and so the generating polytope is a line segment. If the generating polytope of $f(\mathbf{S}^1)$ is a regular $2m$-gon, then according to the above argument, the number of positive points is equal to the number of negative points. As a positive point of the regular $2m$-gon is mapped to a negative point by a means of reflection, the transitive group $\mathcal{G}$ must contain the reflections, and so $\mathcal{G}= D_m$ or D_{2m}. Since $\mathcal{G}$ is Abelian, $\mathcal{G}$ must be D_1 or D_2. Thus, $k = 2$ or $k = 4$. Again the case $k = 4$ is excluded by the existence of a plane that separate positive points from negative points. Thus, $k = 2$ and the generating polytope is

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a line segment. □

4 Special Generating Polytopes

Let $C = f(\mathbf{S}^1)$, $p \in C$, and define $\delta_p : C \longrightarrow \mathbf{R}$ as the smooth function that measures the distance from the point p to any point on C. Then the critical points of δ are the points of the generating frame $\phi(p)$, and all critical points are nondegenerate [4], that is, δ is a Morse function. It was proved in [7] that the number of critical points is finite. In [1], authors showed that the critical points are not focal points of $\mathbf{S}^1$. According to the transitive action on the generating frame and the existence of a plane that separate positive points from negative points, one can conclude that the generating frame is splitted into sets of particular geometric behaviour. We show that this behaviour can be explained by an analytic manner as in the proof of the next theorem.

Theorem 2. *Let $f : \mathbf{S}^1 \longrightarrow \mathbf{R}^n$ be a k-transnormal embedding of $\mathbf{S}^1$ in $\mathbf{R}^n$, $k \geq 2$. Let $C = f(\mathbf{S}^1)$ and $p \in C$. Then*
(1) The generating frame of C at p, $\phi(p)$, is the union of two nonempty disjoint transitive sets A_p, B_p of the same cardinality.
(2) If $X \in \Gamma$, the group of isometries of $\phi(p)$, then either $X(A_p) = A_p$ and $X(B_p) = B_p$ or $X(A_p) = B_p$ and $X(B_p) = A_p$.

Proof: We start with the smooth distance function $\delta_p : C \longrightarrow \mathbf{R}$ as above, which measures the distance from the point p to any point on C. Then points of $\phi(p)$ are no more than being the critical points of δ_p, which are all non-degenerate. Let

$$A_p = \{q \in \phi(p) : \delta_p(q) \text{ is relatively minimum } \}$$

and

$$B_p = \{q \in \phi(p) : \delta_p(q) \text{ is relatively maximum } \}.$$

Using Morse Inequalities [11], and as proved in [4], A_p and B_p have the same cardinality $\frac{k}{2}$. Also A_p and B_p are transitive sets.

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Assume that $q_1, q_2 \in A_p$, then let α be a continuous path in C with initial point $\alpha(0) = q_1$ and terminal point $\alpha(1) = q_2$. If $X \in \Gamma$, then we write $\beta(t) = X(\alpha(t))$ where $t \in [0, 1]$. Consider the possibility that $X(q_1) \in A_p$. In this case there is no focal point of C on the line segment that connects the points q_1 and $X(q_1)$. Using the fact that there is always a continuous variation regarding the index of critical points, there is no focal point of C on the line segment that connects the points $\alpha(t)$ and $\beta(t)$ for all $t \in [0, 1]$. Consquently, there is no focal point of C on the line segment that connects the points q_2 and $X(q_2)$, i.e. $X(q_2) \in A_p$. Thus, $X(A_p) = A_p$ and $X(B_p) = B_p$. If $X(q_1) \in B_p$, then there is one focal point of C on the line segment that connects the points q_1 and $X(q_1)$. Hence there is one focal point on the line segment that connects the points $\alpha(t)$ and $\beta(t)$ for all $t \in [0, 1]$. Particularly, there is one focal point on the line segment that connects the points q_2 and $X(q_2)$, i.e. $X(q_2) \in B_p$. Thus, $X(A_p) = B_p$ and $X(B_p) = A_p$.

Recall that a regular right prism is a right prism with base a regular polygon. Also if $k \geq 3, k \neq 4$, then the group of symmetries of a regular right prism with $2k$ vertices is homomorphic to $D_k \times Z_2$. The next theorem contains a special type of generating polytopes of transnormal embeddings of $\mathbf{S}^1$ in higher Euclidean spaces.

Theorem 3. *Let $f : \mathbf{S}^1 \longrightarrow \mathbf{R}^n$ be a $2k$-transnormal embedding of $\mathbf{S}^1$ in $\mathbf{R}^n$, $k \geq 3$, $n \geq 4$. If $f(\mathbf{S}^1)$ has a regular right prism generating polytope, then k is odd.*

Proof: Again let H be the space of normal planes of $f(\mathbf{S}^1)$, $\pi(H)$ the fundamental group of H and Γ the group of isometries of the generating frame $\phi(p)$. Since $\pi(H) \cong Z$ [10], the homomorphism $h : \pi(H) \longrightarrow \Gamma$ implies that the transitive group $\mathcal{G} = h(\pi(H)) \cong Z$ and should have one generator. If k is even, $k \neq 4$, then the group $D_k \times Z_2$ has no element of order $2k$. Also this holds if $k = 4$ for both the cube and the cuboid. This means that the group of symmetries of such regular right prism does not contain a cyclic subgroup of order $2k$ which acts transitively on the generating frame of $f(\mathbf{S}^1)$. Thus, the generating polytope of $f(\mathbf{S}^1)$ cannot be a regular right

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prism.

When k is odd, $k \geq 3$, assume that R is the rotation of the regular right prism of $2k$ vertices around its axis by an angle $\frac{2\pi}{k}$. Then $R \times \{1\}$ is of order $2k$. This means that the group of isometries of the prism contains a cyclic subgroup of order $2k$ which acts transitively on the generating frame of $f(\mathbf{S}^1)$. Hence if k is odd, $k \geq 3$, then it is possible to have generating polytopes which are regular right prisms.

5 Conclusions

Dihedral groups are a good algebraic tool to study the transnormality of $\mathbf{S}^1$ in higher Euclidean spaces. We showed that generating polytopes of such embeddings followed the transitive action deduced by the fundamental group of the normal plane at the image of $\mathbf{S}^1$. We used a homeomorphism of the fundamental group of the normal planes whose image is a transitive group of isometries of the generating frame of the image of $\mathbf{S}^1$ to deduce special generating polytopes. We took advantage of the orientability of $\mathbf{S}^1$ and the spliting of the generating frame into two sets of the same cardinality, each lies in one of two parallel planes. We showed that if k is odd, $k \geq 3$, then $2k$-transnormal embeddings of $\mathbf{S}^1$ in $\mathbf{R}^n$ exist with regular right prisms generating polytopes.

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