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SOLUTION OF THE BEAM EQUATION USING LAPLACE BERNOULLI WAVELET TRANSFOM METHOD

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Abstract

In the field of engineering, The Beam equation plays a vital role in studying the vibrations and stability of civil structures with mechanical design. In the present work, the Laplace Bernoulli wavelet method is used to solve it. Due to multi resolution representation of the function along with accuracy of the approximate solution and minimal computational time, the method is effective and is one of the alternative methods compared to traditional methods.

Keywords and Phrases: Beam equation, Bernoulli wavelet

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1 Introduction

Today, wavelet theory has received considerable attention in the field of engineering and other disciplines of science. Wavelets are used in signal processing, numerical and system analysis, optimal control theory, etc. The several research articles investigated solution of the beam equation from different perspectives. J.Camacho finds the exact traveling wave solution of a beam equation in [4].H.Wei and S.Ji studied the periodic solutions for a class of semi linear Euler-Bernoulli beam equations with variable coefficients in [6].The solution of Bidirectional dispersive hyperbolic equations by New revival phenomena was given in [3] by G.Farmakis. In[14], P.McKenna and W.Walter studied traveling wave solutions in the nonlinear beam equation. The solution of the nonlinear beam equation by variational proof, via the mountain pass lemma was given in [12,13]. The beam equation is transformed into an ordinary differential equation is discussed in [17]. In [2,15,16] the authors studied the beam equation by numerical and variational iterative methods.

Wavelets is a class of functions that are constructed from the dilation parameter ‘a’ and the translation parameter ‘b’ of the function known as ‘mother wavelet’ and are given by [1,7],

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$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right)$ where a, b are real numbers and $a \neq 0$.

The continuous wavelet transform [1,7,10] of the function f is given by,

$W_f(a,b) = \int_{-\infty}^{\infty} f(t)\, \psi_{a,b}(t)dt$ and

The discrete wavelet transform [7,10] is given by,

$\psi_{k,n}(x) = \frac{1}{\sqrt{|a|}} \psi\left(a_0^k x - nb_0\right)$ where $\psi_{k,n}(x)$ forms the basis of $L^2(\mathbb{R})$.

The following is an outline of the paper structure:

An overlook of Bernoulli wavelets and approximation of function is given in Section 2. The method of solution of both linear and nonlinear beam equation is given in section 3. Section 4 is about application of the Laplace Bernoulli wavelet transform method on two examples. Finally, the conclusion is given in section 5.

2 Bernoulli wavelets [5,8]:

Bernoulli wavelets $\mathrm{B}_{n,m}(x) = \mathrm{B}(k, \check{n}, m, x)$ have been constructed using four arguments $= \check{n} = n-1, n = 1,2,3, \dots, 2^{k-1}$, k is a positive integer and m is the order of the Bernoulli polynomial with normalized time 'x'.

$$\mathrm{B}_{n,m}(x) = \begin{cases} -2^{\frac{k-1}{2}} \mathrm{B}_m(2^{k-1}x - n + 1), & \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}} \\ 0, & otherwise \end{cases}$$

Where $$\mathrm{B}_m(x) = \begin{cases} 1, \; m = 0 \\ \frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}}} \mathfrak{B}_m(x), \; m > 0 \end{cases}$$

And $m = 0,1,2, \dots, M-1$ and $\mathfrak{B}_m(x)$ are known as Bernoulli polynomials[9,11,18] of order m defined on interval [0,1) and are given by

$$\mathfrak{B}_m(x) = \sum_{j=0}^{m} \binom{m}{j} B_{m-j}\, x^j$$

Where $\mathfrak{B}_j = \mathfrak{B}_j(0)$, j=0,1,…,m are the Bernoulli numbers.

For example, $m = 0, \quad \mathfrak{B}_0(x) = 1$

$m = 1, \quad \mathfrak{B}_1(x) = x - \frac{1}{2}$

$m = 2, \quad \mathfrak{B}_2(x) = x^2 - x + \frac{1}{6}$

$m = 3, \quad \mathfrak{B}_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ and so on.

The Bernoulli polynomials, $\mathfrak{B}_m(x)$ forms the basis over the interval [0,1) with the condition

$$\int_0^1 \mathfrak{B}_m(x)\,\mathfrak{B}_n(x)dx = (-1)^{n-1}\frac{(m!)(n!)}{(m+n)!}\mathfrak{B}_{m+n} \qquad m,n \geq 1$$

Operational Matrix and Approximation of function [8,19]

A function $f(x)$ is defined in the interval [0,1] can be approximated in terms of Bernoulli wavelets as $f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{n,m}B_{n,m}(x) = C^T.B(x)$

In truncated form, $f(x) = \sum_{n=1}^{2^k-1}\sum_{m=0}^{M-1} C_{n,m}B_{n,m}(x) = C^T.B(x)$

Where $C_{n,m} = \left(f(x), B_{n,m}(x)\right)$ both C and $B(x)$ are the matrices of size $2^{k-1}M * 1$

$$C = [C_{1,0}, C_{1,1}, \dots, C_{1,M-1,}C_{2,0}, C_{2,1}, \dots, C_{2,M-1}, \dots, C_{2^{k-1},0}, C_{2^{k-1},1}, \dots, C_{2^{k-1},M-1}]^T$$

$$B(x) = [B_{1,0}, B_{1,1}, \dots, B_{1,M-1,}B_{2,0}, B_{2,1}, \dots, B_{2,M-1}, \dots, B_{2^{k-1},0}, B_{2^{k-1},1}, \dots, B_{2^{k-1},M-1}]^T$$

Theorem 2.1[8,9]. Let $B(x)$ be the Bernoulli wavelet vector defined above. The derivative of vector $B(x)$ can be expressed as $\frac{dB(x)}{dx} = DB(x)$.

The matrix D is the $2^kM * 2^kM$ square matrix of derivatives defined as follows

$$D = \begin{bmatrix} F & 0 & \dots & 0 \\ 0 & F & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F \end{bmatrix}$$

Where F is an $M * M$ matrix and $(r,s)^{th}$ the element is defined by

$$F_{r,s} = \begin{cases} 2^k\sqrt{(2r-1)(2s-1)}, & r = 2,3,\dots,M, s = 1,2,\dots,r-1, r+s \text{ } is \text{ } odd \\ 0, & otherwise \end{cases}$$

Corollary 2.2[8,9]. The operational matrix for n^{th} derivative can be obtained from

$\frac{d^nB(x)}{dx^n} = D^nB(x), n = 1,2,\dots$ where D^n is the n^{th} power of D.

3 Method of Solution

Consider the Beam equation in linear form $\zeta_{tt} + \zeta_{4x} = 0 \dots\dots\dots\dots\dots\dots\dots (1)$

Taking the Laplace transform on both sides of Eq.(1) , we get

$$s^2\mathcal{L}(\zeta) - \text{s}\zeta(\text{x},0) - \zeta_t(x,0) + \mathcal{L}(\zeta_{4x}) = 0$$

$$s^2\mathcal{L}(\zeta) = \text{s}\zeta(\text{x},0) + \zeta_t(x,0) - \mathcal{L}(\zeta_{4x})$$

$$\mathcal{L}(\zeta) = s^{-1}\zeta(\text{x},0) + s^{-2}\zeta_t(x,0) - s^{-2}\mathcal{L}(\zeta_{4x})$$

Taking inverse Laplace transform on both sides, we get

$$\zeta = \zeta(\text{x},0) + \text{t}\zeta_t(x,0) - \mathcal{L}^{-1}[s^{-2}\mathcal{L}(\zeta_{4x})] \dots\dots\dots\dots\dots\dots\dots\dots (2)$$

Here,

International Journal of Applied Mathematics
Volume 38 No. 11s, 2025
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$$\mathcal{L}^{-1}[s^{-2}\mathcal{L}(t^n)] = \mathcal{L}^{-1}\left[n!\, s^{-(n+3)}\right]$$

$$= \frac{t^{n+2}}{(n+1)(n+2)} \quad , n = 0,1,2, \ldots$$

Clearly,

$$\mathcal{L}^{-1}[s^2\mathcal{L}(.)] = \int_0^t \int_0^t (.)\, dtdt$$

Define a new operator,

$$\Pi = \mathcal{L}^{-1}[s^{-2}\mathcal{L}(.)]$$

Now using Bernoulli wavelet method, we get

$\zeta(x,t) = C^T B(\mathrm{x,t})$ and $\zeta + t\zeta_t(x,0) = V^T B(x,t)$ … … … … … … … … … … (3)

Substituting Eq.(3) in Eq.(2), we obtain

$$C^T = V^T - (C^T D_x^4)\, P_t^4 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (4)$$

To solve algebraic system (4), we required large computational time. In order to overcome this, we introduce the iteration formula as,

$$\zeta_{n+1} = \zeta(x,0) + t\zeta_t(x,0) - \Pi\left(\frac{\partial^4 \zeta_n}{\partial x^4}\right)$$

Now, we will start iteration with initial value, $\zeta_0 = \zeta(\mathrm{x},0) + \mathrm{t}\zeta_t(x,0)$ we get,

$$C_{n+1}^T = C_0^T - [C_n^T D_x^4] P_t^4$$

Now, we solve the Beam equation in the nonlinear form,

$$\zeta_{tt} + \zeta_{4x} + f(\zeta, \zeta_{x,} \ldots) = 0 \ldots \ldots \ldots \ldots (5)$$

Taking the Laplace transform on both sides,

$$s^2\mathcal{L}(\zeta) - \mathrm{s}\zeta(\mathrm{x},0) - \zeta_t(x,0) + \mathcal{L}\left(\zeta_{4x} + f(\zeta, \zeta_{x,} \ldots)\right) = 0$$

$$s^2\mathcal{L}(\zeta) = \mathrm{s}\zeta(\mathrm{x},0) + \zeta_t(x,0) - \mathcal{L}\left(\zeta_{4x} + f(\zeta, \zeta_{x,} \ldots)\right)$$

$$\mathcal{L}(\zeta) = s^{-1}\zeta(\mathrm{x},0) + s^{-2}\zeta_t(x,0) - s^{-2}\mathcal{L}\left(\zeta_{4x} + f(\zeta, \zeta_{x,} \ldots)\right)$$

Taking inverse Laplace transform on both sides, we get

$$\zeta = \zeta(\mathrm{x},0) + \mathrm{t}\zeta_t(x,0) - \mathcal{L}^{-1}\left[s^{-2}\mathcal{L}(\zeta_{4x}) + f(\zeta, \zeta_{x,} \ldots)\right] \ldots \ldots \ldots (6)$$

Now using Bernoulli wavelet method, we get

$\zeta(x,t) = C^T B(\mathrm{x,t})$, $\zeta + t\zeta_t(x,0) = V^T B(x,t)$ and $f(\zeta) = F^T B(x,t)$ … … … … (7)

Substituting Eq.(7) in Eq.(6), we obtain

International Journal of Applied Mathematics
Volume 38 No. 11s, 2025
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$$\therefore C^T = V^T - (C^T D_x^4 + F^T) P_t^4 \ldots\ldots\ldots\ldots\ldots (8)$$

To solve the above algebraic system, we required a large computational time. To overcome this, we introduce the approximation formula as follows:

$$\zeta_{n+1} = \zeta(x,0) + t\zeta_t(x,0) - \Pi\left(\frac{\partial^4 \zeta_n}{\partial x^4} + f(\zeta_n)\right)$$

Now, we will start iteration with initial value, $\zeta_0 = \zeta(\mathrm{x},0) + \mathrm{t}\zeta_t(x,0)$ we get,

$$C_{n+1}^T = C_0^T - [C_n^T D_x^4 + F_n^T] P_t^4$$

4 Application of Laplace Bernoulli wavelet method

Example 4.1 $\zeta_{tt} + \zeta_{4x} = 0,\ \ 0 < x < 2\pi \ldots\ldots\ldots\ldots (9)$

With boundary conditions $\zeta(0,t) = \zeta(2\pi, t) = 0,\ \ \zeta_{xx}(0,t) = \zeta_{xx}(2\pi,t) = 0$

And Initial Conditions $\zeta(x,0) = \sin(x), \zeta_t(x,0) = 0$

The exact solution is given by $\zeta(x,t) = \sin(x)\cos(t)$.

Now, we solve Eq.(9) by Laplace Bernoulli wavelet transform method, we get

$$C^T = V^T - (C^T D_x^4)\, P_t^4 \ldots\ldots\ldots (10)$$

To solve algebraic system Eq.(10) , the approximation formula as follows

$$\zeta_{n+1} = \zeta(x,0) + t\zeta_t(x,0) - \Pi\left(\frac{\partial^4 \zeta_n}{\partial x^4}\right)$$

Expanding $\zeta(x,t)$ by Bernoulli wavelets with following relation and given initial and boundary conditions,

$$C_{n+1}^T = C_0^T - [C_n^T D_x^4] P_t^4$$

x	t	Exact Solution	Approximate Solution	Absolute Error
0	0	0.0000000000000	0.0000000000000	0.0000000000000
0.502654825	0.502654825	0.0595897113507	0.0595897114378	0.0000000000871
1.005309649	1.005309649	0.3919143992690	0.3919178456269	0.0000034463579
1.507964474	1.507964474	0.9353601116461	0.9353667318261	0.0000066201800
2.010619298	2.010619298	1.2900836738539	1.2900836843615	0.0000000105076
2.513274123	2.513274123	1.0633135104401	1.0633135121346	0.0000000016945
3.015928947	3.015928947	0.2496781771467	0.2496781978453	0.0000000206985

International Journal of Applied Mathematics
Volume 38 No. 11s, 2025
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3.518583772	3.518583772	-0.7103981056490	-0.7103986314529	0.0000005258039
4.021238597	4.021238597	-1.2616568681401	-1.2616563415240	0.0000005266161
4.523893421	4.523893421	-1.1663495270710	-1.1663495142363	0.0000000128347
5.026548246	5.026548246	-0.6571638901489	-0.6571634563413	0.0000004338076
5.52920307	5.52920307	-0.1855337417146	-0.1855335469104	0.0000001948042
6.031857895	6.031857895	-0.0078130501140	-0.0078135641790	0.0000005140650
6.283185307	6.283185307	0.0000000000000	0.0000000000000	0.0000000000000

Table1: Comparison between Exact and Approximate solutions with corresponding error values of example 4.1

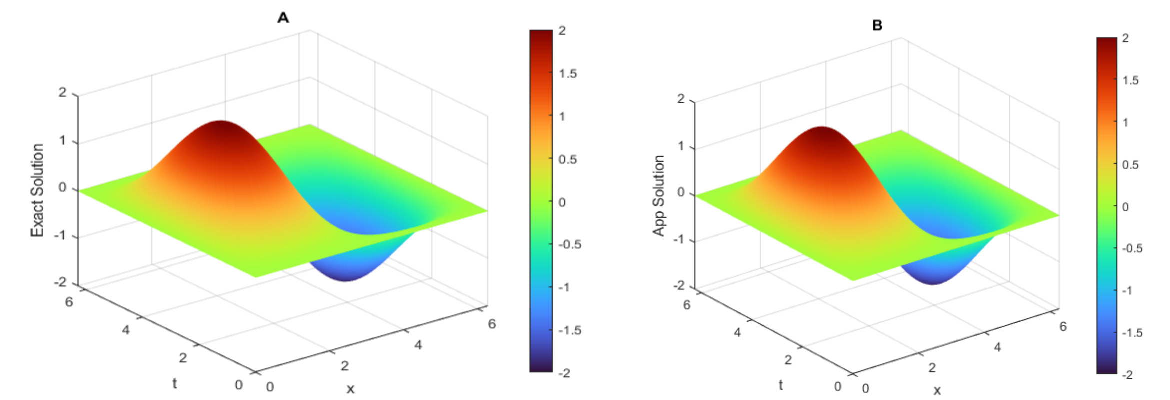

Figure 1: Graphical representation of the exact and approximate solution of example 4.1

Example 4.2 $\zeta_{tt} + \zeta_{4x} = \sin(x)\,,\ \ 0 < x < 2\pi \dots\dots\dots\dots (11)$

With boundary conditions $\zeta(0,t) = \zeta(2\pi, t), \zeta_x(0,t) = \zeta_x(2\pi, t)$

And Initial Conditions $\zeta(x,0) = \zeta_t(x,0) = 0$

The exact solution is given by $\zeta(x,t) = (1-\cos(t)).\sin(x)$.

Now, we solve Eq.(11) by Laplace Bernoulli wavelet transform method, we get

$$C^T = V^T - (C^T D_x^4 + F^T)P_t^4 \dots\dots\dots (12)$$

To solve algebraic system Eq.(12) , the approximation formula as follows

$$\zeta_{n+1} = \zeta(x,0) + t\zeta_t(x,0) - \Pi\left(\frac{\partial^4 \zeta_n}{\partial x^4} + f(\zeta_n)\right)$$

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Expanding $\zeta(x,t)$ by Bernoulli wavelets with following relation and given initial and boundary conditions,

$$C_{n+1}^T = C_0^T - [C_n^T D_x^4 + F_n^T] P_t^4$$

x	t	Exact Solution	Approximate Solution	Absolute Error
0	0	0.0000000000000	0.0000000000000	0.0000000000E+00
0.502654825	0.502654825	0.4221639627510	0.4221641236476	1.6089662352E-07
1.005309649	1.005309649	0.4524135262330	0.4524125641233	-9.6210967987E-07
1.507964474	1.507964474	0.0626666167822	0.0626662546123	-3.6216983010E-07
2.010619298	2.010619298	-0.3852566213879	-0.3852556612349	9.6015300022E-07
2.513274123	2.513274123	-0.4755282581476	-0.4755282301453	2.8002300312E-08
3.015928947	3.015928947	-0.1243449435824	-0.1243448946120	4.8970383096E-08
3.518583772	3.518583772	0.3422735529643	0.3422726446171	-9.0834727651E-07
4.021238597	4.021238597	0.4911436253643	0.4911433147843	-3.1057999966E-07
4.523893421	4.523893421	0.1840622763423	0.1840621751378	-1.0120450517E-07
5.026548246	5.026548246	-0.2938926261462	-0.2938923147862	3.1136000000E-07
5.52920307	5.52920307	-0.4990133642141	-0.4990132514637	1.1275040007E-07
6.031857895	6.031857895	-0.2408768370509	-0.2408761234789	7.1357200038E-07
6.283185307	6.283185307	0.0000000000000	0.0000000000000	0.0000000000E+00

Table2: Comparison between Exact and Approximate solutions with corresponding error values of example 4.2

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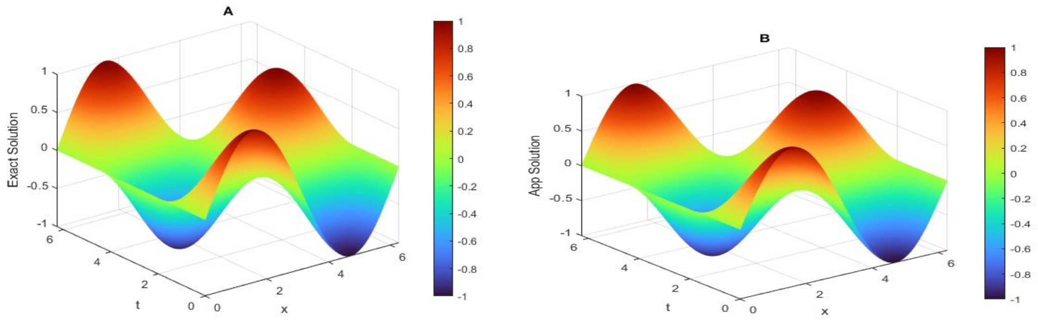

Figure 2: Graphical representation of the exact and approximate solution of example 4.2

5 Conclusion

In the present article, Laplace Bernoulli wavelet transform method is implemented to solve the beam equation in both linear and non-linear form. The method is applied on given illustrative examples and obtained results are prescribed in tabular as well as graphical form. The approximate solutions obtained using the method reflect the accuracy over the exact solution.

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References

[1] L. Debnath and F. A. Shah, Wavelet Transforms and Their Applications, 2nd ed., Birkhäuser, Boston, MA (2015), ISBN: 978-0-8176-8417-4 (hardcover), ISBN: 978-0-8176-8418-1 (eBook), https://doi:10.1007/978-0-8176-8418-1.

[2] Y. S. Choi, K. C. Jen, and P. J. McKenna, The Structure of the Solution Set for Periodic Oscillations in a Suspension Bridge Model, IMA Journal of Applied Mathematics 47, 283–306 (1991), https://doi:10.1093/imamat/47.3.283.

[3] G. Farmakis, J. Kang, P. J. Olver, C. Qu, and Z. Yin, New Revival Phenomena for Bidirectional Dispersive Hyperbolic Equations, Journal of Nonlinear Science 35, 43 (2025), https://doi:10.1007/s00332-025-10136-9.

[4] J. Camacho, M. Bruzon, J. Ramirez, and M. L. Gandarias, Exact Travelling Wave Solutions of a Beam Equation, Journal of Nonlinear Mathematical Physics 18 (2011), 33–49, https://doi:10.1142/S140292511100126X.

[5] D. Chouhan, V. Mishra, and H. M. Srivastava, Bernoulli Wavelet Method for Numerical Solution of Anomalous Infiltration and Diffusion Modeling by Nonlinear Fractional Differential Equations of Variable Order, Results in Applied Mathematics 10, 100146 (2021), https://doi:10.1016/j.rinam.2021.100146

[6] H. Wei and S. Ji, Periodic Solutions for a Class of Semilinear Euler–Bernoulli Beam

Equations with Variable Coefficients, Journal of Dynamics and Differential Equations 37, 237–249 (2025), https://doi:10.1007/s10884-023-10296-w.

[7] G. Hariharan, K. Kannan, and K. R. Sharma, Haar Wavelet Method for Solving Fisher's Equation, Applied Mathematics and Computation 211(2), 284–292 (2009), https://doi:10.1016/j.amc.2008.12.089.

[8] E. Keshavarz, Y. Ordokhani, and M. Razzaghi, Bernoulli Wavelet Operational Matrix of Fractional Order Integration and Its Applications in Solving Fractional Order Differential Equations, Applied Mathematical Modelling 38(24), 6038–6051 (2014), https://doi:10.1016/j.apm.2014.04.064.

[9] D. Rani and V. Mishra, Numerical Inverse Laplace Transform Based on Bernoulli Polynomials Operational Matrix for Solving Nonlinear Differential Equations, Results in Physics 16 (2020), https://doi:10.1016/j.rinp.2019.102836.121-134.

[10] M. Kumar and S. Pandit, Wavelet Transform and Wavelet-Based Numerical Methods: An Introduction, International Journal of Nonlinear Science 13(3), 325–345 (2012), ISSN: 1749-3889 (print), 1749-3897 (online)

[11] D. H. Lehmer, A New Approach to Bernoulli Polynomials, The American Mathematical Monthly 95(10), 905–911 (1988), https://doi:10.1080/00029890.1988.11972114.

[12] Y. Chen and P. J. McKenna, Traveling Waves in a Nonlinearly Suspended Beam: Theoretical Results and Numerical Observations, Journal of Differential Equations 136(2), 325–355 (1997), https://doi:10.1006/jdeq.1996.3155

[13] Y. Chen and P. J. McKenna, Travelling Waves in a Nonlinearly Suspended Beam: Some Computational Results and Four Open Questions, Philosophical Transactions: Mathematical, Physical and Engineering Sciences 355(1732), 2175–2184 (1997), https://doi=10.1098%2Frsta.1997.0116

[14] P. J. McKenna and W. Walter, Travelling Waves in a Suspension Bridge, SIAM Journal on Applied Mathematics 50(3), 703–715 (1990), https://www.jstor.org/stable/2101882

[15] L. D. Humphreys, Numerical Mountain Pass Solutions of a Suspension Bridge Equation, Nonlinear Analysis: Theory, Methods & Applications 28(11), 1811–1826 (1997), https://doi:10.1016/S0362-546X(96)00020-X.

[16] A. C. Lazer and P. J. McKenna, Large-Amplitude Periodic Oscillations in Suspension Bridges: Some New Connections with Nonlinear Analysis, SIAM Review 32(4), 537–578

International Journal of Applied Mathematics
Volume 38 No. 11s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

(1990), https://doi:10.1137/1032120.

[17] S. Doole and S. J. Hogan, A Piecewise Linear Suspension Bridge Model: Nonlinear Dynamics and Orbit Continuation, Dynamics and Stability of Systems 11, 19–47 (1996) https://api.semanticscholar.org/CorpusID:121430305

[18] F. Costabile, F. Dell'Accio, and M. I. Gualtieri, A New Approach to Bernoulli Polynomials, Rendiconti di Matematica e delle Sue Applicazioni, Serie VII 26 (2006).

[19] S. Lal and S. Kumar, Approximation of Functions by Bernoulli Wavelet and Its Applications in Solution of Volterra Integral Equation of Second Kind, Arabian Journal of Mathematics 11, 341–353 (2022), https://doi:10.1007/s40065-021-00351-z.