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ADVANCES IN ALGEBRAIC TOPOLOGY: PERSISTENT HOMOLOGY, HIGHER CATEGORY THEORY, AND APPLICATIONS TO MODERN GEOMETRIC STRUCTURES

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Abstract

This study presents a theoretical synthesis that connects algebraic topology, persistent homology, and higher category theory to provide a unified mathematical perspective for understanding geometric and algebraic structures. The motivation lies in advancing the conceptual foundations of topology by integrating the multi-scale analysis of persistent homology with the hierarchical relational framework of higher categories. The paper proposes a higher categorical interpretation of persistence modules, in which morphisms and higher morphisms encode the evolution and interaction of topological features across filtration levels. A bifunctorial framework is developed to connect filtered complexes with higher morphism spaces, enabling the representation of topological persistence within categorical vector spaces. This structural formulation enriches the study of modern geometric frameworks such as manifolds, orbifolds, and derived geometries, and provides a categorical foundation for modelling transformations in topological quantum field theories. The research emphasizes interpretability, consistency, and abstraction rather than computational experimentation. By embedding persistent homology within higher categorical contexts, algebraic topology gains an extended capacity to model, compare, and classify complex geometric and algebraic relationships, contributing to the broader development of theoretical mathematics.

Keywords: Algebraic Topology, Persistent Homology, Higher Category Theory, Geometric Structures, Topological Invariants.

1. Introduction

Algebraic topology is one of the most powerful fields of modern mathematics, which offers an abstract but powerful way to describe and classify geometric and topological structures. It bridges the discrete and continuous worlds of mathematics by algebraic invariants, including homology and

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cohomology groups, which define spaces in terms of their topological properties, but not their geometric dimension and size or shape [1]. This is the field that takes hunch ideas of shape and relation into formal isometric terms, permitting the study of continuity, deformation, as well as the equivalent in a rigorous way. Algebraic topology has, over time, grown to be more than the classical field of algebraic topology to interrelate with a broad spectrum of mathematical fields, such as geometry, category theory, and computational mathematics [2].

One of the most important modern contributions in this field is persistent homology, which is a notion that builds on the old theory of homology for the analysis of multi-scale phenomena. Persistent homology, a branch of persistent homology, represents the persistence of topological structures like connected components, holes and voids at varying resolutions [2]. Topology is not considered a static structure but the way in which the structure evolves, and the filtrations are used to encode the changes in the topology and show how the space changes when the parameters change. This has been particularly useful in topological data analysis (TDA) whereby geometric or high-dimensional data are analyzed via algebraic invariants. Persistent homology, though frequently used in a computational manner, is theoretical in nature, based on the terminology of algebraic topology and functorial mappings between spaces and spaces and mappings of spaces to spaces and spaces to vector spaces.

Along with these developments, a meta-language of abstract mathematics, higher category theory, has developed. Based on the concepts of common category theory, it introduces morphisms of morphisms, which establish a hierarchy of relationships that are used to model objects and transformations as well as transformations of transformations [3]. The topology and geometry are particularly concerned with this overlay of abstraction, where topological and geometrical mutations tend to possess rich higher-level structures. Higher category theory gives the conceptual means of describing complicated systems of interactions in topology, homotopy, and algebraic geometry by generalizing categorical structures. That way, it extends the scope of the category theory into the realms of contemporary mathematics [4].

Geometric topology is an additional theoretical development that looks at the geometric reflection of topological spaces, especially manifolds and their intrinsic structures [4]. It is the study of the way local geometry may affect global topology, akin to the way abstract topology can be understood in concrete geometry. Contemporary literature focuses on the ability to furnish manifolds with extra geometric structures and unveils the interactions between curvature, symmetry and topological invariance [5]. This unification of algebraic, geometrical and categorical points of view is an expression of a major goal of modern topology: to transcend isolated formalisms and to have a comprehensive vision of mathematical structure.

An important move toward the achievement of this synthesis is provided by the construction of functorial semantics, which views whole mathematical theories as categories and the functors between them as the relations between them [5]. This notion makes algebraic topology a theory of spaces, as well as a theory of how to change systems of spaces, and, thus, generalizes homological views of the classical. It is consistent with the theory of geometric structures on manifolds, whose morphisms are continuous deformations or symmetries that leave some invariants fixed [6]. These concepts have transformed the perception of algebra and geometry of mathematicians, and have given a formal interface where discrete, continuous, and categorical approaches meet.

The structure of homology and cohomology theories was formalized by the foundational axioms of Eilenberg and Steenrod, and now we are able to define topological invariants in a systematic and categorical way [7]. Their axiomatic system was a breakthrough in the process of algebraizing topology since the homological constructions were subject to universal laws of functoriality and naturality. This is what led to additional generalizations of modern mathematics, such as categorical and homological techniques, which generalize these concepts to new abstract settings. These axioms are still important today, being the foundations on which persistent homology and higher categorical structures are both constructed.

In recent studies in mathematical physics and geometry, the significance of unified models has been emphasized, relating topological and algebraic and physical principles. The quest to have a model beyond grand unification, as discussed by Wang, is an example of how abstract algebraic and topological concepts can be used to inform the theory of symmetry, gauge structure and higher-dimensional space [8]. These interdisciplinary activities enhance the belief to the fact that the language of algebraic topology and category theory crosses structural and transformational boundaries and provides universal techniques to grasp the nature of structure and change.

In modern algebra, algebraic reasoning has been generalized into new areas by categorical, homological, and combinatorial approaches developed by researchers like Srivastava, Herzog and Leroy [9]. The underlying compatibility of homological algebra with categorical logic in these approaches gives mathematicians the ability to model more effectively the transformations both between and within the various systems of algebra. These frameworks are natural complements to higher category theory as both are intended to bring out the relational nature of the phenomena of mathematics by abstraction and generalization.

Lastly, the logical form and space are more deeply united with the philosophical aspect of topology and geometry. According to Russell and Potter, geometry is not only based on spatial intuition but also on the logical arrangement of relations [10]. This philosophical observation is echoed in the categorical reformulation of topology, in which the nature of geometry is no longer defined in terms of coordinates, but by the morphisms or equivalences and transformations. With these developments in mind, in this paper, we are going to discuss the conceptual intersection between persistent homology and higher category theory in its larger context of modern algebraic topology. The study attempts to explain the way in which these frameworks work together to deepen our insight into geometric structures by focusing on the theoretical coherence, as opposed to computational demonstration.

Objectives:

1. To develop a conceptual framework that incorporates persistent homology in higher categorical contexts and develop the conceptual undercarriage of algebraic topology
2. To investigate the way this integration contributes to the understanding of the contemporary geometrical structures, the connection of algebraic, categorical, and geometric paradigms in a consistent synthesis of theories

2. Foundational Concepts and Preliminaries

Three related mathematical fields on which this research is based are algebraic topology, persistent homology, and higher category theory. Both of these frameworks have fundamental conceptual resources for the structural and relational aspects of topological and geometric systems. Algebraic topology provides the grammar of space analysis in terms of algebraic invariants. Algebraic topology and persistent homology are analogies of computational ideals that develop ideas of algebraic topography into a computational/filtration paradigm, and mathematical layering and higher category theory provide the abstraction basis of unifying medical operations and relations across multiple levels.

2.1 Algebraic Topology Basics

Algebraic topology is a rigorous association between intuitive concepts and alchemy formality, an exertion of space in which spaces are examined not by their points but by the affiliations, forms that hold in a continuous cause of deformation. One of the key concepts in this field is the simplicial complex, which is a combinatorial model of topological spaces. Simplicial complexes can be used to represent geometric and topological data, and thus represent discrete structures through a collection

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of simplices points, edges, triangles and their higher-dimensional analogues, which are assembled into a larger structure [11]. The geometry of such constructions may be reduced to algebraic ultimate it becomes feasible to carry out the systematic computations as well as get invariants of spaces.

In this context, the homology and cohomology theories emerge as the key concepts of space classification that classify the spaces to an upper level, topologically equivalent. Homology groups Algebraic groups are associated with spaces by their counting of cycles and voids in each dimension, and cohomology is the dual of this, the behaviour of functions and differential forms on the same structures [12]. The interactions of these two theories give a full algebraic account of the notion of connectivity, which is a foundational notion in contemporary topology.

One important aspect of algebraic topology is that it satisfies the Eilenberg Steenrod axioms, which are a formalisation of the properties that any homology theory should have [13]. These axioms, homotopy, exactness, excision and dimension are such that homology is constant in all topological spaces, and that the structural relationships between them remain constant. Their introduction represented a fundamental change of ad hoc techniques to a functorial and unified one in which homology is a mapping between classes of spaces and algebraic structures. The system is axiomatic and this axiomatic system is what defines the essence of topological invariants, as well as provides the basis in the subsequent generalisations of topology, which come in the form of categorical and computational systems.

Algebraic topology is powerful because it is abstract: in this case, one does not deal with the geometry of spaces, but with the algebraic relationships that can exist between them in various situations. Both classical and modern constructions confirm that algebraic topology is the most important field of mathematics, around which geometry, algebra, and category theory meet [14].

2.2 Persistent Homology Framework

Persistent homology extends the principles of algebraic topology by introducing a filtration-based perspective on spaces, an ordered sequence of nested subspaces through which topological features are tracked as a parameter varies. Each stage of this filtration yields a chain complex, whose homology groups describe the creation and annihilation of features such as connected components and holes [15]. The persistence of these features across multiple scales is captured in persistence diagrams, or barcodes, which provide a compact algebraic summary of geometric structure.

From a formal standpoint, persistent homology can be viewed as a functor. $PH: \mathcal{F} \to \text{Vect}$, where $\mathcal{F}$ Represents a category of filtered simplicial complexes, and Vect denotes the category of vector spaces. This functorial nature emphasizes that persistent homology preserves the structure of morphisms, mapping continuous transformations in topological spaces to linear transformations in algebraic spaces [16]. As such, it naturally aligns with categorical reasoning and provides a bridge between discrete computational methods and continuous topological theory.

The conceptual power of persistent homology lies not merely in its computational applicability but in its theoretical coherence with classical topology. The persistence module, defined as a family of vector spaces connected by linear maps reflecting inclusions in a filtration, encapsulates the algebraic evolution of topology over a parameter. By interpreting these modules through categorical structures, persistent homology offers a framework that connects traditional algebraic topology to higher-order abstractions, a theme central to the synthesis explored in this study [15], [16].

2.3 Higher Category Theory Essentials

The category theory starts with the basic concept of objects and morphisms where the connection between the mathematical structures is examined by considering the mappings that maintain the composition and identity [17]. Categories involve formalization of mathematical reasoning that is concerned not with the internal constitution of objects but with transformations between them. Naturally this can be generalized to an even more topological case, where spaces and continuous maps can be modeled in terms of categories, and hence a more robust notion of structural similarity and invariance can be found.

The expanded category theory introduces morphisms of morphism and provides structures that reflect more intricate interrelationships as do topology and geometry. Morphisms between morphisms (so-called 2-morphisms) in a 2-category facilitate the representation of homotopies between continuous maps and higher-dimensional categories extend the concept even further to represent whole systems of transformations [18]. These structures are able to represent fine stratifications of abstraction, which are exactly the sort that arise when one considers relations between topological invariants, persistence modules, and geometric morphisms.

Here, the higher categories give the required structure needed in arranging the connections between algebraic and geometric objects in algebraic topology. The shift of standard to higher categories makes topology a study of spaces into a study of systems of spaces and their interrelations, which is homological and homotopical data being hierarchical. This opinion is beautifully reflected in categorical methods of topology, in which functors, natural transformations, and higher morphisms constitute the logical structure of the contemporary geometric reasoning [19].

The combination of these elementary ideas in algebraic topology, persistent homology, and higher category theory forms the conceptual basis of the creation of a single mathematical system. All these add an essential point of view, where algebraic topology provides the invariants, persistent homology scales them up and higher category theory gives them a coherent structurally universal language.

3. Persistent Homology and Its Algebraic Structure

Persistent homology offers a rigorous mathematical way of measuring the changes of topological characteristics at a variety of different scales in a filtration of spaces. The mathematical formulation is largely based on the purely mathematical origins of persistence and dwells on the structural mapping of spaces and how related are the homology groups to spaces. Principally speaking, persistent homology takes the topology of a space and studies the time-dependence of the space, capturing the birth and death of homological infrastructure as larger algebraic stories [20].

In essence, a filtration is a sequence of topological spaces, that are nested.

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n$$

where each inclusion $X_i \hookrightarrow X_{i+1}$ Induces a corresponding sequence of homology groups connected through linear maps. These sequences form the foundation of the persistence module, defined as a family of vector spaces. $\{H_k(X_i)\}_{i\in I}$ together with homomorphisms $f_{ij}: H_k(X_i) \to H_k(X_j)$ for all $i \leq j$. This structure can be expressed functorially as

$$PH: \mathcal{F} \to \text{Vect},$$

where $\mathcal{F}$ Is the category of filtered simplicial complexes, and Vect is the category of vector spaces over a chosen field [21].

Within this functorial setting, each morphism in $\mathcal{F}$ Corresponds to a continuous inclusion between filtered spaces, while the induced morphism in Vect represents the associated linear transformation between homology groups. This functoriality ensures that the structural information about inclusions and deformations of spaces is preserved within the algebraic representation [22]. Hence, persistent homology transforms geometric evolution into a system of algebraic morphisms, offering a categorical interpretation of shape and connectivity that is invariant under continuous deformations.

The stability theorem is one of the most important conceptual developments in persistent homology which guarantees that when the input data or filtration is slightly perturbed, there will be only a slight change in the resulting persistence diagram. Mathematically, this is a proof that persistence diagrams are continuous functions of the filtration, which do not change the strength of topological invariants [23]. This guarantee can be theoretically justified by such results giving an indication that algebraic tissues used to calculate persistent homology extract intrinsic geometrical structures and not noise or artificial parameter sensitivity.

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Algebraically, persistence modules can be viewed as graded vector spaces, with the grade of a persistence module being an index of filtration or a scale parameter [24]. These graded structures enable homology groups of various levels of filtration to be grouped in one algebraic object, the linear maps are used to record the relation between them. In this context, persistence modules extend the concept of graded algebras, which is a topological environment, to which the grading is the lifespan of features. This idea is formalized in a more recent body of modern theoretical work by the notion of a lifespan functor, which relates to every topological feature two birth and death parameters which jointly encode its persistence in the filtration [20].

Moreover, one may define the categorical interpretation of persistent homology, which allows describing persistence diagrams as not only graphical summaries, but also morphisms in a diagrammatic category. Every persistence diagram may therefore be regarded as a morphism which is the passage between two states of filtration in the functor PH [25]. This opinion focuses on the fact that persistence diagrams are not external diagrams but rather inherent algebraic objects that contain the geometry of change.

Persistence modules may then be further structured in an algebraic way with exact structures determining how sub modules, quotient modules, and homomorphisms relate to each other in the category of persistence modules [26]. Precise sequences of these modules are the reflections of the breaking down of complicated topological transformations into less complicated algebraic parts. This strategy underlines the importance of persistence as an actual algebra theory, as opposed to a heuristic that is a computation.

In order to demonstrate such structural correspondences, Table 1 is a summary of the categorical mapping between topological and algebraic objects in persistent homology.

Table 1. Categorical correspondence between geometric and algebraic components in persistent homology

Topological Concept	Algebraic Counterpart	Categorical Representation
Filtration of spaces	Indexed sequence of vector spaces	Objects in category $\mathcal{F}$
Inclusion of subspaces	Linear map between homology groups	Morphism in Vect
Persistence module	Graded vector space	Functor$PH: \mathcal{F} \rightarrow$ Vect
Persistence diagram	Morphism encoding feature lifespan	Diagrammatic morphism in Vect
Stability of persistence	Continuous dependence on filtration	Natural transformation between functors

Each topological construct has an algebraic counterpart in the framework of categorical theory, which as Table 1 illustrates makes it apparent that not only is persistent homology a computational technique, but it translates geometry to algebra and vice versa. The scaffolding that is afforded by the use of functors, morphisms, and natural transformations is the one in which geometric evolution can be algebraically encoded [27].

The fact that persistent homology is connected to other fields of mathematics, including symplectic geometry and analysis, is yet another indication of its interpretive strength. Persistence has been used to examine invariant properties in geometrical flows, differential equations and metric spaces, where its close relations to geometric analysis are emphasized [28]. Simultaneously, current studies have demonstrated that even persistence barcodes can be given an algebraic and statistical structure, which can be used as elements of vector spaces or manifolds in which higher mathematical contexts can be learned and represented [29].

In this way, the continuity of topological change via algebraic invariants as described by persistent homology is seen as a powerful theory in mathematics that is algebraic and categorical in nature. The development of its formulation as a functor, its formulation in expressing it through the language of graded vector spaces, and through morphisms and exact structures, all of it give it a place among the foundations of modern algebraic topology, a formulation which can be seen to bring together the intuitions of geometry with the language of categories and construals.

4. Integration of Higher Category Theory in Topological Frameworks

Higher category theory is a radical generalization of classical homological algebra, and gives the means to study the properties of relations between morphisms as well as between objects. In a typical category theory, morphisms are between mathematical objects, like spaces or groups, however, higher category theory introduces morphisms between morphisms in a so-called 2-morphism, and additional layers of abstraction, forming multi-dimensional webs of relationships [30]. This framework enables mathematicians to model objects in addition to their direct mappings, but also the mappings between such mappings, which is the view of the changing nature of structures experienced by topology.

The original idea of the introduction of higher categories into topology is the necessity to describe not only the fixed states of spaces, but also the dynamical interaction between them [31]. A larger type, especially an (infinity,n)-category, may represent an infinite number of layers of morphisms, and how one space is connected to another in terms of dimension and scale [30]. In this respect, higher category theory is a supplement to our homotopical and topological thought in that it gives us a language that can represent the continuous transformations as hierarchical algebraic objects.

In this building homotopical categories intertwine the field of higher category theory with that of topology, giving homotopies and continuous deformations of maps as homotopical morphisms [32]. This method brings to light the fact that homotopies themselves, as opposed to being solitary phenomena, may have relationships in terms of 2-morphisms or 3-morphisms. These relations are the systematic ones that are defined by the logical framework of the higher topos theory and allow to categorically model the spatial continuity and transformation [33]. In this context, topological properties and logical thinking can be used simultaneously, and the intuitions of geometry can be translated into formal categorical relations [31].

Another important aspect of this integration is that higher categories possess a monoidal structure. Monoidal and tensor categories represent the way objects form and interact, just as the way simplices form in a simplicial complex [34]. At the level of higher categories, such operations are used to describe the combinations of not only objects but also morphisms and transformations. This analogy is reflected in generalization of the multi-scale relationship experienced in persistent homology topological features changing and coalescing at different filtrations with algebraic stability.

The formal analogy between persistent homology and higher category theory can be seen when it comes to the fact that they both have a similar ability to encode evolution and transformation. The persistent homology of a topological structure traces the occurrence and stability of topological structures under varying levels of filtration, whereas higher categories trace the evolution of morphisms and transformations themselves through hierarchical structures [35]. Both models are based on functorial principles, one of which is a topological transformation to a vector space, and the other is a map of layers of morphisms between the levels of categories.

Homotopy type theory extends this relationship by giving a synthetic formulation of this relationship, with spaces identified as types and logical equivalences as homotopies [36]. This synthetic approach, which pits logic, algebra and geometry with one conceptual framework, is similar in approach to persistent homology which applies the approach of unifying algebraic and topological reasoning. Likewise, in topological quantum field theory (TQFT), constructions that are higher categorical functors are used to describe a transformation between states or spaces across dimensions, and persistence-like behaviour in physical systems [37]. Categorical structures in both instances encode the way systems change in a coherent manner in response to transformation.

In order to illustrate these correspondences, Table 2 maps out the correspondence between the basic concepts of persistent homology and their higher category theory counterparts.

Table 2. Conceptual correspondences between persistent homology and higher category theory

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Concept in Persistent Homology	Analogue in Higher Category Theory	Interpretation
Filtration of spaces	Hierarchical system of morphisms	Multi-level structural scaling
Persistence module	n-category or ∞-category	Nested relationships among invariants
Functor$PH: \mathcal{F} \rightarrow$ Vect	Functor between higher categories	Translation across categorical levels
Persistence diagram	Higher categorical diagram	Encoded transformations between morphisms
Stability theorem	Homotopy coherence condition	Continuity across categorical hierarchies

Each construct of persistent homology, as observed in Table 2, is associated with a natural categorical concept that represents relationships between transformations. The persistence module is an n-category, hierarchical, with algebraic data being arranged in a hierarchical manner, and persistence diagrams being higher categorical diagrams, which describe morphisms of transformations. This analogy serves to emphasize the fact that persistent homology could be regarded as a lower-dimensionalization of a higher categorical structure, which implements multi-scale invariance as functorial composition [35].

Lastly, the semantics of functorial semantics is the one that provides a unification layer to these two domains. It makes precise the relation between topological space change and change in algebraic and logical systems, and guarantees the preservation of structure at the different levels of abstraction [38]. This homologism is part of the conceptual bridge: persistent homology is conceptualizing topological persistence and higher category theory is conceptualizing transformational persistence. Also, they provide a unified theoretical framework to describe the geometry of evolution both algebraically and categorically.

5. Theoretical Framework for Unified Interpretation

The synthesis of persistent homology and higher category theory is a theoretical synthesis that builds a unifying mathematical system that can be used to capture geometric and algebraic evolution using the abstraction of categorical theory. This section presents a conceptual model, which ties filtered complexes of persistent homology to higher morphism spaces of category theory and results in a generalized formulation of the persistent homology functor in a higher categorical one. The framework presented here places emphasis on interpretability, coherence and structural generality, as opposed to computational detail, to show how categorical hierarchies may be used to represent persisting topological structures at more than one level of abstraction [39].

At the heart of this framework lies the reinterpretation of the persistent homology functor as

$$\mathcal{PH}: \mathcal{C}_{fil} \rightarrow \mathcal{HCat}(\text{ Vect }),$$

where $\mathcal{C}_{\text{fil}}$ represents the category of filtered simplicial complexes, and $\mathcal{HCat}(\text{ Vect })$ Denotes the higher categorical vector spaces-vector spaces equipped with morphisms, higher morphisms, and composition rules. This formulation extends the classical functor. $PH: \mathcal{F} \rightarrow$ Vect by embedding it into a hierarchical categorical setting, thus incorporating higher-dimensional relationships among persistence modules. Such a bifunctorial mapping reflects the dual evolution of topological and algebraic structures, allowing one to analyse how geometric features and their interrelations persist across categorical levels [40].

The reason behind this unified model is the categorification of homology, which attempts to generalize homological invariants as objects in higher categories, instead of elements in abelian groups [39]. In this view, the homology structures acquire a further relational sense, morphisms between groups of homologies become the objects of study, and the transformations between the morphisms are the higher-dimensional relationships. This enrichment is natural in the hierarchical enrichment of the layered behaviour of persistent homology, in which topological invariants change continuously over filtrations.

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To formalize this correspondence, filtered chain complexes (C_*, d_*) were considered, where each filtration level C_i yields a homology group $H_k(C_i)$. The collection of these homology groups, along with inclusion-induced homomorphisms, forms a persistence module. Within the higher categorical framework, each persistence module can be viewed as an object in a higher category, with morphisms representing transformations between modules. Higher morphisms then represent natural transformations or equivalences between persistence systems, reflecting the evolution of homological structures across scales [41].

One important aspect of this method is that filtered derived categories are used, in which it is possible to include the effects of curvature and deformation in algebraic and geometric setting [40]. One uses this enrichment of the traditional web of theories of filtrations that aren't merely graded categorical objects making their morphisms best captivate topological persistence and algebraic deformation. Therefore, arriving at a bifunctorial structure, persistent homology and derived categorical strategies intersect to monitor the smooth continuum change of difficult geometrical structures without generating any loss of algebraic structure.

Convolution algebras are also a part of the framework, and are algebraic tools to combine morphisms, between different levels of a category [41]. Persistence modules and morphisms can be built or stacked up to form higher-order invariants using convolution operations and therefore mark out new structures that could not be captured using classical algebraic topology. This method focuses on the functorial consistency, which implies that all the mappings are characterized by the preservation of compositional relations of the filtration process.
In order to summarize these conceptual agreements, Table 3 gives the structural correspondence between persistent homology and higher categorical constructs in the proposed unified model.

Table 3. Structural correspondence between persistent homology and higher categorical constructs

Persistent Homology Component	Higher Category Analogue	Interpretation in Unified Framework
Filtration of complexes $\mathcal{C}_{\text{fil}}$	Graded a higher category	Hierarchical structure of filtered spaces
Persistence module	Object in HCat(Vect)	Encodes evolving homological data
Inclusion maps between filtrations	Morphisms between objects	Structural transformations across scales
Persistence diagrams	Higher morphisms (2-, 3-morphisms)	Relations between persistence modules
Stability theorem	Natural equivalence in a higher category	Functorial continuity under deformation

The framework defines a natural correspondence as shown in Table 3 such that the algebraic extension of topological properties in persistent homology is equal to the morphism structure hierarchy in higher categories. The persistence diagrams are then understood as morphisms at a higher level that record the equivalences between modules as opposed to a visual summary. Equally, the stability theorem in persistent homology has natural conditions of equivalence in higher category theory, which amplifies the property of stability of functorial relations during deformations [42].
Diagrammatic reasoning also helps in this synthesis since it offers a visual and logical formalism to express relationships in higher categories [43]. The use of diagrams within this context depicts not simply a mathematical object but rather a mathematical frame representing transformations and equivalence between objects, it may be given as mathematical expressions. This type of reasoning can be used to increase interpretability, and thus, abstract categorical relationships can be understood in a structural visualization.
This unified theoretical model positions persistent homology within the broader landscape of higher category theory. By formulating a bifunctorial mapping $\mathcal{PH}: \mathcal{C}_{\text{fil}} \to \mathcal{HCat}(\text{ Vect })$It extends classical algebraic topology into a multi-level categorical domain where persistence, deformation, and

equivalence coexist. This synthesis provides a consistent, interpretable, and structurally rich mathematical language for understanding geometric evolution through the lens of categorical abstraction.

6. Conceptual Applications and Future Directions

The coherent system relating persistent homology to higher category theory provides an effective viewpoint into the interpretation and generalization of contemporary geometric objects. In this synthesis, manifolds, orbifolds and stratified spaces can be considered not just as a rather isolated topological object but, instead, in higher categories, wherein categorical relations and categorical renormalisations are represented by morphism and higher morphism. This strategy allows further classification of geometrical structures, including their algebraic invariants as well as their relationship that changes with scale. It generalises the ability of algebraic topology to explain dynamically as well as statically the interactions between geometric and topological properties. With derived and higher geometries such categorical integration others permit the systematic study of complex spaces determined by layered algebraic and topological properties. An example scheme of this is derived schemes, which may be considered using persistent homology as the way their cohomological or deformation properties change. This view brings together homological information and greater categorical structure and forms a language of geometric reasoning which captures continuity as well as abstraction. Similarly, topological quantum field theories (TQFTs) have the advantage of this formulation in that the theory represents the topological change in physical states and configurations of fields in terms of higher categorical functors, which reflect persistence in topology. In the future, such a synthesis provides possible research avenues in categorical modelling of topological persistence of physical systems as well as data geometry. It proposes directions in the construction of computational models of higher categorical persistence, in which algorithms can be used to approximate hierarchical morphisms or functorial transformations between multi-scale data. The abstractness that this model offers facilitates cross-disciplinary innovation as a result of the conceptual clarity that the model offers between pure and applied and computational contexts. Finally, embedding persistent homology into more broad categorical traditions gives algebraic topology a better ability to model, compare, and classify geometric and algebraic structures hence progressing the theoretical base of modern mathematics.

References:

[1]. M. R. Adhikari, *Basic Algebraic Topology and Its Applications*, vol. 615. Springer India, 2016.
[2]. G. Carlsson, "Persistent homology and applied homotopy theory," in *Handbook of Homotopy Theory*, pp. 297–329, Chapman and Hall/CRC, 2020.
[3]. L. Martini, *Internal Higher Category Theory*, 2024.
[4]. B. Martelli, *An Introduction to Geometric Topology*, *arXiv preprint* arXiv:1610.02592, 2016.
[5]. S. A. Solovyov, "Functorial semantics of topological theories," *Iranian Journal of Fuzzy Systems*, vol. 12, no. 5, pp. 1–43, 2015.
[6]. W. M. Goldman, *Geometric Structures on Manifolds*, vol. 227. American Mathematical Society, 2022.
[7]. S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*. Princeton University Press, 2015.
[8]. J. Wang, "Unified model beyond grand unification," *Physical Review D*, vol. 103, no. 10, p. 105024, 2021.
[9]. A. K. Srivastava, I. Herzog, and A. Leroy, *Categorical, Homological and Combinatorial Methods in Algebra*. American Mathematical Society, 2020.
[10]. B. Russell and M. Potter, *An Essay on the Foundations of Geometry*. Routledge, 2022.
[11]. V. Salnikov, D. Cassese, and R. Lambiotte, "Simplicial complexes and complex systems," *European Journal of Physics*, vol. 40, no. 1, p. 014001, 2018.

[12]. K. Shimakawa, K. Yoshida, and T. Haraguchi, "Homology and cohomology via enriched bifunctors," *Kyushu Journal of Mathematics*, vol. 72, no. 2, pp. 239–252, 2018.
[13]. K. Cieliebak and A. Oancea, "Symplectic homology and the Eilenberg–Steenrod axioms," *Algebraic & Geometric Topology*, vol. 18, no. 4, pp. 1953–2130, 2018.
[14]. S. Deo, *Algebraic Topology*. Springer Singapore, 2018.
[15]. E. Jacquard, V. Nanda, and U. Tillmann, "The space of barcode bases for persistence modules," *Journal of Applied and Computational Topology*, vol. 7, no. 1, pp. 1–30, 2023.
[16]. G. Carlsson and M. Vejdemo-Johansson, *Topological Data Analysis with Applications*. Cambridge University Press, 2021.
[17]. M. Ernst, "Category theory and foundations," in *Categories for the Working Philosopher*, pp. 69–89, 2017.
[18]. M. Grandis, *Higher Dimensional Categories: From Double to Multiple Categories*. World Scientific, 2019.
[19]. T. D. Bradley, T. Bryson, and J. Terilla, *Topology: A Categorical Approach*. MIT Press, 2020.
[20]. U. Bauer and M. Schmahl, *Lifespan functors and natural dualities in persistent homology*, *arXiv preprint* arXiv:2012.12881, 2020.
[21]. F. Mémoli and O. B. Okutan, "Quantitative simplification of filtered simplicial complexes," *Discrete & Computational Geometry*, vol. 65, no. 2, pp. 554–583, 2021.
[22]. W. Kim and F. Mémoli, "Generalized persistence diagrams for persistence modules over posets," *Journal of Applied and Computational Topology*, vol. 5, no. 4, pp. 533–581, 2021.
[23]. M. Botnan and M. Lesnick, "Algebraic stability of zigzag persistence modules," *Algebraic & Geometric Topology*, vol. 18, no. 6, pp. 3133–3204, 2018.
[24]. J. P. Tignol and A. R. Wadsworth, "Graded algebra," in *Value Functions on Simple Algebras, and Associated Graded Rings*, Cham: Springer International Publishing, pp. 33–80, 2015.
[25]. A. Z. Al-Yasry, *Graph Homologies and Functoriality*, *arXiv preprint* arXiv:1801.01381, 2018.
[26]. B. Blanchette, T. Brüstle, and E. J. Hanson, *Exact structures for persistence modules*, *arXiv preprint* arXiv:2308.01790, 2023.
[27]. M. Kerber, "Persistent homology: state of the art and challenges," *International Mathematische Nachrichten*, vol. 231, pp. 15–33, 2016.
[28]. L. Polterovich, D. Rosen, K. Samvelyan, and J. Zhang, *Topological Persistence in Geometry and Analysis*, vol. 74. American Mathematical Society, 2020.
[29]. C. D. Hofer, R. Kwitt, and M. Niethammer, "Learning representations of persistence barcodes," *Journal of Machine Learning Research*, vol. 20, no. 126, pp. 1–45, 2019.
[30]. J. E. Bergner, "A survey of models for (∞, n)-categories," in *Handbook of Homotopy Theory*, pp. 263–296, Chapman and Hall/CRC, 2020.
[31]. L. Martini and S. Wolf, *Internal Higher Topos Theory*, *arXiv preprint* arXiv:2303.06437, 2023.
[32]. E. Riehl, *Homotopical Categories: From Model Categories to (∞, 1)-Categories*, *arXiv preprint* arXiv:1904.00886, 2020.
[33]. S. Hazratpour, *A Logical Study of Some 2-Categorical Aspects of Topos Theory*, Doctoral dissertation, University of Birmingham, 2019.
[34]. R. Laugwitz, "The relative monoidal center and tensor products of monoidal categories," *Communications in Contemporary Mathematics*, vol. 22, no. 8, p. 1950068, 2020.
[35]. C. Meusburger, *Introduction to Categorical Homotopy Theory*, 2024.
[36]. M. Shulman, "Homotopy type theory: A synthetic approach to higher equalities," in *Categories for the Working Philosopher*, pp. 36–57, 2017.
[37]. X. Cui, *Higher Categories and Topological Quantum Field Theories*. University of California, Santa Barbara, 2016.
[38]. V. R. Posina, D. N. Ghista, and S. Roy, "Functorial semantics for the advancement of the science of cognition," *Mind and Matter*, vol. 15, no. 2, pp. 161–184, 2017.
[39]. H. Heine, *On the Categorification of Homology*, *arXiv preprint* arXiv:2505.22640, 2025.

[40]. A. Lehmann and W. Lowen, *Filtered Derived Categories of Curved Deformations*, *arXiv preprint* arXiv:2402.08660, 2024.

[41]. J. Harding, C. Walker, and E. Walker, “The convolution algebra,” *Algebra Universalis*, vol. 79, no. 2, p. 33, 2018.

[42]. D. C. Cisinski, *Higher Categories and Homotopical Algebra*, vol. 180. Cambridge University Press, 2019.

[43]. V. Giardino, “Diagrammatic reasoning in mathematics,” in *Springer Handbook of Model-Based Science*, Cham: Springer International Publishing, pp. 499–522, 2017.