

# A Mixed Formulation Approach to the $p$ -Curl Problem: Well-Posedness Analysis

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## Abstract

This paper establishes the existence, uniqueness, and stability of solutions for a mixed variational formulation in three dimensions of the  $p$ -curl problem, a nonlinear model describing the behavior of magnetic fields in type II high-temperature superconductors (HTS). The problem exhibits strong mathematical parallels with the  $p$ -Laplacian. In our previous work, we constructed the Hellinger–Reissner functional for this problem and identified two distinct functional frameworks in which to seek solutions, successfully proving well-posedness for the first case. The present work completes this analysis by establishing well-posedness for the second, more mathematically challenging framework under homogeneous boundary conditions, thereby extending the theoretical foundation for numerical approximations of the solution.

**Keywords:** Superconductivity,  $p$ -curl, mixed formulation, finite element.

## 1. Introduction

Let  $\Omega$  be a compact simply connected domain of  $R^3$  with a smooth boundary  $\partial\Omega$ , which is subdivided into two disjoint components  $\partial\Omega = \Gamma_H \cup \Gamma_E$ , and we assume a divergence free source term  $F : \Omega \rightarrow R^3$ . We aim to prove the existence, uniqueness, and stability of the magnetic field  $\mathbf{H}$  and the electric fields  $\mathbf{E}$  that satisfy the  $p$ -curl problem, defined as follows

$$(1a) \quad \nabla \times E = F \quad \text{in } \Omega ,$$

$$(1b) \quad |\nabla \times H|^{p-2} \nabla \times H = E \quad \text{in } \Omega ,$$

$$(1c) \quad \nabla \cdot H = 0 \quad \text{in } \Omega ,$$

$$(1d) \quad n \times E = 0 \quad \text{in } \Gamma_E ,$$

$$(1e) \quad (n \times H) \times n = 0 \quad \text{in } \Gamma_H .$$

where  $p \in [2, \infty)$  and  $\mathbf{n}$  is the outward normal along the boundary. This problem has been addressed by many authors, including studies on smooth domains by Yin [15, 16], function theoretic aspects explored by Laforest [10] and Amrouche [1], and alternative formulations proposed by Janikova and Slodicka [9]. Additionally, time-transient formulations have been examined in [8]. In previous work, Heise, Kuhn, and Langer addressed the same problem in 3D within  $L^2$  spaces [6]. Furthermore, Prigozhin and Barrett established the well-posedness of the mixed formulation for this problem in 2D using Nédélec elements [3]. The authors recent work [7] introduced the first mixed finite element method in three dimensions. We began by constructing the Hellinger-Reissner functional for this problem, which provided a robust framework for addressing the boundary conditions through a mixed variational approach. Our analysis identified two distinct cases of functional spaces in which to seek the solution. Having established the well-posedness of the mixed finite element method for the first case, ensuring the existence, uniqueness, and stability of the solution, the principal aim of the current work is to extend this analysis by proving the well-posedness for the second case of functional spaces.

## 2. The continuous mixed formulation

Let  $q$  denote the conjugate exponent of  $p$ , defined by the relation  $p^{-1} + q^{-1} = 1$ .

According to equation (1b), the electric field  $\mathbf{E}$  satisfies the following relation

$$|\nabla \times H| = |E|^{\frac{1}{p-1}} = |E|^{q-1},$$

therefore

$$|E|^{q-2} E = \nabla \times H.$$

Before introducing our continuous mixed formulation, we first define the following reflexive Banach spaces for  $s \in [1, \infty[$

$$W^{1,s}(\Omega) := \{ u \in L^s(\Omega) \mid \nabla u \in L^s(\Omega)^3 \},$$

$$W(\text{div}) := \{ u \in L^s(\Omega)^3 \mid \nabla \cdot u \in L^s(\Omega) \},$$

$$W(\text{curl}, \Omega) := \{ u \in L^s(\Omega)^3 \mid \nabla \times u \in L^s(\Omega)^3 \}.$$

The corresponding norms are defined as follows

$$\| u \|_{W^{1,s}} := (\| u \|_{L^s}^s + \| \nabla u \|_{L^s}^s)^{\frac{1}{s}},$$

$$\| u \|_{W(\text{div})} := (\| u \|_{L^s}^s + \| \nabla \cdot u \|_{L^s}^s)^{\frac{1}{s}},$$

$$\| u \|_{W(curl)} := (\| u \|_{L^s}^s + \| \nabla \times u \|_{L^s}^s)^{\frac{1}{s}}.$$

Additionally, we denote the corresponding normal and tangential traces as  $Y_n^s$  and  $Y_t^s$ , which are Banach subspaces of  $L^s(\partial\Omega)$ , and we define the normal and tangential trace operators as follows

$$\begin{aligned} \gamma_n : W^s(div, \Omega) &\rightarrow Y_n^s \\ v &\mapsto n \cdot v, \end{aligned}$$

$$\begin{aligned} \gamma_t : W^s(curl, \Omega) &\rightarrow Y_t^s \\ u &\mapsto n \times u, \end{aligned}$$

and by noting  $r$  the conjugate exponent of  $s$ , we define the weak divergence-free subspace of  $W^s(div, \Omega)$  in the form

$$W^s(div^0, \Omega) := \{ u \in L^s(\Omega)^3 \mid \langle u, \nabla \phi \rangle = 0, \forall \phi \in W_0^{1,r}(\Omega) \}.$$

where  $\langle \cdot, \cdot \rangle$  denotes the  $L^2$  inner product, and  $W_0^{1,r}(\Omega)$  represents the subspace of functions in  $W^{1,r}(\Omega)$  that vanish on the boundary  $\partial\Omega$ . In our previous work [7], after establishing the Hellinger-Reissner functional for our problem, and building upon the work of Farhloul [4], [5], Arnold [2], and Nédélec [12], we proposed to seek the solution in two cases of functional spaces:

**Case 1:** We seek the magnetic field  $\mathbf{H}$  and electric field  $\mathbf{E}$  in the spaces  $M$  and  $X$ , respectively, defined by

$$\begin{aligned} M &:= W^p(div^0, \Omega) \cap W^p(curl, \Omega), \\ X &:= W^q(curl, \Omega). \end{aligned}$$

**Case 2:** We seek the solution  $(\mathbf{H}, \mathbf{E})$  in the spaces

$$\begin{aligned} M &:= W^p(div^0, \Omega) \cap W^p(curl, \Omega), \\ X_0 &:= \{ u \in W^q(curl, \Omega) \mid n \times u = 0 \text{ in } \Gamma_E \}. \end{aligned}$$

The well-posedness of the mixed formulation for the first case was already addressed in [7]. To analyze the second case, we start from equation (2), which leads to the identity

$$\int_{\Omega} |E|^{q-2} E \cdot u \, dx = \int_{\Omega} (\nabla \times H) \cdot u \, dx \quad \forall u \in X_0.$$

Integrating by parts, we obtain

$$\int_{\Omega} |E|^{q-2} E \cdot u \, dx = \int_{\Omega} H \cdot (\nabla \times u) \, dx - \int_{\Omega} [(n \times H \times n)] \cdot (n \times u) \, dx \quad \forall u \in X_0,$$

$$\begin{aligned} \int_{\Omega} |E|^{q-2} E \cdot u \, dx &= \int_{\Omega} H \cdot (\nabla \times u) \, dx - \int_{\Gamma_H} [(n \times H \times n)] \cdot (n \times u) \, dx - \int_{\Gamma_E} [(n \times H \times n)] \\ &\cdot (n \times u) \, dx \quad \forall u \in X_0. \end{aligned}$$

Since  $(n \times H) \times n = 0$  on  $\Gamma_H$  and  $n \times u = 0$  on  $\Gamma_E$ , the boundary terms vanish due to the imposed boundary conditions. On the other hand, from equation (1a), we obtain

$$\int_{\Omega} (\nabla \times E) \cdot v \, dx = \int_{\Omega} F \cdot v \, dx \quad \forall v \in M.$$

Therefore, for a divergence-free source term  $F \in L^q(\Omega)^3$  the mixed continuous formulation in case 2 for the homogeneous boundary conditions aims to find the magnetic field  $\mathbf{H}$  and the electric field  $\mathbf{E}$  in the spaces  $M$  and  $X_0$ , respectively, satisfying

$$\int_{\Omega} |E|^{q-2} E \cdot u \, dx = \int_{\Omega} \nabla \times u \cdot H \, dx \quad \forall u \in X_0,$$

(3)

$$\int_{\Omega} \nabla \times E \cdot v \, dx = \int_{\Omega} F \cdot v \, dx \quad \forall v \in M,$$

To establish the well-posedness of this mixed formulation, we require two key results: the Babuska-Brezzi theorem stated below, and a proof of the inf-sup condition for the spaces  $X_0$  and  $M$ .

**Theorem 2.1.** [13] *Let  $(Y, \|\cdot\|_Y)$  and  $(S, \|\cdot\|_S)$  be two reflexive Banach spaces. Let  $(Y^*, \|\cdot\|_{Y^*})$  and  $(S^*, \|\cdot\|_{S^*})$  be their corresponding dual spaces. Let  $B: Y \rightarrow S^*$  be a continuous linear*

operator and  $B^* : S \rightarrow Y^*$  the dual operator of  $B$ . Let  $K_B = \ker(B)$  be the kernel of  $B$ , and  $\hat{B} : Y / K_B \rightarrow S^*$  the quotient operator associated with  $B$ . The following three properties are equivalent

(i) There exists  $\beta > 0$  such that

$$\beta \leq \inf_{v \in S} \sup_{u \in Y} \frac{\langle Bu, v \rangle}{\|u\|_Y \|v\|_S},$$

(ii)  $B^*$  is an isomorphism and we have

$$\beta \|v\|_S \leq \|B^*v\|_{Y^*} \quad \forall v \in S,$$

(iii)  $\hat{B}$  is an isomorphism from  $(Y / K_B)$  onto  $S^*$

$$\beta \|\dot{u}\|_{Y/K_B} \leq \|\hat{B}\dot{u}\|_{S^*} \quad \forall \dot{u} \in Y / K_B.$$

**Proposition 2.2.** Let  $\Omega$  be a compact and simply connected domain with a boundary  $\partial\Omega$  of class  $C^1$ . Then, there exists  $\beta > 0$  satisfying

$$\beta \leq \inf_{v \in M} \sup_{u \in X_0} \frac{\int_{\Omega} \nabla \times u \cdot v \, dx}{\|v\|_M \|u\|_X}.$$

Proof. Let  $v \in M$ . Since  $v \in L^p(\Omega)^3$ , we have the identity

$$(4) \quad \int_{\Omega} |v|^{p-2} v^q \, dx = \int_{\Omega} |v|^{(p-1)q} \, dx = \int_{\Omega} |v|^p \, dx < \infty.$$

Which implies that,  $|v|^{p-2}v \in L^q(\Omega)$ . By the Helmholtz decomposition (Theorem II.1.1 of [14]), there exist unique  $z \in W^q(\operatorname{div}^0, \Omega)$  and  $\phi \in W_0^{1,q}(\Omega)$  such that

$$(5) \quad |v|^{p-2}v = z + \nabla\phi,$$

with the stability estimate

$$(6) \quad \|z\|_{L^q} + \|\nabla\phi\|_{L^q} \leq C_1 \| |v|^{p-2}v \|_{L^q},$$

for some constant  $C_1 > 0$ . The identity (4) implies that

$$\| |v|^{p-2}v \|_{L^q} = \left( \int_{\Omega} |v|^{p-2}v^q dx \right)^{\frac{1}{q}} = \|v\|_{L^p}^{\frac{p}{q}}.$$

Since  $\frac{p}{q} = p - 1$ , we obtain the bound

$$(7) \quad \|z\|_{L^q} + \|\nabla\phi\|_{L^q} \leq C_1 \|v\|_{L^p}^{p-1}.$$

We have  $\Omega$  is simply connected, and  $z$  is divergence free in a weak sense, then the boundary value problem

$$\begin{aligned} \nabla \times u &= z \text{ in } \Omega, \\ \operatorname{div} u &= 0 \text{ in } \Omega, \\ n \times u &= 0 \text{ in } \partial\Omega, \end{aligned}$$

is well posed, and admits a unique solution  $u \in W^{1,q}(\Omega)^3$  satisfying

$$(8) \quad \|z\|_{L^q} + \|\nabla\phi\|_{L^q} \leq C_1 \|v\|_{L^p}^{p-1}.$$

Moreover, since

$$(9) \quad \int_{\Omega} |\nabla \times u|^q dx = \int_{\Omega} |z|^q dx,$$

We have  $\nabla \times u \in L^q(\Omega)$ , and  $n \times u = 0$  on  $\partial\Omega$ . Consequently,  $u$  belongs to the space

$$(10) \quad X_0 := \{u \in W^q(\operatorname{curl}, \Omega) \mid n \times u = 0 \text{ on } \Gamma_E\}.$$

With the estimate

$$(11) \quad \|u\|_{W(\operatorname{curl})} \leq C_3 \|z\|_{L^q}.$$

This implies that

$$(12) \quad \|u\|_X \leq C_1 C_3 \|v\|_{L^q}^{p-1}.$$

On the other hand, we have

$$\begin{aligned}
 \int_{\Omega} \nabla \times u \cdot v \, dx &= \int_{\Omega} z \cdot v \, dx, \\
 &= \int_{\Omega} (|v|^{p-2} v - \nabla \phi \cdot v) \, dx, \\
 (13) \quad &= \|v\|_{L^p}^p - \int_{\Omega} v \cdot \nabla \phi \, dx. \\
 \int_{\Omega} \nabla \times u \cdot v \, dx &= \|v\|_{L^p}^p + \int_{\Omega} \phi(\nabla \cdot v) \, dx - \int_{\partial\Omega} \phi(n \cdot v) \, dx.
 \end{aligned}$$

Using the boundary condition  $\phi|_{\partial\Omega} = 0$ , we obtain the relation

$$(14) \quad \int_{\Omega} \nabla \times u \cdot v \, dx = \|v\|_{L^p}^p.$$

Combining (14) with (12) gives

$$(15) \quad \frac{\|v\|_{L^p}^p}{c_1 c_3 \|v\|_{L^p}^{p-1}} \leq \frac{\int_{\Omega} \nabla \times u \cdot v \, dx}{\|u\|_X},$$

which simplifies to

$$(16) \quad \frac{1}{c_1 c_3} \|v\|_{L^p} \leq \frac{\int_{\Omega} \nabla \times u \cdot v \, dx}{\|u\|_X},$$

Consequently, for each  $v \in M$  we have the associated  $u$  hence

$$(17) \quad \frac{1}{c_1 c_3} \leq \frac{\int_{\Omega} \nabla \times u \cdot v \, dx}{\|u\|_X \|v\|_{L^p}} \leq \sup_{w \in X_0} \frac{\int_{\Omega} \nabla \times w \cdot v \, dx}{\|w\|_X \|v\|_{L^p}},$$

And since  $v$  is arbitrary, we can derive the inf-sup condition

$$(18) \quad \frac{1}{c_1 c_3} \leq \inf_{v \in M} \sup_{w \in X_0} \frac{\int_{\Omega} \nabla \times w \cdot v \, dx}{\|w\|_X \|v\|_{L^p}},$$

So by taking  $\beta = \frac{1}{c_1 c_3}$ , we get finally

$$(19) \quad \beta \leq \inf_{v \in M} \sup_{w \in X_0} \frac{\int_{\Omega} \nabla \times w \cdot v \, dx}{\|v\|_{L^p} \|w\|_X}.$$

■

**Theorem 2.3.** We consider the following mixed formulation: find  $(E, H) \in X_0 \times M$  such that

$$(20) \quad \int_{\Omega} |E|^{q-2} E \cdot u \, dx = \int_{\Omega} \nabla \times u \cdot H \, dx \quad \forall u \in X_0,$$

$$\int_{\Omega} \nabla \times E \cdot v \, dx = \int_{\Omega} F \cdot v \, dx \quad \forall v \in M.$$

There exists a unique solution to this problem satisfying

$$(21) \quad \| E \|_X + \| H \|_M \leq C^* \left( \| F \|_{L^q} + \| F \|_{L^q}^{\frac{q}{p}} \right).$$

**Proof.** We consider the space

$$K_B = \{ u \in X_0 \mid \int_{\Omega} \nabla \times u \cdot v \, dx = 0 \quad \forall v \in M \}.$$

from proposition 2.2 and theorem 2.1, we have that the operator  $\nabla \times: X_0 \rightarrow M^*$  is an isomorphism from  $X_0 / K_B$  onto  $M^*$ . This means that there exists an element  $\bar{e} \in X_0$  satisfying

$$(22) \quad \int_{\Omega} \nabla \times (e + \bar{e}) \cdot h \, dx = \int_{\Omega} F \cdot h \, dx \quad \forall h \in M, \forall e \in K_B,$$

Furthermore, according to theorem 2.1 (iii), we have the following bound

$$(23) \quad \| \bar{e} \|_X \leq \frac{1}{\beta} \| \nabla \times \bar{e} \|_{M^*} = \frac{1}{\beta} \| F \|_{L^q}.$$

Thus, our mixed problem can be transformed into finding  $e \in K_B$  and  $h \in M$  that satisfy the following equations

$$\begin{aligned} \int_{\Omega} |e + \bar{e}|^{q-2} (e + \bar{e}) \cdot u \, dx &= \int_{\Omega} \nabla \times u \cdot h \, dx, \quad \forall u \in X_0 \\ \int_{\Omega} \nabla \times (e + \bar{e}) \cdot h \, dx &= \int_{\Omega} F \cdot h \, dx, \quad \forall h \in M. \end{aligned}$$

Let's now consider the problem of seeking  $\hat{e} \in K_B$  such that

$$(24) \quad \int_{\Omega} |\hat{e} + \bar{e}|^{q-2} (\hat{e} + \bar{e}) \cdot u \, dx = 0, \quad \forall u \in K_B.$$

As shown in [4, 5, 7], this formulation is the first variation of the functional

$$\mathfrak{J}(e) := \frac{1}{q} \int_{\Omega} |e + \bar{e}|^q \, dx,$$

so we can reformulate the problem (24) into the minimization problem, looking for  $\hat{e} \in K_B$  satisfying

$$(25) \quad \mathfrak{J}(\hat{e} + \bar{e}) = \inf_{u \in K_B} \mathfrak{J}(u + \bar{e}).$$

From the theorem (1.1) of [11] this problem has a unique solution, if the functional  $\mathfrak{J}$  is continuous, strictly convex and coercive, so let's prove this properties immediately.

- For the continuity, we have

$$\begin{aligned} |\mathfrak{J}(u) - \mathfrak{J}(v)| &= |q \| u + \bar{u} \|_{L^q}^q - q \| v + \bar{u} \|_{L^q}^q| \\ &\leq K |q \| u + \bar{u} \|_{L^q}^q - q \| v + \bar{u} \|_{L^q}^q| \\ &\leq K \| u - v \|_{L^q} \leq K \| u - v \|_X. \end{aligned}$$

- For the strict convexity of  $\mathfrak{J}$ , since  $x_q$  is strictly convex and monotonically increasing for positive values of  $x$ , we have for any  $t \in [0, 1]$

$$\begin{aligned} |t v_1 + (1 - t)v_0 + \bar{u}|^q &= |t(v_1 + \bar{u}) + (1 - t)(v_0 + \bar{u})|^q \\ &\leq [t | v_1 + \bar{u} | + (1 - t) | v_0 + \bar{u} |]^q \\ &\leq t | v_1 + \bar{u} |^q + (1 - t) | v_0 + \bar{u} |^q \end{aligned}$$

Therefore, the functional  $\mathfrak{J}$  is continuous and strictly convex. It remains to demonstrate that  $\mathfrak{J}$  is coercive in the space  $K_B$ . In our previous work [7], we proved that for all  $\mathbf{u}$  in the space  $K_B$ , we have  $\| u \|_X = \| u \|_{L^q}$ . Thus, it suffices to observe that for any  $e \in K_B$ , the following inequality holds

$$\frac{1}{q} | \| e \|_X - \| \bar{e} \|_{L^q} |^q = \frac{1}{q} | \| e \|_{L^q} - \| \bar{e} \|_{L^q} |^q \leq \frac{1}{q} \| e + \bar{e} \|_{L^q}^q = \mathfrak{J}(e).$$

and then, by taking a sequence  $\{(e)_n\}_{n \geq 1}$  in  $K_B$ , if the norm  $\|(e)_n\|_X$  tends to infinity, then  $\mathfrak{J}(e)$  tends to infinity, and thus  $\mathfrak{J}$  is coercive in  $K_B$ . Therefore, we can conclude that there exists a unique infimum  $\hat{e} \in K_B$  for  $\mathfrak{J}$ . Moreover, we have

$$(26) \quad \| \hat{e} + \bar{e} \|_{L^q} = (q\mathfrak{S}(\hat{e}))^{\frac{1}{q}} < (q\mathfrak{S}(0))^{\frac{1}{q}} \leq \| \bar{e} \|_{L^q} .$$

In the other hand, from the first equation of the formulation (20), we observe that we need to find  $h \in M$  satisfying

$$(27) \quad \int_{\Omega} \nabla \times u \cdot h \, dx = \int_{\Omega} |\hat{e} + \bar{e}|^{q-2} (\hat{e} + \bar{e}) \cdot u \, dx, \quad \forall u \in X_0.$$

Since the term

$$\int_{\Omega} |\hat{e} + \bar{e}|^{q-2} (\hat{e} + \bar{e}) \cdot u \, dx,$$

is linear with respect to the second argument, then the right-hand side of (27) belongs to  $(X_0)^*$ , we have from theorem 2.1 (ii) that  $\nabla \times M \rightarrow (X_0)^*$  is an isomorphism, with the following bound

$$(28) \quad \| \hat{h} \|_M \leq \frac{1}{\beta} \| \nabla \times \hat{h} \|_{X^*},$$

Therefore

$$(29) \quad \| \hat{h} \|_M \leq \frac{1}{\beta} \| \hat{e} + \bar{e} \|_{L^q}^{\frac{q}{p}}.$$

By combining equations (22), (24), and (27), we show that our mixed formulation has a unique solution  $(\hat{e} + \bar{e}) \in X_0 \times M$  satisfying the bound

$$\begin{aligned} \| (\hat{e} + \bar{e}) \|_X + \| \hat{h} \|_M &= \left( \| \hat{e} + \bar{e} \|_{L^q} + \| \nabla \times (\hat{e} + \bar{e}) \|_{L^q}^q \right)^{\frac{1}{q}} + \| \hat{h} \|_M, \\ &\leq \left( \| \bar{e} \|_{L^q}^q + \| \nabla \times \bar{e} \|_{L^q}^q \right)^{\frac{1}{q}} + \frac{1}{\beta} \| \bar{e} \|_{L^q}^{\frac{q}{p}}, \\ &\leq \frac{1}{\beta} \| F \|_{L^q} + \left( \frac{1}{\beta} \right)^{\frac{q}{p}+1} \| F \|_{L^q}^{\frac{q}{p}}. \end{aligned}$$

Thus, with  $E = \hat{e} + \bar{e}$  and  $H = \hat{h}$ , by taking  $C^* = \max \left\{ \frac{1}{\beta}; \left( \frac{1}{\beta} \right)^{\frac{q}{p}+1} \right\}$  we obtain

$$\| E \|_X + \| H \|_M \leq C^* \left( \| F \|_{L^q} + \| F \|_{L^q}^{\frac{q}{p}} \right)$$

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