International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Fuzzy-Soft Neutrosophic Bitopological Spaces and Dual Continuities

Ms. A. Sivaranjani[1], Dr.S.Sudha[2]
[1]Student - M.Sc.Mathematics (Integrated)- 4th year
Department of Mathematics
Hindustan Institute of Technology and Science, Chennai - 603103
Email: ranjanianand230201@gmail.com
[2]Assistant Professor (SG)
Department of Mathematics,
Hindustan Institute of Technology and Science, Chennai - 603103
Email: sudha.aarpitha@gmail.com

Abstract

This paper introduces the concept of *fuzzy-soft neutrosophic bitopological spaces (FSNBTS)*, combining the theories of fuzzy sets, soft sets, neutrosophic sets, and bitopological structures. We define two distinct fuzzy-soft neutrosophic topologies on a common universe and study their mutual relationships through dual continuity. Basic properties, inclusion relations, and corresponding closure and interior operators are investigated. Characterizations of dual continuous functions, dual open maps, and dual compactness are derived. The presented framework generalizes fuzzy-soft topology, neutrosophic topology, and classical bitopological spaces as particular cases. Illustrative examples, propositions, and theorems establish the consistency and independence of the introduced axioms.

Keywords: Fuzzy-soft topology; Neutrosophic set; Bitopological space; Dual continuity; Hybrid topology; Soft set theory.

1 Introduction

Topological generalizations have long served as foundations for reasoning under uncertainty. Fuzzy topology [1] extends openness through graded membership; soft topology [2] incorporates parameterization; neutrosophic topology [3] introduces indeterminacy; and bitopology [4] handles dual topological structures on the same set. Despite progress in these areas, little work exists combining bitopological frameworks with fuzzy-soft neutrosophic environments. The coexistence of two fuzzy-soft neutrosophic topologies offers a mechanism to model dual uncertainty systems within a unified logical space. This paper aims to fill this theoretical gap.

Received: August 05, 2025

2 Preliminaries

Definition 2.1 (Fuzzy Set)**.** *A fuzzy set A on a universe X is a function $\mu_A : X \to [0,1]$.*

Definition 2.2 (Soft Set)**.** *A soft set over X with parameter set E is a mapping $F : E \to \mathcal{P}(X)$.*

Definition 2.3 (Neutrosophic Set)**.** *A neutrosophic set N on X is characterized by three functions $T_N, I_N, F_N : X \to [0,1]$, representing truth, indeterminacy, and falsity degrees.*

Definition 2.4 (Fuzzy-Soft Neutrosophic Set)**.** *A fuzzy-soft neutrosophic set F_E over X and E is a mapping*

$$F_E : E \to \mathcal{N}(X), \quad F_E(e)(x) = (T_{F_E(e)}(x), I_{F_E(e)}(x), F_{F_E(e)}(x)).$$

Union, intersection, and complement are defined componentwise using max–min operations for T, I, and F.

3 Fuzzy-Soft Neutrosophic Topology

Definition 3.1. *A family $\tau \subseteq \mathfrak{FSN}(X, E)$ is a fuzzy-soft neutrosophic topology on X if:*

1. $\tilde{0}_E, \tilde{1}_E \in \tau$,
2. *The union of any subfamily of τ belongs to τ,*
3. *The intersection of any finite subfamily of τ belongs to τ.*

The pair (X, τ) is called a fuzzy-soft neutrosophic topological space (FSNTS).

4 Fuzzy-Soft Neutrosophic Bitopological Space

Definition 4.1. *Let X be non-empty and E a set of parameters. A fuzzy-soft neutrosophic bitopological space (FSNBTS) is a triple (X, τ_1, τ_2) where τ_1 and τ_2 are two distinct fuzzy-soft neutrosophic topologies on X.*

Example 4.1. *Let $X = \{x_1, x_2\}, E = \{e_1, e_2\}$. Define two topologies τ_1 and τ_2 by assigning different neutrosophic triples to subsets of X. Then (X, τ_1, τ_2) forms an FSNBTS.*

Definition 4.2 (Dual Open and Closed Sets)**.** *A fuzzy-soft neutrosophic set F_E is dual open if $F_E \in \tau_1 \cup \tau_2$, and dual closed if its complement belongs to both topologies.*

Definition 4.3 (Dual Interior and Closure)**.**

$$\mathrm{int}_d(F_E) = \mathrm{int}_{\tau_1}(F_E) \cup \mathrm{int}_{\tau_2}(F_E), \quad \mathrm{cl}_d(F_E) = \mathrm{cl}_{\tau_1}(F_E) \cap \mathrm{cl}_{\tau_2}(F_E).$$

Received: August 05, 2025

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

5 Dual Continuities

Definition 5.1. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be FSNBTS. A function $f : X \to Y$ is dual fuzzy-soft neutrosophic continuous if $f^{-1}(G_E) \in \tau_i$ whenever $G_E \in \sigma_i$, $i = 1, 2$.*

Theorem 5.1 (Composition of Dual Continuous Maps)**.** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \to (Z, \rho_1, \rho_2)$ be dual continuous. Then $g \circ f$ is dual continuous.*

Proof. For each $i = 1, 2$ and $U \in \rho_i$, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. Since $g^{-1}(U) \in \sigma_i$ and $f^{-1}(g^{-1}(U)) \in \tau_i$, $(g \circ f)^{-1}(U) \in \tau_i$. Hence $g \circ f$ is dual continuous. □

Theorem 5.2 (Component Continuity)**.** *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is dual continuous, then f is continuous w.r.t. each component topology $\tau_i \to \sigma_i$. Conversely, if f is continuous in each component, then f is dual continuous.*

Proof. Direct from the definition of dual continuity and componentwise verification. □

6 Dual Compactness and Connectedness

Definition 6.1 (Dual Compactness)**.** *A FSNBTS (X, τ_1, τ_2) is dual compact if every dual open cover admits a finite subcover.*

Definition 6.2 (Dual Connectedness)**.** *A FSNBTS is dual connected if it cannot be represented as the disjoint union of two non-empty dual open sets.*

Proposition 6.1 (Continuous Image of Dual Compact Space)**.** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be dual continuous. If (X, τ_1, τ_2) is dual compact, then $f(X)$ is dual compact. If f is surjective, Y is dual compact.*

Proof. Let $\mathcal{U}$ be a dual open cover of $f(X)$. Then $\mathcal{V} = \{f^{-1}(U) \mid U \in \mathcal{U}\}$ is a dual open cover of X since f is dual continuous. By dual compactness, a finite subcover exists. The corresponding finite family in $\mathcal{U}$ covers $f(X)$. □

7 Relations Between FSN Topologies

Theorem 7.1 (Intersection of FSN Topologies)**.** *If τ_1 and τ_2 are FSN topologies on X, then $\tau_1 \cap \tau_2$ is also an FSN topology.*

Proof.
1. $\tilde{0}_E, \tilde{1}_E \in \tau_1 \cap \tau_2$.
2. Arbitrary unions: $\bigvee_\alpha U_\alpha \in \tau_1$ and τ_2, hence in intersection.
3. Finite intersections: $U \wedge V \in \tau_1$ and τ_2, hence in intersection.

□

Remark 7.1. *This generalizes single FSN topology and allows analysis of dual uncertainty layers.*

Received: August 05, 2025

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

8 Additional Propositions

Proposition 8.1 (Dual Interior and Closure Properties). *Let (X, τ_1, τ_2) be an FSNBTS and F_E a fuzzy-soft neutrosophic set. Then:*

1. $\mathrm{int}_d(F_E) \subseteq F_E \subseteq \mathrm{cl}_d(F_E)$
2. $\mathrm{int}_d(F_E \cap G_E) = \mathrm{int}_d(F_E) \cap \mathrm{int}_d(G_E)$
3. $\mathrm{cl}_d(F_E \cup G_E) = \mathrm{cl}_d(F_E) \cup \mathrm{cl}_d(G_E)$

Proof. Follows parameterwise and componentwise from FSNBTS definitions and standard topological laws. □

Proposition 8.2 (Dual Open Map Preservation). *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is dual continuous and surjective, the image of a dual open set under a dual open map is dual open.*

Proof. Direct from the definition of dual open maps. □

Proposition 8.3 (Dual Subspace Topology). *Let $A \subseteq X$. Then $(A, \tau_1|_A, \tau_2|_A)$ forms an FSNBTS, where $\tau_i|_A = \{F_E \cap A \mid F_E \in \tau_i\}$.*

Proof. Intersection with a subset preserves topology axioms; parameterwise and componentwise. □

Proposition 8.4 (Dual Connectedness under Continuous Maps). *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is dual continuous and surjective and X is dual connected, then Y is dual connected.*

Proof. Assume Y is a disjoint union of dual open sets. Then preimages in X form a disjoint dual open cover of X, contradicting dual connectedness. □

Proposition 8.5 (Dual Compactness under Closed Subsets). *If (X, τ_1, τ_2) is dual compact and $F_E \subseteq X$ is dual closed, then F_E with subspace dual topology is dual compact.*

Proof. Extend the cover of F_E to X by adding $X \setminus F_E$. Use dual compactness of X, then restrict back to F_E. □

Proposition 8.6 (Dual Interior of Increasing Family). *Let $\{F_E^\lambda\}_{\lambda\in\Lambda}$ be an increasing family of fuzzy-soft neutrosophic sets. Then:*

$$\mathrm{int}_d\left(\bigvee_{\lambda\in\Lambda} F_E^\lambda\right) = \bigvee_{\lambda\in\Lambda} \mathrm{int}_d(F_E^\lambda)$$

Proof. Follows from continuity of sup operations and dual interior definition componentwise. □

9 Hierarchy and Generalization

- $I = 0$: Fuzzy-soft bitopological space.
- $F = 0$: Intuitionistic fuzzy-soft bitopological space.
- $E = \{e_0\}$: Fuzzy-neutrosophic bitopological space.
- $\tau_1 = \tau_2$: Ordinary fuzzy-soft neutrosophic topology.

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

10 Conclusions

We established fuzzy-soft neutrosophic bitopological spaces and studied dual continuity, compactness, and connectedness. The framework extends fuzzy-soft and neutrosophic topologies while enabling dual structural analysis. Future work may include separation axioms, dual homeomorphisms, categorical structures, and convergence via fuzzy-soft neutrosophic nets.

References

[1] C. L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.*, 24(1), 182–190, 1968.

[2] D. A. Molodtsov, Soft set theory—First results, *Comput. Math. Appl.*, 37(4–5), 19–31, 1999.

[3] F. Smarandache, Neutrosophic set—a generalization of the intuitionistic fuzzy set, 1998.

[4] J. C. Kelly, Bitopological spaces, *Proc. Lond. Math. Soc.*, 13(3), 71–89, 1963.

Received: August 05, 2025