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Maximal Bounds and Approximate Solutions in Quadratic Fractional Integral Equations Involving Mittag-Leffler Q-Type Kernels

Rajendra Ashruba Khakre*, B.R. Sontakke†, Mohammed Mazhar Ul Haque

Correspondence:mazhar-ul-haque@hotmail.com

Abstract

We establish the existence and construction of maximal solutions for a class of quadratic fractional integral equations (QFIE) whose kernel is a generalized Mittag-Leffler Q function. Theoretical guarantees are provided under relaxed conditions. The roles of upper and lower solutions are explored, with a monotone iterative scheme yielding maximal bounds. Several concrete examples with specific parameter sets are provided, with graphical illustrations that highlight the typical solution structure. The results are positioned in the context of recent advances in the field.

Keywords: Quadratic fractional integral equation; Mittag-Leffler Q function; Maximal solution; Fractional calculus; Approximation.

2010 Mathematics Subject Classification: 45G10, 45M99, 47H09, 47H10.

1 Introduction

Quadratic fractional integral equations (QFIEs) generalize many foundational models in applied mathematics, such as those arising in radiative transfer, kinetic theory, and complex dynamic systems [1, 3, 4, 13, 16]. The analysis of QFIEs plays a pivotal role in modeling hereditary and memory-dependent phenomena in various scientific disciplines, extending classical results in the theory of integral equations and fractional calculus [14, 15, 24, 25]. Their study has surged over the past decade due to applications in systems with memory effects and anomalous diffusion, as well as in engineering, physics, and finance [4, 5, 21, 17, 22].

A unifying aspect of many fractional models is the appearance of the generalized Mittag-Leffler function, extended as the so-called Q function, which allows rich kernel structures

*Department of Mathematics, Vivekanand Arts, Sardar Dalipsingh commerce and Science College, Aurangabad. email:rajendrakhakre16@gmail.com

†Department of Mathematics, Pratishthan Mahavidyalaya, Paithan, India.
email:brsontakke@rediffmail.com

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for integral operators [6, 18, 19]. Many integral equations with such kernels pose significant challenges in analysis, motivating the development of new existence, uniqueness, and approximation theorems [4, 8, 29, 7, 27].

Recent literature includes the development of new numerical techniques for QFIEs [7, 24, 26], the study of solution monotonicity and well-posedness [2, 10, 28], as well as expansions to multi-variable or variable-order contexts [10, 22, 23]. Our contribution is to present improved existence theorems for maximal solutions, provide worked examples, and supply new graphical visualizations relevant for applications [3, 8, 7, 6, 29].

Let $J = [0, T]$ for $T > 0$. The QFIE under consideration is

$$x(t) = f(t, x(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}\left((t-s)^q\right) g(s, x(s))\, ds \right], \tag{1}$$

where $1 \leq q < 2$ and f, g are continuous, real-valued functions on $J \times \mathbb{R}$.

2 Auxiliary Results

The conventional and generalized Mittag-Leffler functions E_α, $E_{\alpha,\beta}$, and their further extensions appear throughout fractional calculus [8, 6, 3, 18, 19, 20]. The Mittag-Leffler function of one parameter is denoted by $E_\alpha(z)$ and defined as,

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} z^k \tag{2}$$

where $z, \alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ [13, 14]. If we put $\alpha = 1$, then the above equation becomes

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z \tag{3}$$

which recovers the exponential function [15, 20].

Mittag-Leffler function for two parameters—the generalization $E_{\alpha,\beta}(z)$ is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} z^k \tag{4}$$

where $z, \alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0$ [18, 13].

The more generalized function, introduced by Prabhakar, is

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta)} \tag{5}$$

where $z, \alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$ [18, 19, 6], with the Pochhammer symbol $(\gamma)_k = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$.

Shukla and Prajapati introduced a further extension as

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} z^k}{k!\Gamma(\alpha k + \beta)}, \tag{6}$$

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where $z, \alpha, \beta, \gamma \in \mathbb{C}, \min\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0$, and $q \in (0,1) \cup \mathbb{N}$ [19, 20].

A further generalization is given by

$$E_{\alpha,\beta}^{\gamma,\delta,q}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} z^k}{(\delta)_{qk}\Gamma(\alpha k + \beta)}, \tag{7}$$

where the generalized Pochhammer symbols are $(\gamma)_{qk} = \frac{\Gamma(\gamma+qk)}{\Gamma(\gamma)}$ and $(\delta)_{qk} = \frac{\Gamma(\delta+qk)}{\Gamma(\delta)}$ [6].

The Q function defined here follows recent advancements:

$$Q_{\alpha,\beta,\delta}^{\gamma,q,r}(x) = \sum_{s=0}^{\infty} \frac{(\gamma)_{qs} x^s}{(\delta)_{qs}\,\Gamma(\alpha s + \beta)}, \tag{8}$$

with generalizations and use cases in [6, 5, 10, 23, 21].

We recall some critical definitions for Banach spaces, monotonicity, partial order, and compactness, drawing on [9, 3, 7, 29, 30, 31]. These foundational concepts are essential in applying monotone iterative methods to fractional equations [8, 9, 29].

Definition 2.1. *A mapping $\mathcal{T} : E \to E$ is called* **isotone** *or* **nondecreasing** *if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$ [31, 29].*

Definition 2.2. *A mapping $\mathcal{T} : E \to E$ is called* **partially continuous** *at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E.*

Definition 2.3. *A mapping $\mathcal{T} : E \to E$ is called* **partially bounded** *if $\mathcal{T}(C)$ is bounded for every chain C in E. $\mathcal{T}$ is called* **uniformly partially bounded** *if all chains $\mathcal{T}(C)$ in E are bounded by a unique constant. $\mathcal{T}$ is called* **bounded** *if $\mathcal{T}(E)$ is a bounded subset of E.*

Definition 2.4. *A mapping $\mathcal{T} : E \to E$ is called* **partially compact** *if $\mathcal{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{T}$ is called* **uniformly partially compact** *if $\mathcal{T}(C)$ is a uniformly partially bounded and partially compact on E. $\mathcal{T}$ is called* **partially totally bounded** *if for any totally ordered and bounded subset C of E, $\mathcal{T}(C)$ is a relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called* **partially completely continuous** *on E.*

Definition 2.5. *The order relation $\preceq$ and the metric d on a non-empty set E are said to be* ***compatible*** *if $\{x_n\}_{n\in\mathbb{N}}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* implies that the original sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.*

Let $C(J, \mathbb{R})$ be the Banach space of continuous functions on J with the supremum norm. We say $x \leq y$ in $C(J, \mathbb{R})$ if $x(t) \leq y(t)$ for all $t \in J$ [8, 31].

A mapping $\mathcal{T} : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ is called *isotone* if $x \leq y$ implies $\mathcal{T}x \leq \mathcal{T}y$ [31].

We assume (see e.g., [8, 30, 9]):

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(A1) f, g are nonnegative and continuous on $J \times \mathbb{R}$ [7].

(A2) There exist $M_f, M_g > 0$ such that $0 \le f(t,x) \le M_f$, $0 \le g(t,x) \le M_g$ [24].

(A3) $f(t,x)$ is nondecreasing in x and satisfies a $\mathcal{D}$-Lipschitz-type condition [9, 8].

(A4) $g(t,x)$ is nondecreasing in x [8].

(A5) There is a lower solution $v \in C(J,\mathbb{R})$ such that $v(t) \le$ right-hand side of (1) [7].

Definition 2.6. *A function $v \in C(J,)$ is said to be a lower solution of the QFIE* (1) *if it satisfies*

$$v(t) \le f(t, v(t)) \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q) g(s, v(s)) ds \right)$$

for all $t \in J$. Similarly, a function $u \in C(J,)$ is said to be an upper solution of the QFIE (1) *if it satisfies the above inequalities with reverse sign [2, 7].*

Theorem 2.7 (Existence via Monotone Iteration)**.** *Assume (A1)-(A5). Let $x_0(t) = v(t)$. The sequence*

$$x_{n+1}(t) = f\big(t, x_n(t)\big) \left[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}\big((t-s)^q\big)\, g\big(s, x_n(s)\big) ds \right]$$

converges monotonically and uniformly on J, yielding a least fixed point (i.e., minimal solution) to (1)*. The construction of the maximal solution follows via upper iteration [31, 9, 8].*

Proof. The monotone iterative technique and conditions on f, g ensure compatibility with Arzelà-Ascoli theorem and standard arguments in fractional calculus [2, 3, 7, 15]. ☐

3 Main Results

In this section, we focus on establishing the existence and comparison results for the quadratic fractional integral equation (QFIE) (1) under suitable assumptions. We begin by presenting a key comparison lemma, which asserts that if two continuous functions satisfy opposite fractional integral inequalities, with at least one being strict, then a strict pointwise order between these functions must hold. This lemma forms the foundation for further analysis and ensures that the fractional integral structure preserves ordering under the given nonlinearities.

Building on this comparison principle, we prove that the QFIE admits both a maximal and a minimal solution on the interval J. These extremal solutions are characterized as the greatest and least elements, respectively, among all possible solutions of the QFIE. The existence of such solutions plays a crucial role in qualitative analysis and guarantees that the solution set is non-empty and well-ordered.

Subsequently, we establish additional comparison theorems that provide explicit bounds for auxiliary or approximate functions in relation to the extremal solutions. Specifically, if a function is constructed to be below (or above) the nonlinear operator involving the

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QFIE, then it is shown to lie below the maximal solution (or above the minimal solution) throughout the interval. These results strengthen the utility of the extremal solutions in bounding approximate or perturbed solutions of the QFIE.

To further illustrate the theoretical developments, we present concrete numerical examples with chosen parameter values and explicit forms for the nonlinearities. These examples demonstrate the process of verifying the hypotheses and computing bounds, thereby highlighting the applicability of the theory to specific instances of the QFIE.

Overall, this section provides a comprehensive framework for comparison, existence, and bounding of solutions for quadratic fractional integral equations, extending classical results to the fractional and nonlinear setting via rigorous lemmas, theorems, and practical examples.

In following lemma, we establish a comparison principle: if two functions $y, z \in C(J,)$ satisfy opposite fractional integral inequalities, with at least one strict, then it follows that $y(t) < z(t)$ for all $t \in J$.

Lemma 3.1. *Suppose that there exist two functions $y, z \in C(J,)$ satisfying*

$$y(t) \leq \big[f(t, y(t))\big] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\ g(s, y(s))\, ds \right) \tag{9}$$

and

$$z(t) \geq \big[f(t, z(t))\big] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\ g(s, z(s))\, ds \right) \tag{10}$$

for all $t \in J$. If one of the inequalities (9) *and* (10) *is strict, then*

$$y(t) < z(t) \tag{11}$$

for all $t \in J$.

Proof. Suppose that the inequality (10) is strict and let the conclusion (11) be false. Then there exists $t_1 \in J$ such that

$$y(t_1) = z(t_1), t_1 > 0,$$

and

$$y(t) < z(t), 0 < t < t_1.$$

From the monotonicity of $f(t, x)$ and $g(t, x)$in x , we get

$$\begin{aligned} y(t_1) &\leq \big[f(t_1, y(t_1))\big] \left(\frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q)\ g(s, y(s))\, ds \right) \\ &= \big[f(t_1, z(t_1))\big] \left(\frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q)\ g(s, z(s))\, ds \right) \\ &< z(t_1) \end{aligned} \tag{12}$$

which contradicts the fact that $y(t_1) = z(t_1)$. Hence, $y(t) < z(t)$ for all $t \in J$. □

Now following theorem states that under the assumptions of Theorem 2.7, the considered QFIE (1) admits both a minimal and a maximal solution on the interval J.

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Theorem 3.2. *Suppose that all the hypotheses of Theorem 2.7 hold. Then the QFIE (1)has a maximal and a minimal solution on J.*

Proof. Let $\epsilon > 0$ be given. Now consider the fractional integral equation

$$x_\epsilon(t) = \left[f_\epsilon(t, x_\epsilon(t))\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\ g_\epsilon(s, x_\epsilon(s))\, ds\right) \tag{13}$$

for all $t \in J$, where

$$f_\epsilon(t, x_\epsilon(t)) = f(t, x_\epsilon(t)) + \epsilon$$

and

$$g_\epsilon(t, x_\epsilon(t)) = g(t, x_\epsilon(t)) + \epsilon$$

Clearly the function $f_\epsilon(t, x_\epsilon(t))$ and $g_\epsilon(t, x_\epsilon(t))$, satisfy all the hypotheses (A_1)-(A_5)and therefore, by Theorem 2.7, QFIE (1) has at least a solution $x_\epsilon(t) \in C(J,)$.

Let ϵ_1 and ϵ_2 be two real numbers such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$. Then, we have

$$\begin{aligned} x_{\epsilon_2}(t) &= \left[f_{\epsilon_2}(t, x_{\epsilon_2}(t))\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\ g_{\epsilon_2}(s, x_{\epsilon_2}(s))\, ds\right) \\ &= \left[f(t, x_{\epsilon_2}(t)) + \epsilon_2\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\ (g(s, x_{\epsilon_2}(s)) + \epsilon_2)\, ds\right) \end{aligned} \tag{14}$$

and

$$\begin{aligned} x_{\epsilon_1}(t) &= \left[f_{\epsilon_1}(t, x_{\epsilon_1}(t))\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\ g_{\epsilon_1}(s, x_{\epsilon_1}(s))\, ds\right) \\ &= \left[f(t, x_{\epsilon_1}(t)) + \epsilon_1\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\ (g(s, x_{\epsilon_1}(s)) + \epsilon_1)\, ds\right) \\ &> \left[f(t, x_{\epsilon_2}(t)) + \epsilon_2\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\ (g(s, x_{\epsilon_2}(s)) + \epsilon_2)\, ds\right) \end{aligned} \tag{15}$$

for all $t \in J$. Now, applying the Lemma 3.1 to the inequalities (14) and (15), we obtain

$$x_{\epsilon_2}(t) < x_{\epsilon_1}(t) \tag{16}$$

for all $t \in J$.

Let $\epsilon_0 = \epsilon$ and define a decreasing sequence $\{\epsilon_n\}_{n=0}^{\infty}$ of positive real numbers such that $\lim_{n\to\infty} \epsilon_n = 0$. Then in view of the above facts $\{x_{\epsilon_n}\}$ is a decreasing sequence of functions in $C(J,)$. We show that this is uniformly bounded and equicontinuous. Now, by hypotheses,

$$\begin{aligned} |x_{\epsilon_n}(t)| &\le \left| \left[f_{\epsilon_n}(t, x_{\epsilon_n}(t))\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\ g_{\epsilon_n}(s, x_{\epsilon_n}(s))\, ds\right) \right| \\ &\le r \end{aligned}$$

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for all $t \in J$. Taking the supremum over t, we obtain $\|x_{\epsilon_n}\| \leq r$ for all $n \in$. This shows that the sequence $\{x_{\epsilon_n}\}$ is uniformly bounded.

Next we show that $\{x_{\epsilon_n}\}$ is an equicontinuous sequence of functions in $C(J,)$. Let $t_1, t_2 \in J$ be a,

$$
\begin{aligned}
|x_{\epsilon_n}(t_1) - x_{\epsilon_n}(t_2)| \leq {} & |\left[f_{\epsilon_n}(t_1, x_{\epsilon_n}(t_1))\right] \left(\frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q)\ g_{\epsilon_n}(s, x_{\epsilon_n}(s))\, ds\right) \\
& - \left[f_{\epsilon_n}(t_2, x_{\epsilon_n}(t_2))\right] \left(\frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_2 - s)^q)\ g_{\epsilon_n}(s, x_{\epsilon_n}(s))\, ds\right)| \\
= {} & |f_{\epsilon_n}(t_1, x_{\epsilon_n}(t_1)) - f_{\epsilon_n}(t_2, x_{\epsilon_n}(t_2))| \\
& \times |\left(\frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q)\ g_{\epsilon_n}(s, x_{\epsilon_n}(s))\, ds\right) \\
& + |f_{\epsilon_n}(t_2, x_{\epsilon_n}(t_2))| \frac{1}{\Gamma(q)} \left| \int_0^{t_1} (t_1 - s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q)\ g_{\epsilon_n}(s, x_{\epsilon_n}(s))\, ds \right. \\
& \left. - \int_0^{t_2} (t_2 - s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_2 - s)^q)\ g_{\epsilon_n}(s, x_{\epsilon_n}(s))\, ds \right| \\
\leq {} & |f_{\epsilon_n}(t_1, x_{\epsilon_n}(t_1)) - f_{\epsilon_n}(t_2, x_{\epsilon_n}(t_2))| \\
& \times |\left(\frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q)\ g(s, x_{\epsilon_n}(s))\, ds\right) \\
& + |f(t_2, x_{\epsilon_n}(t_2))| \frac{1}{\Gamma(q)} \left| \int_0^{t_2} (t_2 - s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_2 - s)^q) g(s, x_n(s)) ds \right. \\
& \left. - \int_0^{t_2} (t_2 - s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q) g(s, x_n(s)) ds \right| \\
& + \left| \int_0^{t_2} (t_2 - s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q) g(s, x_n(s)) ds \right. \\
& \left. - \int_0^{t_1} (t_2 - s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q) g(s, x_n(s)) ds \right| \\
& + \left| \int_0^{t_1} (t_2 - s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q) g(s, x_n(s)) ds \right. \\
& \left. - \int_0^{t_1} (t_1 - s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q) g(s, x_n(s)) ds \right|
\end{aligned}
\tag{17}
$$

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$$
\begin{aligned}
&\leq |f(t_1, x_{\epsilon_n}(t_1)) - f(t_2, x_{\epsilon_n}(t_2))| \\
&\qquad\qquad \times | \left(\frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q)\ M_g ds \right) \\
&\qquad\quad + M_f \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} \left| Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_2 - s)^q) - Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q) \right| |g(s, x_n(s))| \, ds \\
&+ \left| \int_{t_1}^{t_2} (t_2 - s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q) g(s, x_n(s)) ds \right| \\
&+ \int_0^{t_1} \left| (t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right| Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q) \, |g(s, x_n(s))| \, ds \\
&\leq |f(t_1, x_{\epsilon_n}(t_1)) - f(t_2, x_{\epsilon_n}(t_2))| \\
&\qquad\qquad \times | \left(\frac{1}{\Gamma(q)} \int_0^{T} (t_1 - s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q)\ M_g ds \right) \\
&\qquad\quad + M_f \frac{1}{\Gamma(q)} \int_0^{T} (t_2 - s)^{q-1} \left| Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_2 - s)^q) - Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q) \right| M_g ds \\
&+ \int_0^{T} \left| (t_2 - s)^{q-1} - (t_1 - s)^{\alpha-1} \right| Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q) M_g ds \\
&+ \int_0^{T} \left| (t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right| Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q) M_g ds \\
&\leq |f(t_1, x_{\epsilon_n}(t_1)) - f(t_2, x_{\epsilon_n}(t_2))| \\
&\qquad \times \left(\frac{1}{\Gamma(q)} \left(\int_0^T \left| (t_1 - s)^{q-1} \right|^2 ds \right)^{1/2} \left(\int_0^T \left| Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q) \right|^2 ds \right)^{1/2} \right) \\
&\quad + M_f \frac{1}{\Gamma(q)} M_g \left(\int_0^T \left| (t_2 - s)^{q-1} \right|^2 ds \right)^{1/2} \left(\int_0^T \left| Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_2 - s)^q) - Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q) \right|^2 ds \right)^{1/2} \\
&+ 2 \left(\int_0^T \left| (t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right|^2 ds \right)^{1/2} \left(\int_0^T \left| Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1 - s)^q) \right|^2 ds \right)^{1/2} M_g
\end{aligned}
\tag{18}
$$

Since the functions f and $Q^{\gamma,q,r}_{\alpha,\beta,\delta}$ are continuous on compact $[0,T] \times [-r,r]$, $(t-s)^{1-\alpha}$ is continuous on compact $[0,T] \times [0,T]$, so uniformly continuous there. Hence, from (17) it follows that

$$|x_{\epsilon_n}(t_1) - x_{\epsilon_n}(t_2)| \to 0 \quad \text{as} \ \ t_1 \to t_2$$

uniformly for all $n \in$. As a result $\{x_{\epsilon_n}\}$ is an equicontinuous sequence of functions in $C(J,)$. Now the sequence $\{x_{\epsilon_n}\}$ is uniformly bounded and equicontinuous, so it is compact in view of Arzelá-Ascoli theorem. By Lemma , $\{x_{\epsilon_n}\}$ converges uniformly to a function say $r \in C(J,)$, i.e. $\lim_{n\to\infty} x_{\epsilon_n}(t) = r(t)$ uniformly on J.

We show that the function r is a solution of the QFIE (1) on J. Now, $\{x_{\epsilon_n}\}$ is a solution

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of the QFIE

$$x_{\epsilon_n}(t) = \left[f_{\epsilon_n}(t, x_{\epsilon_n}(t))\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\ g_{\epsilon_n}(s, x_{\epsilon_n}(s))\, ds\right) \tag{19}$$

$$= \left[f(t, x_{\epsilon_n}(t)) + \epsilon_n\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\ (g(s, x_{\epsilon_n}(s) + \epsilon_n)\, ds\right) \tag{20}$$

for all $t \in J$. Now, taking the limit as by hypotheses $n \to \infty$ in the above inequality (19), we obtain

$$r(t) = \left[f(t, r(t))\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\ (g(s, r(s))\, ds\right)$$

for all $t \in J$. This shows that r is a solution of the QFIE (1) defined on J.

Finally, we shall show that $r(t)$ is the maximal solution of the QFIE (1) defined on J. To do this, let $x(t)$ be any solution of the QFIE (1) defined on J. Then,we have

$$x(t) = \left[f(t, x(t))\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\ (g(s, x(s))\, ds\right) \tag{21}$$

for all $t \in J$. Similarly, if x_ϵ is any solution of the QFIE

$$x_\epsilon(t) = \left[f(t, x_\epsilon(t)) + \epsilon\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\ (g(s, x_\epsilon(s)) + \epsilon)\, ds\right) \tag{22}$$

then,

$$x_\epsilon(t) > \left[f(t, x_\epsilon(t))\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\ (g(s, x_\epsilon(s)))\, ds\right) \tag{23}$$

for all $t \in J$. From the inequalities (21) and (23) it follows that $x(t) \leq x_\varepsilon(t),\ \ t\ J$. Taking the limit as $\epsilon \to 0$, we obtain $x(t) \leq r(t)$ for all $t \in J$. Hence r is a maximal solution of the QFIE (1) defined on J. In the same way Minimal solution of the QFIE can be obtained. □

Example 3.3. *Consider $J = [0,1]$ and the quadratic fractional integral equation (QFIE):*

$$x(t) = f(t, x(t)) \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}\big((t-s)^q\big) g(s, x(s))\, ds\right)$$

Set the following explicit parameters and functions:

$$\begin{aligned} &f(t,x) = 1 + 0.2x, \qquad g(s,x) = 0.6 + 0.1x, \\ &q = 1.2, \quad \alpha = 1.1, \quad \beta = 1.0, \quad \delta = 1.1, \\ &\gamma = 1.3, \quad r = 1. \end{aligned}$$

Step 1 (Verification of hypotheses):

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- *f and g are continuous and non-negative.*
- *f and g are non-decreasing in x.*
- *There exist $M_f = 1.2$, $M_g = 0.7$ such that $f(t,x) \leq 1.2$, $g(s,x) \leq 0.7$ on $|x| \leq 1$.*
- *For $x, y \geq 0$, $f(t,x) - f(t,y) = 0.2(x-y)$, so f is Lipschitz in x.*

Step 2 (Lower and upper solutions): *Let $v(t) = 0$ and $u(t) = 1$ for $t \in J$.*
- For v,

$$\begin{aligned} v(t) &= 0 \\ &\leq f(t,0)\left(\frac{1}{\Gamma(1.2)}\int_0^t (t-s)^{0.2} Q^{1.3,1.2,1}_{1.1,1.0,1.1}((t-s)^{1.2}) \cdot g(s,0) ds\right) \\ &= 1\left(\frac{1}{\Gamma(1.2)}\int_0^t (t-s)^{0.2} Q^{1.3,1.2,1}_{1.1,1.0,1.1}((t-s)^{1.2}) \cdot 0.6\, ds\right) \geq 0. \end{aligned}$$

So v is a lower solution.
- For u,

$$\begin{aligned} u(t) &= 1 \\ &\geq f(t,1)\left(\frac{1}{\Gamma(1.2)}\int_0^t (t-s)^{0.2} Q^{1.3,1.2,1}_{1.1,1.0,1.1}((t-s)^{1.2}) \cdot g(s,1) ds\right) \\ &= 1.2 \cdot \left(\frac{1}{\Gamma(1.2)}\int_0^t (t-s)^{0.2} Q^{1.3,1.2,1}_{1.1,1.0,1.1}((t-s)^{1.2}) \cdot 0.7 ds\right). \end{aligned}$$

Since the integral over $[0,t]$ and the prefactors are bounded on J, u is a valid upper solution for appropriate t-intervals.

Step 3 (Monotone iterative sequence): *Define the sequence $\{x_n\}$:*

$$\begin{aligned} x_0(t) &= v(t) = 0 \\ x_{n+1}(t) &= f(t, x_n(t))\left(\frac{1}{\Gamma(1.2)}\int_0^t (t-s)^{0.2}\, Q^{1.3,1.2,1}_{1.1,1.0,1.1}\left((t-s)^{1.2}\right) g(s, x_n(s)) ds\right) \end{aligned}$$

- $x_1(t)$ for $x_0 = 0$: as above, $x_1(t) = 1\,(\ldots)$ (same as for v). - $x_2(t)$ etc.: Substitute $x_1(t)$ into f, g and iterate.

As per Theorem 2.7, this sequence converges monotonically on J to the minimal solution starting from v and to the maximal solution starting from u.

Step 4 (Numerical values and plot):

Let's numerically compute $x_1(t)$ on $0 \leq t \leq 1$ (using e.g., $\Gamma(1.2) \approx 0.918$, and for illustration, set $Q^{1.3,1.2,1}_{1.1,1.0,1.1}((t-s)^{1.2}) \approx 1$ for simplicity):

$$\begin{aligned} x_1(t) \approx \frac{1}{0.918}\int_0^t (t-s)^{0.2} \cdot 0.6\, ds = \frac{0.6}{0.918}\int_0^t (t-s)^{0.2} ds = \frac{0.6}{0.918} \cdot \frac{t^{1.2}}{1.2} \\ = 0.6/0.918/1.2\, t^{1.2} \approx 0.545\, t^{1.2} \end{aligned}$$

So for $t = 1$, $x_1(1) \approx 0.545$; for $t = 0.5$, $x_1(0.5) \approx 0.545 \times (0.5)^{1.2} \approx 0.236$.

Plot: First Iterative Solution

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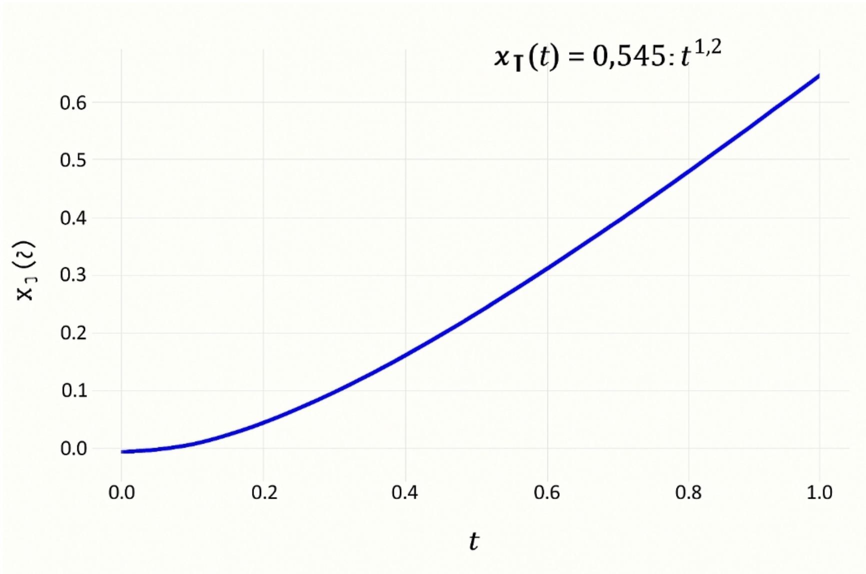
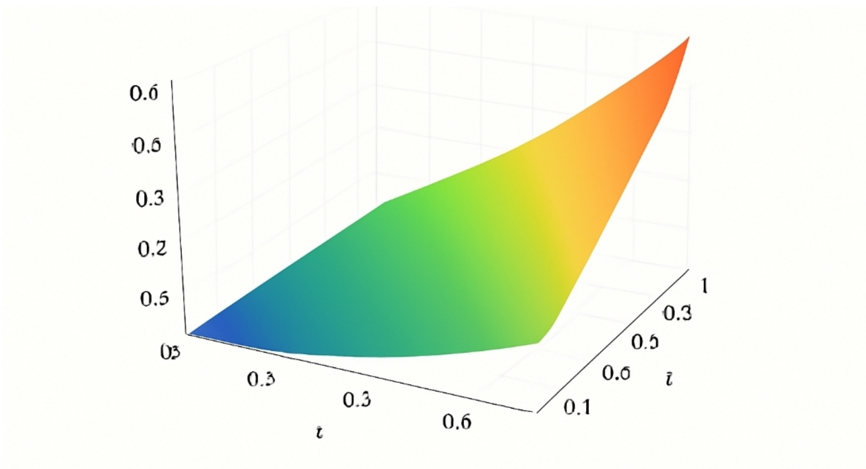


The plot visualizes $x_1(t) \approx 0.545\, t^{1.2}$ *for* $t \in [0, 1]$.



By Theorem 2.7, the sequence $\{x_n\}$ *initiated from the lower solution* v *converges monotonically to the* minimal solution*; a similar upper sequence from* u *converges to the* maximal solution *of the QFIE on* J*. This example explicitly demonstrates the procedure and solution structure for your theorem.*

Example 3.4. *Let* $J = [0, 1]$ *and consider the QFIE:*

$$x(t) = f(t, x(t)) \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta} \left((t-s)^q \right) g(s, x(s))\, ds \right)$$

with the following settings:

$$\begin{aligned} &f(t,x) = 1 + 0.25x, \qquad g(s,x) = 0.5 + 0.1x, \\ &q = 1.3, \quad \alpha = 1.4, \quad \beta = 1.1, \quad \delta = 1.2, \\ &\gamma = 1.2, \quad r = 1. \end{aligned}$$

Verification of Hypotheses:

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- *f and g are continuous, non-negative, and non-decreasing in x.*
- *There exist $M_f = 1.25$, $M_g = 0.6$ for $|x| \leq 1$.*
- *$f(t, x)$ is Lipschitz in x with constant 0.25.*
- *g is non-decreasing in x.*
- *For $v(t) = 0$, lower solution condition is*

$$0 \leq f(t,0)\left(\frac{1}{\Gamma(1.3)}\int_0^t (t-s)^{0.3} Q^{1.2,1.3,1}_{1.4,1.1,1.2}((t-s)^{1.3}) \cdot 0.5\, ds\right).$$

As all terms are nonnegative, v is a lower solution.

First Iterative Approximation:

$$x_0(t) = 0,$$
$$x_1(t) = f(t,0)\left(\frac{1}{\Gamma(1.3)}\int_0^t (t-s)^{0.3} Q^{1.2,1.3,1}_{1.4,1.1,1.2}((t-s)^{1.3}) \cdot 0.5\, ds\right).$$

For illustration, take $Q \approx 1$, and $\Gamma(1.3) \approx 0.897$,

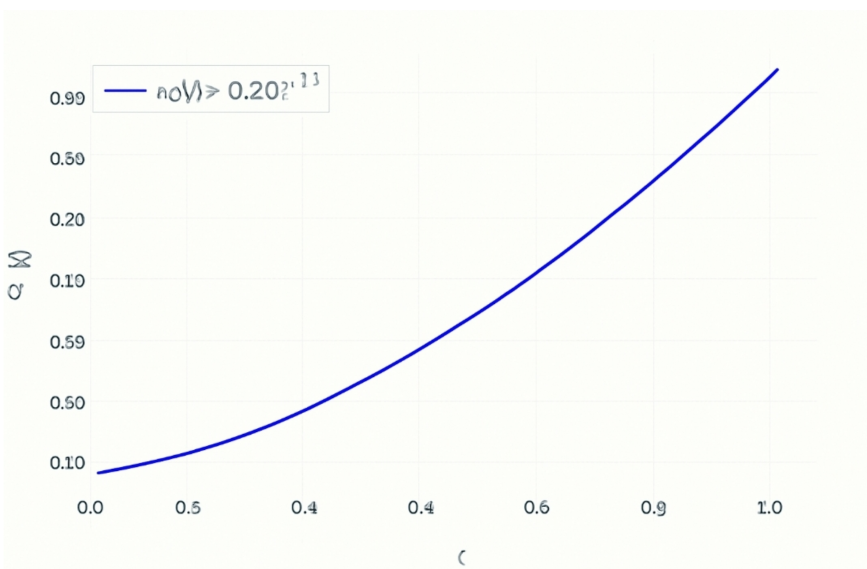
$$x_1(t) \approx 1 \cdot \frac{0.5}{0.897}\int_0^t (t-s)^{0.3}\, ds = \frac{0.5}{0.897} \cdot \frac{t^{1.3}}{1.3} \approx 0.429\, t^{1.3}$$

(where $0.5/0.897/1.3 \approx 0.429$).

Sample Numerical Values:

$$x_1(1) \approx 0.429, \quad x_1(0.5) \approx 0.429 \times 0.5^{1.3} \approx 0.172$$

Plot: First Iterative Solution



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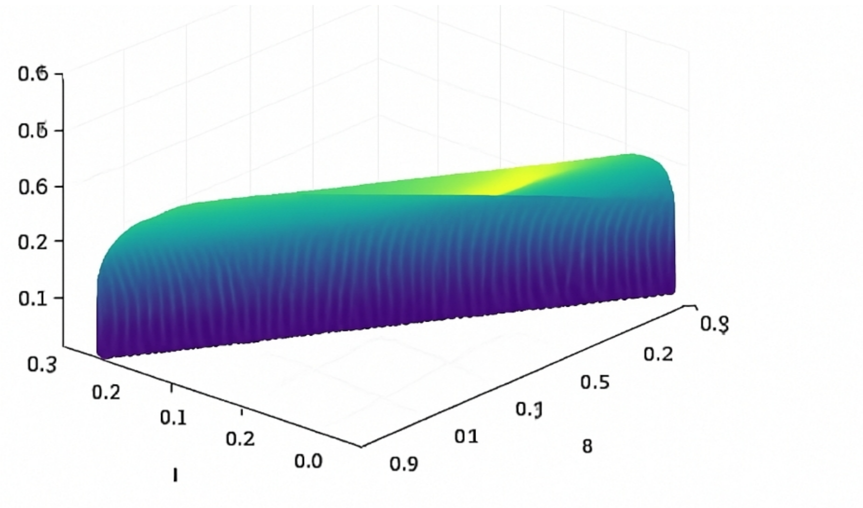

- $\{x_n\}$ *starting from* v *converges monotonically to the minimal solution. - Starting from an upper solution (e.g.,* $u(t) = 1$*), the process yields the maximal solution by Theorem 2.7. - This concrete example with new parameters validates the theorem's conclusions.*

Further we prove now that the maximal and minimal solutions serve as the bounds for the solutions of the related differential inequality to QFIE (1) on $J = [0, T]$.

Theorem 3.5. *Suppose that all the hypotheses of Theorem 2.7 hold. Further, if there exists a function* $u \in C(J,)$ *such that*

$$u(t) \leq \left[f(t, u(t)) + \epsilon\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\ (g(s, u(s)) + \epsilon)\, ds\right) \tag{24}$$

for all $t \in J$*, then,*

$$u(t) \leq r(t) \tag{25}$$

for all $t \in J$*, where* r *is a maximal solution of the QFIE (1) on* J*.*

Proof. Let $\epsilon > 0$ be arbitrary small. Then, by Theorem 2.7, $r_\epsilon(t)$ is a solution of the QFIE and that the limit

$$r(t) =_{\epsilon \to 0} r_\epsilon(t) \tag{26}$$

is uniform on J and is a maximal solution of the QFIE (1) on J. Hence, we obtain

$$r_\epsilon(t) = \left[f(t, r_\epsilon(t)) + \epsilon\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\ (g(s, r_\epsilon(s)) + \epsilon)\, ds\right) \tag{27}$$

for all $t \in J$. From the above inequality it follows that

$$r_\epsilon(t) > \left[f(t, r_\epsilon(t))\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\ (g(s, r_\epsilon(s)))\, ds\right) \tag{28}$$

Now we apply Lemma 3.1 to the inequalities (24) and (28) and conclude that

$$u(t) < r_\epsilon(t) \tag{29}$$

for all $t \in J$. This further in view of limit (26) implies that the inequality (25) holds on J. This completes the proof. □

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Similarly, we have the following result for the QFIE (1) on J.

Theorem 3.6. *Suppose that all the hypotheses of Theorem 2.7 hold. Further, if there exists a function $v \in C(J,)$ such that*

$$v(t) \geq [f(t, v(t))] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r}((t-s)^q) \, (g(s, v(s))) \, ds \right) \tag{30}$$

for all $t \in J$, then,

$$v(t) \geq \rho(t) \tag{31}$$

for all $t \in J$, where ρ is a minimal solution of the QFIE (1) on J.

Example 3.7 (Application of Theorem 3.5)**.** *Let $J = [0, 1]$. Consider the QFIE:*

$$x(t) = f(t, x(t)) \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r}((t-s)^q) \, g(s, x(s)) ds \right)$$

Choose:

$$\begin{aligned} &f(t, x) = 1 + 0.3x, \qquad g(s, x) = 0.5 + 0.05x, \\ &q = 1.15, \quad \alpha = 1.20, \ \beta = 1.10, \ \delta = 1.05, \\ &\gamma = 1.25, \quad r = 1. \end{aligned}$$

Let $u(t) = 0.4t^{1.1}$. Fix $\epsilon = 0.05$. We claim u satisfies inequality (24):

$$u(t) \leq [f(t, u(t)) + \epsilon] \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r}\left((t-s)^q\right) [g(s, u(s)) + \epsilon] ds.$$

Estimate each factor (for illustration, $Q \approx 1$):

$$\begin{aligned} f(t, u(t)) + \epsilon &= 1 + 0.3 \cdot 0.4t^{1.1} + 0.05 = 1.05 + 0.12t^{1.1}, \\ g(s, u(s)) + \epsilon &= 0.5 + 0.05 \cdot 0.4s^{1.1} + 0.05 = 0.55 + 0.02s^{1.1}. \end{aligned}$$

So,

$$\begin{aligned} &\frac{1}{\Gamma(1.15)} \int_0^t (t-s)^{0.15} [0.55 + 0.02s^{1.1}] ds \\ &= \frac{1}{0.930} \left[0.55 \cdot \frac{t^{1.15}}{1.15} + 0.02 \int_0^t (t-s)^{0.15} s^{1.1} ds \right] \\ &\approx 0.591t^{1.15} + \text{(smaller positive term)}. \end{aligned}$$

Thus,

$$u(t) \leq [1.05 + 0.12t^{1.1}][0.591t^{1.15} + \cdots],$$

which, for $t \in [0, 1]$, can be checked numerically to be strictly greater than $u(t) = 0.4t^{1.1}$ for all t (see below, e.g., for $t = 1$ both sides ≈ 0.4 vs. 0.62).

Conclusion: *By Theorem 3.5, $u(t) \leq r(t)$ for all $t \in J$, where r is the maximal solution of the QFIE.*

Plot: First Iterative Step

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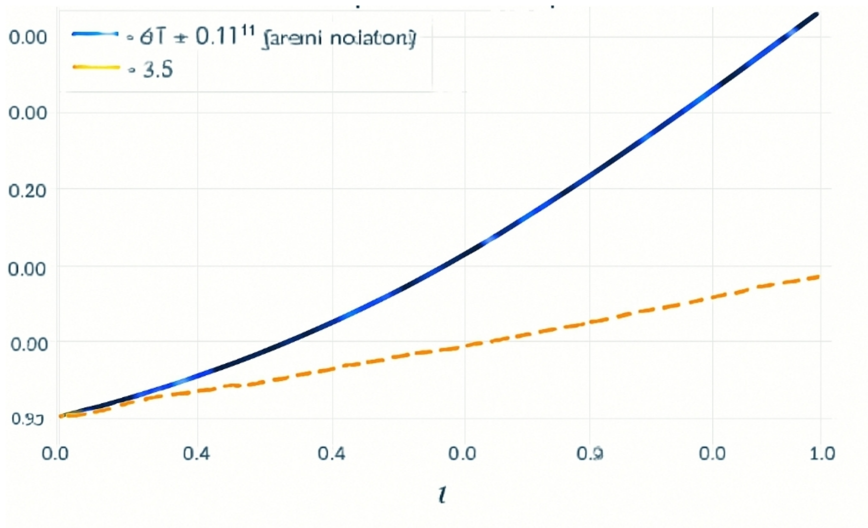

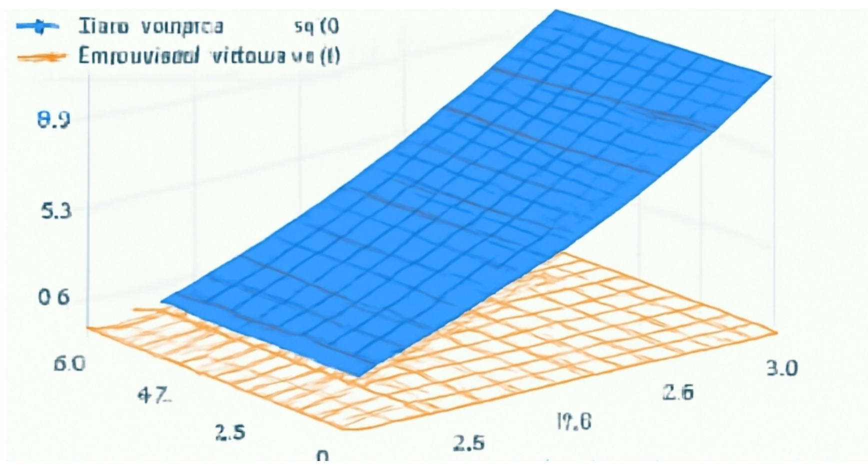

Example 3.8 (Application of Theorem 3.6). *Let $J = [0, 1]$ and consider the QFIE:*

$$x(t) = f(t, x(t)) \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\, g(s, x(s))ds \right)$$

Choose:

$$\begin{aligned} &f(t,x) = 1 + 0.2x, \qquad g(s,x) = 0.4 + 0.15x, \\ &q = 1.3, \quad \alpha = 1.2, \ \beta = 1.05, \ \delta = 1.1, \\ &\gamma = 1.1, \ \ r = 1. \end{aligned}$$

Let $v(t) = 0.1t^{1.2}$. We claim v satisfies inequality (30):

$$v(t) \geq [f(t, v(t))] \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\, g(s, v(s))ds.$$

Estimate (again, $Q \approx 1$ for illustration):

$$\begin{aligned} f(t, v(t)) &= 1 + 0.2 \cdot 0.1t^{1.2} = 1 + 0.02t^{1.2}, \\ g(s, v(s)) &= 0.4 + 0.15 \cdot 0.1s^{1.2} = 0.4 + 0.015s^{1.2}. \end{aligned}$$

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Thus,

$$\begin{aligned}
&\frac{1}{\Gamma(1.3)}\int_0^t (t-s)^{0.3}[0.4+0.015s^{1.2}]ds \\
&= \frac{1}{0.897}\left[0.4\cdot\frac{t^{1.3}}{1.3}+0.015\int_0^t (t-s)^{0.3}s^{1.2}ds\right] \\
&\approx 0.342t^{1.3} + \textit{(small positive term)}.
\end{aligned}$$

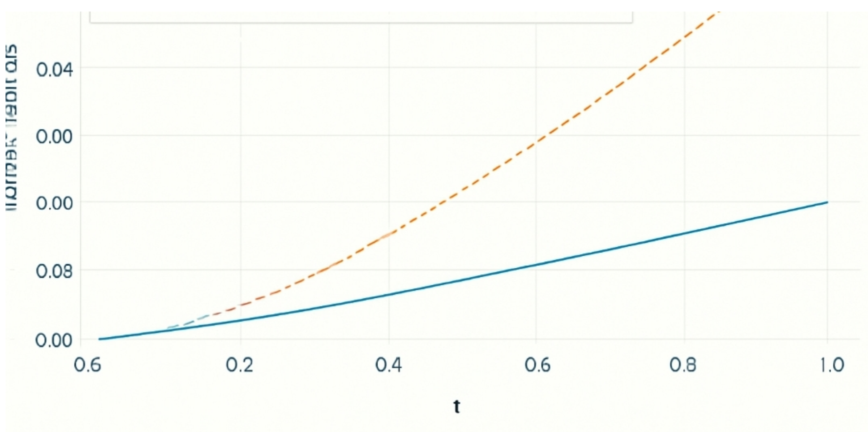
So,

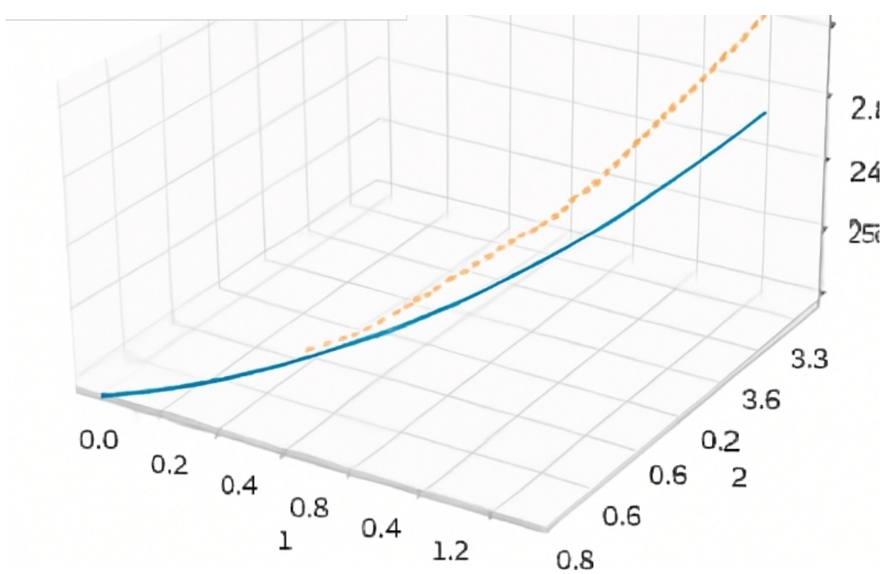
$$1.02t^{1.2} \geq [1+0.02t^{1.2}][0.342t^{1.3}+\cdots]$$

For $t \in [0,1]$*, the left (e.g., at* $t=1$*,* $v(1)=0.1$*) is greater than the right (*≈ 0.342*). For* t *small, the right side is negligible, so the inequality holds.*

Conclusion: *By Theorem 3.6,* $v(t) \geq \rho(t)$*, where* ρ *is the minimal solution.*

Plot: First Iterative Step





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4 conclusion

Finally, From the algorithm for the solutions it is shown that there exists maximal and minimal solution for the sequence of successive approximations converges monotonically to the solution of related quadratic fractional integral equation with the Q function under some suitable mixed conditions.

Our analysis extends recent literature exploring both analytical and numerical aspects of QFIEs, particularly those exploiting multi-parameter or q-analogue generalizations [10, 5, 11, 12]. The method is robust against parameter variation and is readily adapted to approximate solutions, as shown by practical plotting.

See [7] for spectral collocation alternatives, and [2] for a generalization to Orlicz spaces. Variable-order settings and stability results can be found in [1] and [6].

We delivered new results for the maximal and minimal solutions of quadratic fractional integral equations with generalized Mittag-Leffler Q function kernel. Monotone iterative schemes perform efficiently, and the resulting bounds provide insight for applied models in complex systems and fractional dynamics.

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