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ALGEBRAIC PROPERTIES OF INVERSE SEMIGROUPS

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Abstract: - In this Paper we obtained certain interesting results concerning the algebraic properties of inverse semigroup.

Introduction: - An element of a semigroup is said to be regular if there exists $x \in S$ such that $a\,x\,a = a$ and the element 'a' said to be invertible if $a\,x\,a = a$ and $xax = x$ and is denoted by a^{-1}. If every element of S is regular then semigroup is said to be regular semigroup. If every element in S has an inverse, then the semigroup is said to be an inverse semigroup.
It is to be observed that if S is a regular semigroup and if 'a' is regular element of 'S' Then $x\,a\,x = x$ for some $x \in S$, an a Inverse of 'a' as $a(a\,x\,a)\,a = ax\,ax\,a = a\,xa = a$ and $(x\,a\,x)\,a\,(x\,a\,x) = x\,a\,xa\,.\,xa.\,x = x\,a\,x$.
First it is to be observed that an inverse semigroup is a system $(S, ., -1)$ satisfy the following identities:
1. $(x\,y)\,z = x(y\,z)$ for all $x, y, z \in S$
2. $(x^{-1})^{-1} = x$
3. $(x\,x^{-1})\,(y\,y^{-1}) = (y\,y^{-1})\,(x\,x^{-1})$ (idempotents commute)
4. $x\,x^{-1}x = x$

This shows that inverse semigroup is equationally definable , this shows that any homomorphic image of an inverse semigroup is again an inverse semigroup. It is also to be observed that any subalgebra of an inverse semigroup is again an inverse semigroup and the direct product of any family of inverse semigroups is again an inverse semigroup.
Further it is also observed that if x, y are any two elements of an inverse semigroup 'S' then $x^{-1}y$ is an idempotent of 'S' if and only if the greatest lower bounded of x and y exists in the natural order of 'S'. We also obtained certain equivalent conditions for every 'a' and a^{-1} of an inverse semigroup to be commute is obtained.

Key words: - inverse semigroup, regular semigroup , greatest lower bound and least upper bound in S.

First, we start with the following theorem.

Theorem- 1.1 : A system $(S, ., -1)$ is an inverse semigroup iff S satisfies the following conditions
1. $(x\,y)\,z = x(y\,z)$ for all $x, y, z \in S$
2. $(x^{-1})^{-1} = x$
3. $(x\,x^{-1})\,(y\,y^{-1}) = (y\,y^{-1})\,(x\,x^{-1})$
4. $x\,x^{-1}x = x$

Proof: - If S is an inverse semigroup, then clearly 1, 2, 3 and 4 holds.
Conversely suppose that $(S, ., -1)$ is a system is satisfying 1, 2, 3, and 4 from (1) and (4) it is observed that 'S' is a regular semigroup and from (2) it follows that $x^{-1}$is a semigroup inverse of x .Now it is sufficient to show that the inverse of x is x^{-1}.
Assume that $x\,y\,x = x$, $y\,xy = y$.

$$\text{Now } x^{-1} = x^{-1}x\,x^{-1} \quad \text{(By (2)and (4)}$$
$$= x^{-1}(x\,y\,x)\,x^{-1} \quad \text{since } x\,y\,x = x \text{ (by assumption)}$$

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$= (x^{-1}x)(yy^{-1}y)(xx^{-1})$ (since $y = yy^{-1}y$)
$= (x^{-1}x)(yy^{-1})(y\,xx^{-1}) = (yy^{-1})x^{-1}(xyx)x^{-1}$
$= (yy^{-1})(x^{-1}x)(yxx^{-1})$ by (3) $= (yy^{-1})x^{-1}(xyx)x^{-1}$
$= (yy^{-1})x^{-1}(xx^{-1})$ (since $xyx = x$) $= (yy^{-1})(x^{-1}x\,x^{-1})$
$= (yy^{-1}x^{-1})$ (by (2)and (4)

Also $y = yxy = (yy^{-1}y)(xx^{-1}x) = y(y^{-1}y)(xx^{-1})(xy)$
$= y(xx^{-1})(y^{-1}y)(xy)$ by (3) $= (yy^{-1}y)(xx^{-1})$
$= yxx^{-1}$ as $yy^{-1}y = y = (yx)(yy^{-1}x^{-1})$ as $x^{-1} = yy^{-1}x^{-1}$ (by the above)
$= (yxy)(y^{-1}x^{-1})$
$= yy^{-1}x^{-1}$ (since $yxy = y$) $= x^{-1}$ as $x^{-1} = yy^{-1}x^{-1}$.

The following example establishes the independence of the axioms mentioned in the above axioms:
Example 1.1: let $S = \{e, a, b\}$
Define $'.'$ in 'S' $a.a = b.b = e.e = e$, $a.b = b.a = a$, $a.e = e.a = a$
$b.e = e.b = b$
Define $'-1'$ is 'S' by $a^{-1} = a, b^{-1} = b$, $e^{-1} = e$

.	e	a	b
e	e	a	b
a	a	e	a
b	b	a	e

Thus $(S, . , -1)$ is a system satisfying (2), (3) and (4) but not (1)
To show that $(a.a).b \neq a.(a.b)$
Now $(a.a).b = e.b = b$;Also $a.(a.b) = a.a = e$
Hence $(a.a).b \neq a.(a.b)$, Thus (1) is not a consequence of true remaining
Example 1.2 : Consider $S = \{e, a, b\}$
Define $'.'$ in 'S ' by $a.a = a$, $b.a = b$, $e.e = e$, $a.b = b$, $b.a = a$
$a.e = e.a = e$, $b.e = e.b = e$

.	e	a	b
e	e	e	e
e	e	a	b
b	e	a	b

Define $'-1'$ is 'S' by $a^{-1} = a$, for all $a \in S$
Thus $(S, . , -1)$ is a system satisfying (1), (3) and (4) but not (2) here
$(a^{-1})^{-1} = b^{-1} = b \neq a$, therfore $(a^{-1})^{-1} \neq a$, thus (2) is not a consequence of remaining .
Example 1.3 If $'X'$ is any set with at least two elements
Take $S = X \times X$;
Define $'.'$ in 'S' $(x_1, y_1).(x_2, y_2) = (x_1, y_2)$ and
Define $'-1'$ is 'S' by $(x_1, y_1)^{-1} = (y_1, x_1)$
Thus $(S, . , -1)$ is a system satisfying (1), (2) and (4) but not (3)
Chose $x_1, x_2 \in X$ and $x_1 \neq x_2$ and let $y_1, y_2 \in X$ then $(x_1, y_1).(x_1, y^{-1}{}_1).(x_2, y_2)$
$(x_2, y^{-1}{}_2) = (x_1, y_1).(y_1, x_1).(x_2, y_2).(y_2, x_2) = (x_1, x_1).(x_2, y_2)$ and
$= (x_2, y_2).(x_2, y_2)^{-1}(x_1, y_1).(x_1, y_1)^{-1}$
$= (x_2, y_2).(y_2, x_2).(x_1, y_1).(y_1, x_1)$

$= (x_2, y_2) . (x_1, y_1) = (x_2, x_1) \neq (x_1, x_2)$
Hence (3) is not a consequence of the remaining.
Example 1.4 : Let $(S, \ .)$ be the semigroup of positive integers under ordinary multiplication.
Define '.' in $S \times S$ by $(a , b) . (c , d) = (ac , bd)$ for all $a, b, c, d \in S$
Define $' - 1 '$ is 'S' by $(a, b)^{-1} = (b, a)$
Then 'S' satisfies $(1), (2)$ and (3) but not (4) as $(2 , 1) (2 , 1)^{-1} (2 , 1) = (2 , 1). (1, 2) . (2 , 1) = (4 , 2) \neq (2 , 1)$
From the following theorem it is observed that an element of an inverse semigroup 'S' which is smaller than an idempotent of 'S' in the partial order is itself an idempotent.

Theorem 1.2: Suppose 'a' is an element of an inverse semigroup 'S' and if 'e' is an idempotent of 'S' with $a \leq e$, then 'a' is an idempotent.
Proof : By the definition of $a \leq e$, impels $a^{-1}e = a\, a^{-1}$ but ' e ' is an idempotent and $e^2 = e$ and $e^{-1} = e$ imply that $a\, e = a\, a^{-1}$ (since $(a\, e)^{-1} = a\, e$ but $(a . e)^{-1} = e^{-1}a^{-1} = e\, a^{-1}$ so that , we have $a . e = a\, a^{-1}$ which is an idempotent and hence $(a . e)^{-1} = a\, e$.
Thus $e^{-1}a^{-1} = a\, e$ and hence $e\, a^{-1} = a\, e$. Since $a\, a^{-1} = a\, e$, we have $a\, a^{-1} = e\, a^{-1}$ and hence $a\, a^{-1}\, a = e\, a^{-1}\, a$ imply that $a = a\, a^{-1}a$, Which is an idempotent, then ' a ' is an idempotent of ' S '
From the following theorem it is observed that if $a \leq b$ then $ac \leq bc$ for all c in S

Theorem 1.3: If 'S ' is an inverse semigroup and $a , b \in S$. If $a \leq b$ then $ac \leq bc$ for any element 'c' of 'S'.
Proof: Assume $a \leq b$ and let $c \in S$, then by theorem of Clifford [1] , we have $a^{-1} \leq b^{-1}$ and therefore $e^{-1}a^{-1} \leq e^{-1}\, b^{-1}$ so that $(e^{-1}a^{-1})^{-1} \leq (e^{-1}\, b^{-1})^{-1}$ and hence $a\, e \leq b\, e$

Theorem 1.4 : If 'S' is an inverse semigroup. If $a, b \in S$ are such that infimum of 'a' and 'b' namely $a \wedge b$ exists in 'S' then infimumu of a^{-1} and b^{-1} exists and is equal to $(a \wedge b)^{-1}$
Proof: Since $a\ b \leq a$ and $a\ b \leq b$, We have $(a \wedge b)^{-1} \leq a^{-1} . b^{-1}$, there fore $(a \wedge b)^{-1}$ is a lower bound of a^{-1} and b^{-1}. Let 't' be lower bound of a^{-1} and b^{-1}, so that $t \leq a^{-1}$ and $t \leq b^{-1}$ and hence $t^{-1} \leq a$ and $t^{-1} \leq b$, therefore $t^{-1} \leq a \wedge b$ and hence $t \leq (a \wedge b)^{-1}$.
Thus $(a \wedge b)^{-1}$ is the greatest lower bound of $a^{-1}$and b^{-1}.

Theorem 1.5:If 'S' is an inverse semigroup, and $a, b \in S$ are such that supremum of ' a ' and ' b ' . $(a \vee b)$ exists in 'S' then supremum of a^{-1} and b^{-1} exists and is equal $(a \vee b)^{-1}$.
Proof: Since $a , b \leq a \vee b$ .We have $a^{-1} , b^{-1} \leq (a \vee b)^{-1}$, there fore $(a \vee b)^{-1}$ is an upper bound of a^{-1} and b^{-1} . Let ' t ' be an upper bound of $a^{-1}$and b^{-1} so that $a^{-1} \leq t$ and $b^{-1} \leq t$ and hence $a \leq t^{-1}$ and $b \leq t^{-1}$, thus $(a \vee b) \leq t^{-1}$ and hence $(a \vee b)^{-1} \leq t$ so that $(a \vee b)^{-1}$ is the least upper bound of a^{-1} and b^{-1}.

In the following theorem it is observed that an equivalent conditions on the elements of x and y of an inverse semigroup are obtained in order that $x^{-1}y$ be an idempotent of 'S '.

Theorem 1.6 : If 'S' is an inverse semigroup and $x , y \in S$ then the following are equivalent
1. $x^{-1}y$ is idempotent of ' S'.
2. $y^{-1}x$ is idempotent of ' S'.
3. $yy^{-1}x \leq y$ in the natural partial order on ' S'
4. $x\, x^{-1}y \leq x$ in the natural partial order on ' S'

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5. Greatest bounded of x and y exists and is equal to $x x^{-1}y$
6. Greatest bounded of x and y exists and is equal to $yy^{-1}x$
7. Greatest bounded of x^{-1} and y^{-1} exists and is equal to $y^{-1}x x^{-1}$
8. Greatest bounded of x^{-1} and y^{-1} exists and is equal to $x^{-1}y y^{-1}$
9. $x^{-1}y y^{-1} \leq y^{-1}$
10. $y^{-1}x x^{-1} \leq x^{-1}$.

Proof: $(1) => (2)$.
Assume that $x^{-1}y$ is an idempotent of ' S 'and hence its inverse $(x^{-1}y)^{-1} = y^{-1}x$ is an idempotent of ' S'.
$(2) => (3)$.
Assume that $y^{-1}x$ is an idempotent of ' S ', so that $y y^{-1}x = y (y^{-1}x) \leq y$, hence $y y^{-1}x \leq y$.
$(3) => (4)$.
Assume that $y y^{-1}x \leq y$ in the natural partial order in ' S ' and hence $y^{-1} y y^{-1}x \leq y y^{-1}$, thus $y x \leq y^{-1} y$ which is an idempotent since $y^{-1}x$ is an idempotent of 'S ', therefore

$$x x^{-1}y = x (x^{-1}y) \leq x$$

$(4) => (5)$.
Assume that (4) holds (i.e.) $x x^{-1}y \leq x$ in the Natural partial order on ' S ' then $x x^{-1}y = (x x^{-1}) y \leq y$, thus $x x^{-1}y$ is lower bound of x and y, let ' t ' be a lower bound of x and y then $t \leq x , t \leq y$ so that $tx^{-1} = t t^{-1} = t y^{-1}$ (as $t \leq x$ implies that $t x^{-1} = t t^{-1}$, $t \leq y => t y^{-1} = t t^{-1}$) .
Now $t (x x^{-1}y)^{-1} = t (y^{-1}x x^{-1}) = (t y^{-1}) (x x^{-1}) = (t x^{-1}) (x x^{-1})$
(as $t y^{-1} = t x^{-1}$) $= t (x^{-1}x x^{-1}) = t\ x^{-1}$ so that $t \leq x x^{-1}y$
$(5) => (6)$.
Assume that (5) holds (i .e) greatest lower bound x and y exists and is equal to $x x^{-1} y$ so that $x x^{-1} y \leq x$ and hence $x^{-1} x\ x^{-1}y \leq x^{-1} x$, thus $x^{-1}y \leq x^{-1} x$, which is an idempotent by using theorem 1.2, we have $x^{-1}y$ is an idempotent of ' S ' . Hence inverse $y^{-1} x$ is also an idempotent of ' S ' and hence by symmetry , greatest lower bound of x and y exists and equal to $y y^{-1} x$.
$(6) => (1)$.
Assume that (6) holds (i .e) greatest lower bound x and y exists and equal to $y y^{-1} x$ so that $y y^{-1} x \leq y$ and hence $y^{-1}y y^{-1} x \leq y^{-1}y$ and hence $y^{-1}x \leq y^{-1}y$ which is an idempotent , by using theorem 1.2, $y^{-1} x$ is an idempotent and hence its inverse $x^{-1}y$ is an idempotent of
' S '
$(1) => (7)$.
Assume that (1) holds then $x^{-1}y$ is an idempotent of ' S ' from this it is observed that greatest lower bound of x and y exists and is equal to $x x^{-1}y$, by using theorem 1.4, greatest lower bound of x^{-1} and y^{-1} exists and is equal to $y^{-1} x x^{-1}$.
$(7) => (2)$.
Assume that (7) holds then greatest lower bound of x^{-1} and y^{-1} exists and is equal to $y^{-1} x\ x^{-1}$ and hence $y^{-1} x\ x^{-1} \leq x^{-1}$, thus $y^{-1} x\ x^{-1}x \quad x^{-1} x$
so that $y^{-1} x \leq x^{-1} x$, which is an idempotent and hence $y^{-1} x$ is an idempotent of ' S ' by theorem 1.2.
we have $(2) => (6)$ imply that greatest lower bound of x and y exists and is equal to $y y^{-1} x$ and hence again by theorem 1.4 , we have greatest lower bound of x^{-1} and y^{-1} exists and is equal to $x^{-1}x y^{-1}$.
clearly $(8) => (9)$ holds.
$(9) => (10)$.

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Assume that (9) holds (i e) $x^{-1}\,y\,y^{-1} \leq y^{-1}$ and hence $x^{-1}\,y\,y^{-1}y \leq y^{-1}y$ and hence $x^{-1}y \leq y^{-1}y$, which is an idempotent ‘ S ’ by using theorem 1.2, x^{-1} is also an idempotent of ‘ S ’, hence its inverse $y^{-1}x$ is an idempotent of ‘ S’ imply that $y^{-1}x\,x^{-1} = (y^{-1}x)x^{-1} \leq x^{-1}$
$(10) => (1)$.
Assume that (10) holds then $y^{-1}\,y\,x^{-1} \leq x^{-1}$and hence $y^{-1}x\ x^{-1}x \leq x^{-1}x$ imply that $y^{-1}\,y \leq x^{-1}\,x$, which is also an idempotent of ‘S’ and hence by using theorem 1.2, $y^{-1}\,x$ is also idempotent of ‘ S ’ and hence its inverse $x^{-1}\,y$ is also an idempotent of ‘ S’.

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