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BINOMIAL TRANSFORMATIONS OF CERTAIN PRODUCTS OF RISING AND FALLING FACTORIALS

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Abstract

In this paper, we evaluate two binomial transformations of a certain product of rising and falling factorials and its alternating analogous. We derive recurrence for these transformations and derive an explicit formula for one of them.

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1 Introduction

Let the sequence $\{b_n\}$ is given. The sequence $\{h_n\}$ defined by

$$h_n = \sum_{k=0}^{n} \binom{n}{k} b_k,$$

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is called as the *binomial transform* of the sequence $\{b_n\}$.

For more details about the binomial transform of a sequence and especially the relationships between the exponential generating functions of a given sequence and its binomial transform, we refer to the works [2, 3] and their references list.

For a positive integer m, the falling and rising factorial are defined by

$$\begin{aligned} x^{\underline{m}} &= x(x-1)...(x-m+1), \\ x^{\overline{m}} &= x(x+1)...(x+m-1), \end{aligned}$$

respectively, with $x^{\underline{0}} = x^{\overline{0}} = 1$.

In [4], Spivey studied binomial transformation of certain sequences and especially he gave some results on the binomial transformations of falling factorials. He gave some relationships between the binomial transformation of a sequence $\{b_k\}$ and $\{\Delta b_k\}$, where Δ is the difference operator defined by $\Delta b_k = b_{k+1} - b_k$. Now we recall the following two results (Theorems 1 and 2 in [4]): Firstly, for all integer n, if $\{g_n\}$ is the binomial transform of the sequence $\{a_k\}$, then

$$\{g_{n+1} - g_n - a_0 [n = -1]\}$$

is the binomial transform of the sequence $\{a_{k+1}\}$.

Let $\{a_k\}$ and $\{b_k\}$ be sequences such that $\Delta b_k = a_k$. Secondly, if $\{g_n\}$ and $\{h_n\}$ are the binomial transform of these sequences $\{a_k\}$ and $\{b_k\}$, respectively, then

$$g_n = h_{n+1} - 2h_n - b_0[n = -1].$$

By the result given in [4] and some known facts, Spivey showed that

$$\sum_{k=0}^{n} \binom{n}{k} k^{\underline{m}} = n^{\underline{m}} 2^{n-m}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k k^{\underline{m}} = (-1)^m m! \, [n = m] \, ,$$

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which are the identities 11 and 17, respectively in [4].

In this paper we will obtain recurrence relations for the binomial and alternating binomial transformation of a sequence constructed as a certain product of falling and rising factorials. Further we derive an explicit formula for the alternating binomial transform of the sequence by solving its recurrence relation.

2 The main results

We give some new results including binomial transformations of certain product of rising and falling factorials and their alternating analogous. Let

$$b_k = \frac{1}{2m-1}\frac{k^{\underline{m}}k^{\overline{m}}}{k} = \frac{1}{2m-1}\left(k+m-1\right)^{\underline{2m-1}}.$$

Theorem 1. *If $h_n = \sum_{k=0}^{n}\binom{n}{k}b_k$, then the sequence $\{h_n\}$ satisfies the recurrence relation*

$$(n+1-m)h_{n+1} = (3n+1)h_n - 2nh_{n-1}.$$

Proof. Let $a_k = \Delta b_k$ and $g_n = \sum_{k=0}^{n}\binom{n}{k}a_k$. Then by Theorem 2 in [4], we have

$$g_n = h_{n+1} - 2h_n - b_0[n=-1].$$

Since $b_0 = 0$, we get

$$g_n = h_{n+1} - 2h_n. \tag{1}$$

On the other hand, we have

$$a_k = \Delta b_k = (k+m-1)^{\underline{2m-2}}$$

and

$$(k-m+1)a_k = (2m-1)b_k.$$

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Combining these, we have

$$\begin{aligned} h_n &= \sum_{k=0}^{n} \binom{n}{k} b_k = \frac{1}{2m-1} \sum_{k=0}^{n} \binom{n}{k} (k-m+1) a_k \\ &= \frac{1}{2m-1} \sum_{k=0}^{n} \binom{n}{k} k a_k - m \sum_{k=0}^{n} \binom{n}{k} a_k + \sum_{k=0}^{n} \binom{n}{k} a_k \\ &= \frac{1}{2m-1} \left(\sum_{k=0}^{n} \binom{n}{k} k a_k - m g_n + g_n \right). \end{aligned}$$

Since $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, we write

$$\sum_{k=0}^{n} \binom{n}{k} k a_k = \sum_{k=1}^{n} \binom{n-1}{k-1} n a_k = n \sum_{k=0}^{n-1} \binom{n-1}{k} a_{k+1}.$$

By Theorem 1 in [4], we write

$$\sum_{k=0}^{n} \binom{n}{k} k a_k = n \left(g_n - g_{n-1} - a_0 \left[n = 0 \right] \right).$$

Since $a_0 = 0$, we write

$$\sum_{k=0}^{n} \binom{n}{k} k a_k = n(g_n - g_{n-1}).$$

Thus we conclude

$$h_n = \frac{1}{2m-1} \left(n(g_n - g_{n-1}) - m g_n + g_n \right). \tag{2}$$

By (1) and (2), we obtain the recursion for the sequence $\{h_n\}$

$$(n+1-m)h_{n+1} = (3n+1)h_n - 2n h_{n-1},$$

as claimed. $\square$

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Since the recurrence of $\{h_n\}$ depend on both n and m, we could present its solution according to the cases of m by the following table:

m	Recurrence	Solution
1	$nh_{n+1} = (3n+1)h_n - 2nh_{n-1}$	$h_n = n2^{n-1}$
2	$(n-1)h_{n+1} = (3n+1)h_n - 2nh_{n-1}$	$h_n = n^{\underline{2}}2^{n-3}\frac{(n+4)}{3}$
3	$(n-2)h_{n+1} = (3n+1)h_n - 2nh_{n-1}$	$h_n = n^{\underline{3}}2^{n-5}\frac{(32+13n+n^2)}{5}$
4	$(n-3)h_{n+1} = (3n+1)h_n - 2nh_{n-1}$	$h_n = n^{\underline{4}}2^{n-7}\frac{(n+3)(n+8)(n+16)}{7}$

As an alternating analogue of Theorem 1, we give our second result.

Theorem 2. *Let $\{b_k\}$ be defined as before and $c_k = (-1)^k b_k$ and $s_n = \sum_{k=0}^{n} \binom{n}{k} c_k$. Then the sequence $\{s_n\}$ satisfies the recurrence*

$$(n+1-m)s_{n+1} = (n+1-2m)s_n.$$

Also if

$$d_k = \frac{1}{(2m-2)!}c_k \quad \text{and} \quad v_n = \sum_{k=0}^{n} \binom{n}{k} d_k,$$

then for $n \geq m$

$$v_{n+1} = \binom{m-1}{n+1-m}(-1)^{n+1}.$$

Proof. Let $a_k = \Delta c_k = \frac{-2k-1}{k-m+1}c_k$. Thus

$$(-2k-1)c_k = (k-m+1)a_k.$$

By Theorem 2 in [4], we have

$$r_n = \sum_{k=0}^{n} \binom{n}{k} a_k = s_{n+1} - 2s_n - c_0[n=-1].$$

Since $c_0 = 0$, we get

$$r_n = s_{n+1} - 2s_n. \tag{3}$$

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Also

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(-2k-1)c_k &= -2\sum_{k=0}^{n}\binom{n}{k}kc_k-\sum_{k=0}^{n}\binom{n}{k}c_k\\ &= -2n(s_n-s_{n-1})-s_n. \end{aligned} \tag{4}$$

On the other hand we have

$$(-2k-1)c_k=(k-m+1)a_k,$$

which, after some computations, gives us

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(-2k-1)c_k &= \sum_{k=0}^{n}\binom{n}{k}(k-m+1)a_k\\ &= (n-m+1)r_n-nr_{n-1}. \end{aligned} \tag{5}$$

By combining (3), (4) and (5), we have

$$(n+1-m)s_{n+1}=(n+1-2m)s_n, \tag{6}$$

as claimed.

For the sequence $\{v_n\}$, we note that if $0\le n\le m-1$, then $v_n=0$ and $v_m=(-1)^m$. On the other hand, since $d_k=\frac{1}{(2m-2)!}c_k$, we get

$$v_n=\sum_{k=0}^{n}\binom{n}{k}d_k=\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(2m-2)!}c_k=\frac{1}{(2m-2)!}s_n,$$

so v_n satisfies the recursion in 6. Since

$$s_{n+1}=\frac{(n+1-2m)}{(n+1-m)}s_n,$$

for $n\ge m$, in $n-m+1$ times step later we obtain

$$s_{n+1}=\frac{(n+1-2m)}{(n+1-m)}\frac{(n-2m)}{(n-m)}...\frac{(m+1-2m)}{(m+1-m)}s_m.$$

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So

$$v_{n+1} = \frac{(n+1-2m)}{(n+1-m)} \frac{(n-2m)}{(n-m)} \cdots \frac{(1-m)}{1} v_m,$$

which, by $v_m = (-1)^m$, equals

$$\frac{(m-(n-m+1))}{(n+1-m)} \frac{(m-(n-m))}{(n-m)} \cdots \frac{(m-1)}{1} (-1)^{n+1},$$

which gives us

$$v_{n+1} = \binom{m-1}{n+1-m} (-1)^{n+1}.$$

□

Here we shall note that the sequence $\{h_n\}$ defined in Theorem 1 has the form

$$\frac{1}{2m-1} n^{\underline{m}} 2^{n-(2m-1)} p_m(n),$$

where $p_m(n)$ is the polynomial in n of degree $m-1$, satisfied the recurrence relation

$$p_{m+1}(n) = (4n+4m)p_m(n) - (4n-2)p_m(n-1) + (n-m-1)p_m(n-2).$$

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