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GENERAL SOLUTION OF THE OPERATOR $\diamondsuit_C^k$ AND FOURIER ANALYSIS OF ITS CONVOLUTIONS

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Abstract

This paper studies the general solution of

$$\diamondsuit_C^k u(x) = f(x),$$

where $\diamondsuit_C^k$ is an operator associated with the diamond operator, iterated k times on $\mathbb{R}^n$. We first derive an explicit representation of the solution in terms of convolutions of fundamental kernels, applicable to both homogeneous and inhomogeneous cases. We then present a Fourier analysis of $\diamondsuit_C^k$, showing that its Fourier symbol is

$$\sigma_{\diamondsuit_C^k}(\xi) = \left(\frac{|\xi'|^4}{C^4} - |\xi''|^4\right)^k,$$

and obtain a closed-form Fourier transform of the fundamental solution. These results provide a concise characterization of iterated ultra hyperbolic operators in both spatial and frequency domains, unify convolution identities with Fourier multipliers, and extend existing results in distribution theory and partial differential equations.

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1 Introduction

In this section, we introduce a new class of operators known as the operator $\diamondsuit_C^k$, iterated k times. We investigate various structural properties of these operators and study some of their relations. Kananthai [5] studied the diamond operator, which is defined by

$$\begin{aligned}\diamondsuit_1^k &= \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k\\ &= \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right) - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right]^k \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right) + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right]^k\\ &= \Box_1^k \triangle_1^k,\end{aligned}$$

with $p+q=n$.

He also showed that the function

$$u(x) = (-1)^k R^H_{2k,1}(x) * R^e_{2k,1}(x)$$

is the unique fundamental solution for the operator $\diamondsuit_1^k$, where $*$ denotes convolution, $R^H_{2k,1}(x) = R^H_{2k,C}(x)$, and $R^e_{2k,1}(x) = R^e_{2k,C}(x)$ with $C=1$, as defined in (2.2) and (2.6), respectively. That is,

$$\diamondsuit_1^k \left((-1)^k R^H_{2k,1}(x) * R^e_{2k,1}(x)\right) = \delta, \quad x \in \mathbb{R}^n. \tag{1.1}$$

For $k=1$, the operator $\diamondsuit_1$ can be expressed as

$$\diamondsuit_1 = \triangle_1 \Box_1 = \Box_1 \triangle_1,$$

where $\Box_1$ is the ultra-hyperbolic operator,

$$\Box_1 = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right) - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right], \tag{1.2}$$

and $\triangle_1$ is the Laplace operator,

$$\triangle_1 = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right) + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right], \tag{1.3}$$

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with $p+q=n$.

Kananthai [6] studied the product of ultra-hyperbolic operators related to elastic waves for the equation

$$\square_{C_1}^k \square_{C_2}^k u(x) = \delta. \tag{1.4}$$

The fundamental solution of (1.4) is given by

$$u(x) = S_{2k}^H(x) * T_{2k}^H(x),$$

where

$$\square_{C_1}^k = \left[\frac{1}{C_1^2}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right) - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right]^k, \tag{1.5}$$

and

$$\square_{C_2}^k = \left[\frac{1}{C_2^2}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right) - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right]^k. \tag{1.6}$$

Sritanratana and Kananthai [9] studied the product of non-linear diamond operators related to elastic waves, defined by

$$\diamondsuit_{C_1}^k = \left[\frac{1}{C_1^4}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k,$$

and

$$\diamondsuit_{C_2}^k = \left[\frac{1}{C_2^4}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k,$$

of the form

$$\diamondsuit_{C_1}^k \diamondsuit_{C_2}^k u(x) = f\left(x, \triangle_{C_1}^{k-1}\square_{C_1}^k \square_{C_2}^k u(x)\right),$$

where $u(x)$ is related to the elastic wave equation depending on the values of p, q, and k.

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Bupasiri and Nonlaopon [3] studied the equation

$$\diamondsuit_C^k u(x) = \sum_{r=0}^{t} a_r \diamondsuit_C^r \delta, \tag{1.7}$$

where $\diamondsuit_C^k$ is the operator related to the diamond operator, defined by

$$\begin{aligned}
\diamondsuit_C^k &= \left[\frac{1}{C^4}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \\
&= \left[\frac{1}{C^2}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right) - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right]^k \left[\frac{1}{C^2}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right) + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right]^k \\
&= \Box_C^k \triangle_C^k,
\end{aligned}$$

with C a positive real number.

Moreover, Papakhao and Nonlaopon [8] showed that

$$u(x) = [R^H_{2(k-1),C}]^{(m)} + R^H_{2k,C}(x) * f(x)$$

is the general solution of the equation

$$\Box_C^k u(x) = f(x),$$

where $f(x)$ is a generalized function and

$$\Box_C^k = \left[\frac{1}{C^2}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right) - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right]^k. \tag{1.8}$$

The purpose of this article is to find the general solution to the equation

$$\diamondsuit_C^k u(x) = f(x), \tag{1.9}$$

using convolutions of generalized functions, where $p+q = n$, $x \in \mathbb{R}^n$, C is a positive constant, and $f(x)$ is a given generalized function.

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The operator $\diamondsuit_C^k$ is defined by

$$\begin{aligned}\diamondsuit_C^k &= \left[\frac{1}{C^4}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \\ &= \left[\frac{1}{C^2}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right) - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right]^k \left[\frac{1}{C^2}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right) + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right]^k \\ &= \Box_C^k \triangle_C^k, \end{aligned} \tag{1.10}$$

where

$$\triangle_C^k = \left[\frac{1}{C^2}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right) + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right]^k \tag{1.11}$$

is the operator related to the Laplace operator. Finally, we emphasize the role of the Fourier transform in our analysis. The Fourier transform provides a powerful tool to convert differential operators into multiplication operators in the frequency domain, thereby simplifying the study of fundamental solutions and convolution identities. In particular, for the operator $\diamondsuit_C^k$, the Fourier transform allows us to represent the solution as a product of the Fourier symbols and to investigate the convolution of Riesz kernels efficiently. This approach not only yields explicit representations of solutions but also clarifies the spectral properties of the operator, which are essential for proving the inversion formulas presented in the main results.

2 Preliminaries

Key properties of the *ultra-hyperbolic* and *Riesz elliptic* kernels needed for the sequel are summarized here.

Definition 2.1. Let $x = (x_1, x_2, \dots, x_n)$ be a point in the n-dimensional space $\mathbb{R}^n$, and define

$$u = C^2(x_1^2 + x_2^2 + \cdots + x_p^2) - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{2.1}$$

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where $p+q=n$. Let $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ denote the interior of the forward cone, and let $\overline{\Gamma}_+$ denote its closure. The following function was introduced by Nozaki ([7], p. 72):

$$R^H_{\alpha,C}(x) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+, \\ 0, & \text{if } x \notin \Gamma_+. \end{cases} \tag{2.2}$$

The function $R^H_{\alpha,C}(x) = R^H_{\alpha,1}(x)$ is called the *ultra-hyperbolic kernel of Marcel Riesz*. Here α is a complex parameter and n is the dimension of the space. The constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)}, \tag{2.3}$$

where p is the number of positive terms in u as defined in (2.1). We assume supp $R^H_{\alpha,C}(x) \subset \overline{\Gamma}_+$. The function $R^H_{\alpha,C}(x)$ is an ordinary function if $\operatorname{Re}\alpha \geq n$, and it is a distribution if $\operatorname{Re}\alpha < n$.

If $p=1$, then (2.2) reduces to the function $M_{\alpha,C}(u)$, defined by

$$M_{\alpha,C}(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, & \text{if } x \in \Gamma_+, \\ 0, & \text{if } x \notin \Gamma_+, \end{cases} \tag{2.4}$$

where $u = C^2x_1^2 - x_2^2 - \cdots - x_n^2$ and $H_n(\alpha) = \pi^{\frac{n-1}{2}} 2^{\alpha-1} \Gamma\left(\frac{\alpha-n+2}{2}\right)$. The function $M_{\alpha,1}(u)$ is called the *hyperbolic kernel of Marcel Riesz*.

Definition 2.2. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, and define

$$v = C^2(x_1^2 + x_2^2 + \cdots + x_p^2) + x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2. \tag{2.5}$$

Define the function

$$R^e_{\alpha,C}(x) = \frac{v^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, \tag{2.6}$$

where α is any complex number and the constant $H_n(\alpha)$ is given by

$$H_n(\alpha) = \frac{\pi^{1/2} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}. \tag{2.7}$$

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The function $R^e_{\alpha,C}(x) = R^e_{\alpha,1}(x)$ is called the *elliptic kernel of Marcel Riesz.*

Definition 2.3 (Fourier Transform)**.** Let $f \in L^1(\mathbb{R}^n)$, the space of integrable functions on $\mathbb{R}^n$. The *Fourier transform* of f is defined by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx, \tag{2.8}$$

where $x = (x_1, \ldots, x_n), \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and

$$\xi \cdot x = \sum_{j=1}^{n} \xi_j x_j$$

is the standard inner product on $\mathbb{R}^n$.

The *inverse Fourier transform* is given by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi\cdot x} \widehat{f}(\xi)\, d\xi. \tag{2.9}$$

If f is a distribution with compact support, the Fourier transform is defined in the sense of distributions by (see [10, Theorem 7.4-3])

$$\widehat{f}(\xi) = \mathcal{F}\{f\}(\xi) := \frac{1}{(2\pi)^{n/2}} \langle f(x), e^{-i\xi\cdot x}\rangle, \tag{2.10}$$

where $\langle\cdot,\cdot\rangle$ denotes the distributional pairing.

Lemma 2.4 ([3])**.** *Let $\triangle^k_C u(x) = \delta$ for $x \in \mathbb{R}^n$, where $\triangle^k_C$ is the operator associated to the Laplace operator iterated k times, defined by* (1.11)*. Then*

$$u(x) = (-1)^k R^e_{2k,C}(x)$$

is the fundamental solution of the operator $\triangle^k_C$, where

$$R^e_{2k,C}(x) = \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k}\pi^{n/2}\Gamma(k)} |v|^{2k-n}. \tag{2.11}$$

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Lemma 2.5 ([2]). *Let* $\Box_C^k u(x) = \delta$ *for* $x \in \Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0$ *and* $u > 0\}$, *where* $\Box_C^k$ *is the operator associated to the ultra-hyperbolic operator iterated* k *times, defined by* (1.2). *Then*

$$u(x) = R^H_{2k,C}(x)$$

is the fundamental solution of $\Box_C^k$, *where*

$$R^H_{2k,C}(x) = \frac{u^{\frac{2k-n}{2}}}{K_n(2k)} = \frac{\left(C^2(x_1^2 + \cdots + x_p^2) - x_{p+1}^2 - \cdots - x_{p+q}^2\right)^{\frac{2k-n}{2}}}{K_n(2k)}, \tag{2.12}$$

with

$$K_n(2k) = \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{2+2k-n}{2}\right)\Gamma\left(\frac{1-2k}{2}\right)\Gamma(2k)}{\Gamma\left(\frac{2+2k-p}{2}\right)\Gamma\left(\frac{p-2k}{2}\right)}. \tag{2.13}$$

Lemma 2.6 ([3]). *Let* $\Diamond_C^k u(x) = \delta$ *for* $x \in \mathbb{R}^n$. *Then*

$$u(x) = (-1)^k R^e_{2k,C}(x) * R^H_{2k,C}(x)$$

is the fundamental solution of the operator $\Diamond_C^k$, *where* $\Diamond_C^k$ *is the operator associated to the diamond operator iterated* k *times, defined by* (1.10), *and* $R^e_{2k,C}(x)$, $R^H_{2k,C}(x)$ *are defined by* (2.11) *and* (2.12), *respectively. Moreover,* $(-1)^k R^e_{2k,C}(x) * R^H_{2k,C}(x)$ *is a tempered distribution.*

It is straightforward to show that

$$R^e_{-2k,C}(x) * R^H_{-2k,C}(x) = (-1)^k \Diamond_C^k \delta,$$

for k a non-negative integer.

Lemma 2.7 ([1]). $R^e_{2k,C}(x)$ *is a homogeneous distribution of order* $(2k - n)$. *In particular, it is a tempered distribution.*

Lemma 2.8 ([1]). *(The convolutions of tempered distributions)*

(a) $(\triangle_C^k \delta) * u(x) = \triangle_C^k u(x)$, *where* $u(x)$ *is any tempered distribution.*

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(b) *Let* $R^e_{2k,C}(x)$ *and* $R^e_{2m,C}(x)$ *be defined by* (2.6). *Then* $R^e_{2k,C}(x) * R^e_{2m,C}(x)$ *exists and is a tempered distribution.*

(d) *Let* $R^e_{2k,C}(x)$ *and* $R^e_{2m,C}(x)$ *be defined by* (2.6). *If* $R^e_{2k,C}(x) * R^e_{2m,C}(x) = \delta(x)$, *then* $R^e_{2k,C}(x)$ *is an inverse of* $R^e_{2m,C}(x)$ *in the convolution algebra, denoted by* $R^e_{2k,C}(x) = R^{e*-1}_{2m,C}(x)$.

Lemma 2.9 ([8]). *(The convolution of tempered distributions)*

(a) $(\Box^k_C\delta) * u(x) = \Box^k_C u(x)$, *where* $u(x)$ *is any tempered distribution.*

(b) *Let* $R^H_{2k,C}(x)$ *and* $R^H_{2m,C}(x)$ *be defined by* (2.2). *Then* $R^H_{2k,C}(x) * R^H_{2m,C}(x)$ *exists and is a tempered distribution.*

(c) *Let* $R^H_{2k,C}(x)$ *and* $R^H_{2m,C}(x)$ *be defined by* (2.2). *Then*

$$R^H_{2k,C}(x) * R^H_{2m,C}(x) = R^H_{2k+2m,C}(x),$$

where k *and* m *are non-negative integers.*

Lemma 2.10 ([4]). *Given a hypersurface* P, *we have*

$$P\delta^{(m)}(P) + m\delta^{(m-1)}(P) = 0,$$

where $\delta^{(m)}$ *denotes the* m*-th derivative of the Dirac delta distribution.*

Lemma 2.11 ([9]). *Consider the equation*

$$\triangle^k_C u(x) = 0, \tag{2.14}$$

where $\triangle^k_C$ *is the operator associated to the Laplace operator iterated* k *times, as defined in* (1.11), *and* $x \in \mathbb{R}^n$. *Then*

$$u(x) = (-1)^{k-1}[R^e_{2(k-1),C}(x)]^{(m)}$$

defined by (2.6) *with* m *derivatives, is a solution of* (2.14), *where* $m = \frac{n-4}{2}$, $n \geq 4$, *and* n *is an even dimension.*

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Lemma 2.12 ([8]). *Consider the equation*

$$\Box_C^k u(x) = 0, \tag{2.15}$$

where $\Box_C^k$ *is the operator associated to the ultra-hyperbolic operator iterated* k *times, as defined in* (1.2), *and* $x \in \mathbb{R}^n$. *Then*

$$u(x) = [R^H_{2(k-1),C}(x)]^{(m)}$$

defined by (2.2) *with* m *derivatives, is a solution of* (2.15), *where* $m = \frac{n-4}{2}$, $n \geq 4$, *and* n *is an even dimension.*

Lemma 2.13. *Consider the equation*

$$\Diamond_C^k u(x) = 0, \tag{2.16}$$

where $\Diamond_C^k$ *is the operator associated to the diamond operator iterated* k *times, as defined in* (1.10), *and* $u(x)$ *is an unknown generalized function. Then*

$$u(x) = (-1)^k R^e_{2k,C}(x) * [R^H_{2(k-1),C}(x)]^{(m)}$$

is a solution of (2.16), *where* $[R^H_{2(k-1),C}(x)]^{(m)}$ *and* $R^e_{2k,C}(x)$ *are defined by* (2.2) *and* (2.6), *respectively, with* m *derivatives.*

Proof. The operator $\Diamond_C^k$ can be expressed as the product of the operators $\Box_C^k$ and $\triangle_C^k$, i.e.,

$$\Diamond_C^k u(x) = \Box_C^k \triangle_C^k u(x) = 0.$$

By Lemma 2.12, we have

$$\triangle_C^k u(x) = [R^H_{2(k-1),C}(x)]^{(m)}.$$

Convolving both sides with $(-1)^k R^e_{2k,C}(x)$, we obtain

$$(-1)^k R^e_{2k,C}(x) * \triangle_C^k u(x) = (-1)^k R^e_{2k,C}(x) * [R^H_{2(k-1),C}(x)]^{(m)}.$$

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By Lemma 2.4, we have

$$\begin{aligned}\triangle_C^k\big((-1)^k R^e_{2k,C}(x) * u(x)\big) &= \delta * u(x) = u(x)\\ &= (-1)^k R^e_{2k,C}(x) * [R^H_{2(k-1),C}(x)]^{(m)}.\end{aligned}$$

Hence,

$$u(x) = (-1)^k R^e_{2k,C}(x) * [R^H_{2(k-1),C}(x)]^{(m)}.$$

□

3 Main Results

Before presenting the main theorem, we recall that the operator $\diamondsuit_C^k$, defined as the composition of the operators associated to the ultra-hyperbolic operator $\square_C^k$ and the Laplace operator $\triangle_C^k$, plays a central role in the study of higher-order partial differential equations in even-dimensional spaces. The analysis of such operators often requires the use of fundamental solutions expressed in terms of the generalized kernels $R^e_{2k,C}(x)$ and $R^H_{2k,C}(x)$, which allow the construction of solutions for both homogeneous and inhomogeneous equations.

The following theorem provides a general formula for the solution of the iterated diamond equation with an arbitrary generalized source term $f(x)$, combining the effects of the homogeneous part and the particular solution induced by $f(x)$.

Theorem 3.1. *Consider the equation*

$$\diamondsuit_C^k u(x) = f(x), \tag{3.1}$$

where $\diamondsuit_C^k = \square_C^k \triangle_C^k$ is the operator iterated k times, defined in (1.10)*, C is a positive constant, $f(x)$ is a generalized function, $u(x)$ is an unknown generalized function, $x \in \mathbb{R}^n$, and n is even. Then the general solution of* (3.1) *is*

$$u(x) = \left((-1)^k R^e_{2k,C}(x) * [R^H_{2(k-1),C}(x)]^{(m)}\right) + \left((-1)^k R^e_{2k,C}(x) * R^H_{2k,C}(x) * f(x)\right), \tag{3.2}$$

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where $\left[R^H_{2(k-1),C}(x)\right]^{(m)}$ *and* $R^e_{2k,C}(x)$ *are defined in* (2.2) *and* (2.6), *respectively, with* m *derivatives.*

Proof. Convolving both sides of (3.1) with $(-1)^k R^e_{2k,C}(x) * R^H_{2k,C}(x)$, we obtain

$$\left((-1)^k R^e_{2k,C}(x) * R^H_{2k,C}(x)\right) * \diamondsuit^k_C u(x) = (-1)^k R^e_{2k,C}(x) * R^H_{2k,C}(x) * f(x). \tag{3.3}$$

By Lemma 2.13, we have

$$\diamondsuit^k_C \left((-1)^k R^e_{2k,C}(x) * R^H_{2k,C}(x)\right) * u(x) = (-1)^k R^e_{2k,C}(x) * R^H_{2k,C}(x) * f(x), \tag{3.4}$$

$$\delta * u(x) = (-1)^k R^e_{2k,C}(x) * R^H_{2k,C}(x) * f(x). \tag{3.5}$$

Hence,

$$u(x) = (-1)^k R^e_{2k,C}(x) * R^H_{2k,C}(x) * f(x) \tag{3.6}$$

is a solution of (3.1).

For the homogeneous equation $\diamondsuit^k_C u(x) = 0$, a solution is

$$u(x) = (-1)^k R^e_{2k,C}(x) * [R^H_{2(k-1),C}(x)]^{(m)} \tag{3.7}$$

by Lemma 2.13. Therefore, the general solution of (3.1) is

$$\begin{aligned} u(x) = {} & \left((-1)^k R^e_{2k,C}(x) * [R^H_{2(k-1),C}(x)]^{(m)}\right) \\ & + \left((-1)^k R^e_{2k,C}(x) * R^H_{2k,C}(x) * f(x)\right). \end{aligned} \tag{3.8}$$

This completes the proof. □

By setting $C = 1$, (3.1) becomes the diamond equation

$$\diamondsuit^k_1 w(x) = f(x), \tag{3.9}$$

where $\diamondsuit^k_1$ is the diamond operator iterated k times, $f(x)$ is a generalized function, and $w(x)$ is an unknown generalized function. From (3.6), a solution is

$$w(x) = (-1)^k R^e_{2k,1}(x) * R^H_{2k,1}(x) * f(x), \tag{3.10}$$

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where $R^H_{2k,1}(x) = R^H_{2k}(x)$ and $R^e_{2k,1}(x) = R^e_{2k}(x)$ are defined in (2.2) and (2.6), respectively. The general solution of the diamond equation is

$$w(x) = \left((-1)^k R^e_{2k}(x) * [R^H_{2(k-1)}(x)]^{(m)}\right) + \left((-1)^k R^e_{2k}(x) * R^H_{2k}(x) * f(x)\right). \tag{3.11}$$

For the homogeneous equation $\triangle^k_C w(x) = 0$, a solution is

$$w(x) = (-1)^{k-1}[R^e_{2(k-1),C}(x)]^{(m)} \tag{3.12}$$

by Lemma 2.11. Hence, the general solution of

$$\triangle^k_C w(x) = f(x) \tag{3.13}$$

is

$$w(x) = (-1)^{k-1}[R^e_{2(k-1),C}(x)]^{(m)} + (-1)^k R^e_{2k,C}(x) * f(x). \tag{3.14}$$

Setting $C = 1$, (3.13) becomes the iterated Laplace equation

$$\triangle^k_1 w(x) = f(x), \tag{3.15}$$

with general solution

$$w(x) = (-1)^{k-1}[R^e_{2(k-1)}(x)]^{(m)} + (-1)^k R^e_{2k}(x) * f(x). \tag{3.16}$$

Theorem 3.2 (Fourier Transform of $\diamondsuit^k_C$). *Let $n = p + q$ so that $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$. Write $x = (x', x'')$ with $x' = (x_1, \ldots, x_p) \in \mathbb{R}^p$ and $x'' = (x_{p+1}, \ldots, x_{p+q}) \in \mathbb{R}^q$. Similarly, let $\xi = (\xi', \xi'')$ with $\xi' = (\xi_1, \ldots, \xi_p)$, $\xi'' = (\xi_{p+1}, \ldots, \xi_{p+q})$, so that*

$$|\xi'|^2 = \sum_{i=1}^{p} \xi_i^2, \qquad |\xi''|^2 = \sum_{j=p+1}^{p+q} \xi_j^2.$$

Define the $\diamondsuit^k_C$ operator

$$\diamondsuit^k_C := \left[\frac{1}{C^4}\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k, \tag{3.17}$$

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acting on $\mathcal{S}'(\mathbb{R}^n)$. Suppose $u(x) = (-1)^k\, R^e_{2k,C}(x) * R^H_{2k,C}(x)$ is a fundamental solution of $\diamondsuit^k_C$. Then, in the sense of tempered distributions, the following hold:

1. *The Fourier symbol (multiplier) of $\diamondsuit^k_C$ is*

$$\sigma_{\diamondsuit^k_C}(\xi) = \left(\frac{|\xi'|^4}{C^4} - |\xi''|^4\right)^k. \tag{3.18}$$

Remark 3.3. The function $\sigma_{\diamondsuit^k_C}(\xi)$ is called the *Fourier symbol* or *Fourier multiplier* of $\diamondsuit^k_C$, encoding how the operator acts in the frequency domain via

$$\mathcal{F}\{\diamondsuit^k_C u\}(\xi) = \sigma_{\diamondsuit^k_C}(\xi)\,\widehat{u}(\xi).$$

2. *The Fourier transform of the fundamental solution u is*

$$\widehat{u}(\xi) = \frac{1}{\sigma_{\diamondsuit^k_C}(\xi)} = \frac{1}{\left(\frac{|\xi'|^4}{C^4} - |\xi''|^4\right)^k}, \tag{3.19}$$

which implies

$$\widehat{R^e_{2k,C}}(\xi)\,\widehat{R^H_{2k,C}}(\xi) = \frac{(-1)^k}{\left(\frac{|\xi'|^4}{C^4} - |\xi''|^4\right)^k}. \tag{3.20}$$

3. *The identity*

$$R^e_{-2k,C}(x) * R^H_{-2k,C}(x) = (-1)^k\, \diamondsuit^k_C\delta \tag{3.21}$$

is equivalent to

$$\widehat{R^e_{-2k,C}}(\xi)\,\widehat{R^H_{-2k,C}}(\xi) = (-1)^k\,\sigma_{\diamondsuit^k_C}(\xi) = (-1)^k\left(\frac{|\xi'|^4}{C^4} - |\xi''|^4\right)^k. \tag{3.22}$$

Proof. We prove the theorem in several steps.

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Step 1: Fourier transform of the operator.
Let $u \in \mathcal{S}'(\mathbb{R}^n)$. By the definition of the Fourier transform on tempered distributions, the derivative ∂_{x_i} is mapped to $i\xi_i$. Applying this to (3.17), we obtain

$$\mathcal{F}\{\diamondsuit_C^k u\}(\xi) = \sigma_{\diamondsuit_C^k}(\xi)\,\widehat{u}(\xi),$$

where $\sigma_{\diamondsuit_C^k}(\xi)$ is given by (3.18).

Step 2: Fourier transform of the fundamental solution.
Since u is a fundamental solution, we have

$$\diamondsuit_C^k u(x) = \delta.$$

Taking the Fourier transform of both sides gives

$$\sigma_{\diamondsuit_C^k}(\xi)\,\widehat{u}(\xi) = 1 \quad \Rightarrow \quad \widehat{u}(\xi) = \frac{1}{\sigma_{\diamondsuit_C^k}(\xi)},$$

which proves (3.19).

Step 3: Fourier transform of the convolution representation.
Using the representation

$$u(x) = (-1)^k R^e_{2k,C}(x) * R^H_{2k,C}(x),$$

and the fact that the Fourier transform maps convolutions to pointwise products, we obtain

$$\widehat{R^e_{2k,C}}(\xi)\,\widehat{R^H_{2k,C}}(\xi) = \frac{(-1)^k}{\left(\frac{|\xi'|^4}{C^4} - |\xi''|^4\right)^k}.$$

Step 4: Verification of the inverse identity.
Finally, the identity

$$R^e_{-2k,C}(x) * R^H_{-2k,C}(x) = (-1)^k \diamondsuit_C^k \delta$$

is equivalent, under the Fourier transform, to

$$\widehat{R^e_{-2k,C}}(\xi)\,\widehat{R^H_{-2k,C}}(\xi) = (-1)^k\,\sigma_{\diamondsuit_C^k}(\xi),$$

which confirms (3.21). □

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4 Applications

Theorem 3.2 provides a Fourier analytic framework for handling the $\diamondsuit_C^k$ as the operator associated with the diamond operator iterated k times. Its Fourier symbol representation allows direct computation of solutions in the frequency domain, and the convolution formulas yield explicit solutions in the spatial domain. Typical applications include:

1. **Generalized source terms.** For the diamond equation

$$\diamondsuit_1^k w(x) = f(x),$$

where $f(x) \in \mathcal{S}'(\mathbb{R}^n)$, the solution admits the explicit form

$$w(x) = (-1)^k R_{2k}^e(x) * R_{2k}^H(x) * f(x) + (-1)^k R_{2k}^e(x) * [R_{2(k-1)}^H(x)]^{(m)}.$$

This formula enables computations with Dirac measures, polynomial-type sources, and compactly supported distributions.

2. **Green's function representation.** When $f(x) = \delta(x - x_0)$, the solution reduces to a shifted convolution kernel,

$$w(x) = (-1)^k R_{2k}^e(x) * R_{2k}^H(x - x_0) + (-1)^k R_{2k}^e(x) * [R_{2(k-1)}^H(x)]^{(m)},$$

which provides the explicit Green's function of the iterated operator.

3. **Boundary-value problems.** The decomposition into homogeneous and particular parts facilitates the treatment of boundary conditions. The homogeneous contribution

$$(-1)^k R_{2k}^e(x) * [R_{2(k-1)}^H(x)]^{(m)}$$

can be tuned to enforce boundary constraints, while the convolution with $f(x)$ accounts for the inhomogeneous forcing.

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5 Conclusion

We have established the Fourier transform characterization of the operator $\Diamond_C^k$ and derived explicit formulas for its fundamental solutions. The Fourier symbol

$$\sigma_{\Diamond_C^k}(\xi) = \left(\frac{|\xi'|^4}{C^4} - |\xi''|^4\right)^k$$

encodes the operator's action in the frequency domain, while the convolution representation with kernels $R^e_{2k,C}(x)$ and $R^H_{2k,C}(x)$ provides a constructive method for solutions in the spatial domain.

These results unify the frequency-domain and distributional perspectives, yielding:

1. explicit Green's functions for the diamond operator iterated k times,

2. systematic handling of both homogeneous and inhomogeneous problems,

3. a framework extendable to related higher-order operators.

Specializing to $C = 1$ recovers the classical diamond and Laplace equations, confirming consistency with standard theory. Overall, the Fourier approach enriches the analytical toolkit for higher-order PDEs in mathematical physics and applied analysis.

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References

[1] S. Bupasiri, On the Weak Solution of the Compound Laplace Equation, *Far East Journal of Applied Mathematics*, **47**, No 2 (2010), 95–104.

[2] S. Bupasiri and K. Nonlaopon, On the Weak Solution of Compound Equations Related to the Ultra-Hyperbolic Operators, *Far East Journal of Applied Mathematics*, **35**, (2009), 129–139.

[3] S. Bupasiri and K. Nonlaopon, On the Solution of the n-Dimensional Operator Related to the Diamond Operator, *Far East Journal of Mathematical Sciences*, **45**, (2010), 69–80.

[4] I.M. Gelfand and G.E. Shilov, *Generalized Functions*, Vol. 1, Academic Press, New York (1964).

[5] A. Kananthai, On the Solutions of the n-Dimensional Diamond Operator, *Applied Mathematics and Computation*, **88**, (1997), 27–37.

[6] A. Kananthai, On the Product of the Ultra-Hyperbolic Operator Related to the Elastic Wave, *Applied Mathematics and Computation*, **4**, No 6 (1999), 88–91.

[7] Y. Nozaki, On Riemann-Liouville Integral of Ultra-Hyperbolic Type, *Kodai Mathematical Seminar Reports*, **6**, (1964), 69–87.

[8] J. Papakhao and K. Nonlaopon, On the General Solution of the Operator $\square_c^k$ Related to the Ultra-Hyperbolic Operator, *Applied Mathematical Sciences*, **5**, No 53 (2011), 2629–2638.

[9] G. Sritanratana and A. Kananthai, On the Product of the Non-Linear Diamond Operators Related to the Elastic Wave, *Applied Mathematics and Computation*, **147**, No 1 (2004), 79–88.

[10] A.H. Zemanian, *Distribution Theory and Transform Analysis*, McGraw-Hill, New York (1964).

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