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INVESTIGATION OF MAX RADIAL NUMBER IN TREES

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Abstract

Let $G\big(V(G), E(G)\big)$ be a simple connected graph. For any set $S \subseteq V$, the S-radial set, $B_R(S)$, is defined as the set of vertices $u \in V\backslash S$ which are at a radial distance from some vertex $v \in S$. The Max-radial number of G is defined as $S\ max\{|B_R(S)| - |S|\}$ and it is denoted by $\partial_R(G)$. In this paper, we characterize the family of trees with extreme and almost extreme max-radial number. Also, we investigate how this parameter varies with other graph parameters and obtain some interesting results.

Keywords: *Eccentricity, Radius, Diameter, S-radial set, ∂_R-set, Max-Radial number.*

AMS Subject Classification code (2020): 05C(Primary)

1 INTRODUCTION

In this paper, we consider only *connected, simple, undirected and finite* graph. For basic notations and terminology, one can refer in [6]. For any vertex v in a graph G, the *degree*, $deg\ (v)$, of v is the number of edges incident with v. A vertex v is called a *full vertex* if it is adjacent to all other vertices. The minimum and maximum degree among all the vertices of G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. A graph G is *acyclic* if G contains no cycle. A graph G is said to be a *tree* if G is connected and acyclic.

Let u, v be any two vertices of G. The *distance*, $d(u, v)$ between u and v, is the length of the shortest (u, v)-path in G. For any vertex $u \in V(G)$, the *eccentricity*, $e(u)$, of u is the distance of a vertex farthest from u. That is, $e(u) = max\ \{d(u, v\}: v \in V(G)\}$. The *radius* of G is the minimum eccentricity of the graph and is denoted by $rad(G)$. The *diameter* of G is the maximum eccentricity of G and is denoted by $diam(G)$. For further reference on distance in graphs, one can refer [5].

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Getting inspired by the concept of differential in graphs, the study on max-radial number of a graph has been initiated.

In [4], we have discussed about an application of the max-radial number of graphs in game theory. For any subset S of vertex set V, the S-radial set, $B_R(S)$, is defined as the set of vertices $u \in V \backslash S$ which are at a distance of $rad(G)$ from any vertex $v \in S$. The Max-radial number of G is defined as $S\ max\{|B_R(S)| - |S|\}$ and it is denoted by $\partial_R(G)$. Max-Radial number of special classes of graphs has been discussed in [3].

In this paper, we characterize the family of trees with extreme max radial number. Also, we obtain the relationship between the max-radial number and other graph theoretical parameters.

2 MAIN RESULTS

We know that for any graph G, $\partial_R(G)$ lies between 0 and $n-2$. In trees also, the bound is maintained, since $\partial_R(P_{2n}) = 0$ and $\partial_R(K_{1,n-1}) = n-2$. In this section, we characterize the family of trees with max-radial number equal to and almost equal to its upper and lower bound.

Proposition 2.1 For any tree T, $\partial_R(T) = 0$ if and only if $T \cong P_{2n}$.

Proof Suppose T be any tree other than even path with $\partial_R(T) = 0$. If T has a vertex v of degree three, then fix a vertex $w \in N(v)$. Now, let $X = \{u\}$ where $d(u, w) = r$. For this singleton set X, $|B_R(X)| \geq 2$. Since $deg\ (v) \geq 3$ and $\partial_R(X) \geq 1$ implies $\partial_R(T) > 0$, which is a contradiction. Therefore, $\Delta(T) \leq 2$. Also, for odd paths, $\partial_R(T) = 1$. Hence we conclude that $T \cong P_{2n}$. And the converse is obvious. ■

For $n \geq 5$, let P_{2n}^{+} denote a tree obtained from P_{2n} by attaching two pendent vertices one at each of the vertices $v_{n+\lceil\frac{n}{2}\rceil-1}$ and $v_{n-\lfloor\frac{n}{2}\rfloor+1}$. For example, P_{10}^{+} is shown in Figure 2.1.

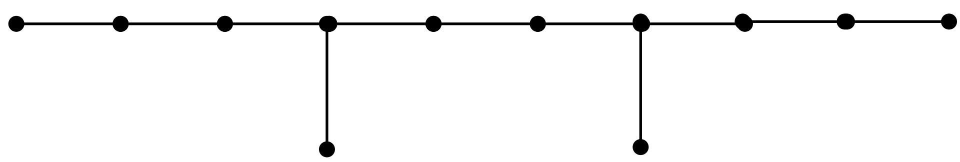

Figure 2.1

Also, let P_{2n}^{*}, $n \geq 2$ denote a family of trees obtained from P_{2n} by attaching a pendent vertex at any of its vertices. For reference, the trees in P_6^{*} are shown in Figure 2.2. Note that $P_{2n+1} \in P_{2n}^{*}$.

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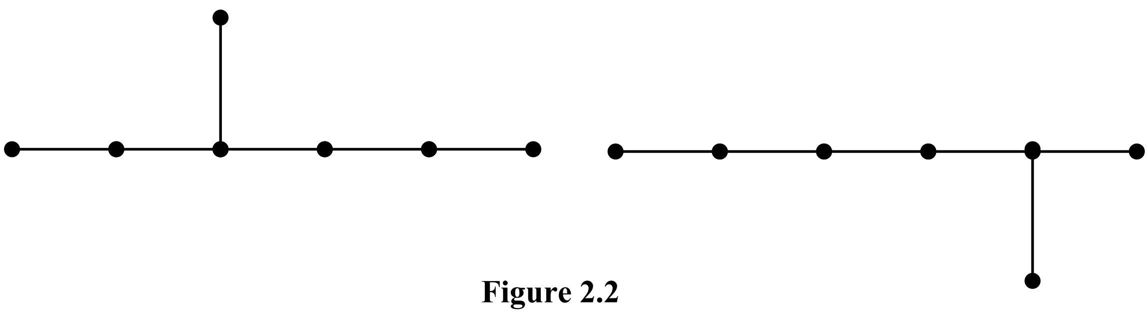

Figure 2.2

We can easily verify that $\partial_R(P_{2n}^{+}) = 1$ and $\partial_R(T) = 1$ if $T \in P_{2n}^{*}$.

Theorem 2.2 For any tree T, $\partial_R(T) = 1$ if and only if T is an odd path or T is isomorphic to any tree in P_{2n}^{*} or T is isomorphic to P_{2n}^{+}.

That is, $\partial_R(T) = 1$ iff $T \cong P_{2n+1}\ or\ P_{2n}^{+}\ or\ T'$ where $T' \in P_{2n}^{*}$.

Proof Let T be any tree with $\partial_R(T) = 1$. Suppose P denotes a diametral path in T.

Case (i) If P is of odd length , then let $X = \{v\ /\ v\ is\ central\ vertex\ of\ P\}$. Now $|B_R(X)| = 2$, implies $\partial_R(X) = 1$. In this case, no vertex in P is of degree 3. Otherwise as discussed in the above proposition, max radial number gets increased by one. Therefore, $T \cong P_{2n+1}$.

Case (ii) P is of even length. Atleast one vertex in P must be of degree greater than two, Since $\partial_R(P_{2n}) = 0$. Suppose w be a vertex in P with degree atleast 3. Note that if w is attached to any path of length two or if $deg\ deg\ (w\ ) \geq 4$. Then there exists ∂_R-set X in T with $\partial_R(X) > 1$, which is not possible. Therefore, $deg\ deg\ (w)\ = 3$ and in particular, w is attached to only one pendent vertex. As a continuation of the same argument, if w is the only vertex of degree 3 in P, then $T \cong P_{2n}^{+}$. Otherwise, there can be atmost two vertices of degree three in P. In particular, they must be the vertices $v_{r+\lceil\frac{r}{2}\rceil-1}$ and $v_{r-\lfloor\frac{r}{2}\rfloor+1}$ (where $2r$ = length of P) in P, so that the max radial number never exceeds one. Then $T \cong P_{2n}^{+}$. And the converse is obvious. ∎

We have already noted that $\partial_R(G) = n - 2$ if and only if $\Delta(G) = n - 1$. Hence a tree with max radial number $n - 2$ is a star and star only. Also, no graph exists with max radial number $n - 3$[ ]. Let us now characterize trees with max radial number $n - 4$.

Let P_3^{*} denote the family of trees obtained from P_3 by attaching n_1, n_2, n_3 pendent vertices respectively at each of its vertices, where $n_1 \geq 0, n_2 \geq 0, n_3 \geq 0$ (not both n_1 & $n_3 = 0$). Some trees in the P_3^{*}-family are shown in Figure 2.3.

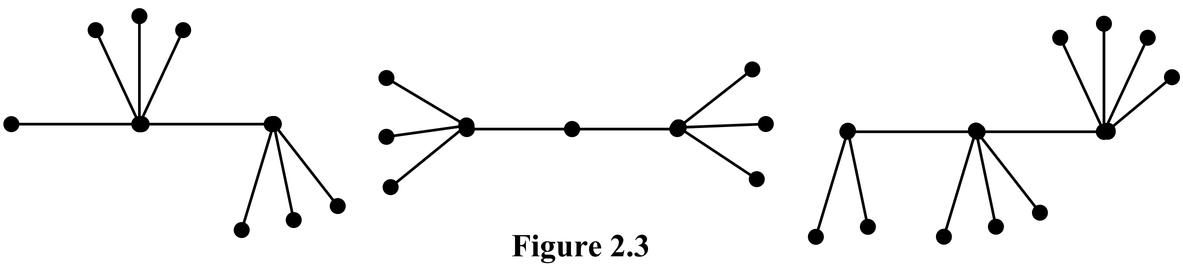

Figure 2.3

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Since n_1 & n_3 are not both simultaneously zero, stars do not belong to P_3^*-family.

Bistars $B(m,n)$ denote the family of trees obtained by attaching m,n pendant vertices respectively at each of the vertices of K_2.

Theorem 2.3 For any tree T, $\partial_R(T) = n-4$ if and only if $T \cong B(m,n)$ (not both $m=n=0$) or $T \cong T'$, where $T' \in P_3^*$.

Proof Let T be any tree with $\partial_R(T) = n-4$. We know that for any subset $S \subseteq V(T), |B_R(S)| \neq n-1$, which is possible only for star. Therefore, $|B_R(S)| = n-2$ *or* $n-3$ for any $S \subseteq V(T)$. This forces that $|S| = 2$ *or* 1 respectively. Let P denote a diametral path in T.

If possible, let length of $P \geq 5$. Then, no matter if the vertices of S (one or both) are in P or not, $B_R(S)$ contains atmost four vertices of P, which implies, $|B_R(S)| < n-3$. Hence $\partial_R(S) < n-4$ for any $S \subseteq V(T)$, which is a contradiction. Therefore, we conclude that P is of length atmost 4. Also, since $\partial_R(T) = n-4$, $len(P) > 2$. Then P is a length 3 or 4.

Any tree of diameter 3 or 4 must be a $B(m,n)$ (not both $m,n=0$) or a member of P_3^*-family. The theorem follows as the converse is obvious. ■

Corollary 2.4 For any tree T, $\partial_R(T) = n-4$ if and only if $diam\ (T) = 3$ *or* 4.

3 Existence Theorems

In this section, we obtain some more bounds for max radial number using other graph parameters. Also we present some existence theorems with desired value for given parameters.

Proposition 3.1 $\Delta - 2$ forms a lower bound for max radial number of trees. That is, $\partial_R(T) \geq \Delta - 2$.

Proof Let T be any tree with maximum degree Δ. Let v be a vertex with $deg\ deg\ (v) = \Delta$. Choose a neighbour w of v such that $e(w) = e(v)+1$. Now, let $S = \{u\}$ where u is a vertex in T such that $d(u,w) = rad(T)$. Therefore, $|B_R(S)| \geq \Delta - 1$, which implies, $\partial_R(T) \geq \Delta - 2$. ■

Note 3.2 The above inequality is sharp. The following proposition stands in favour of this.

A *broom graph* $B_{n,m}$ is a graph of order n with a path P_m and pendant vertices that are all adjacent to the path's origin or terminus. For reference, the Broom graph $B_{7,4}$ is shown in Figure 2.4.

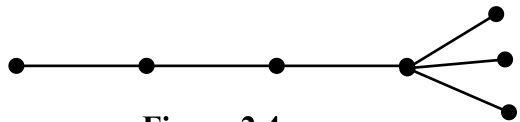

Figure 2.4

Proposition 3.3 For any given integer $m \geq 3$, there exists a tree with maximum degree m and $\partial_R(T) = m-2$.

Proof Consider the Broom graph $B_{m+2,3}$. Therefore, $\Delta(B_{m+2,3}) = m$ and $\partial_R(B_{m+2,3}) = m-2$. ■

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Even more, we can prove the existence of trees with diameter and max radial number to be same value.

Theorem 3.4 There exists a family of trees with any given integer $m \geq 3$ as its diameter and max radial number.

Proof When m is odd, consider $T \cong B_{2m+1,m}$. When m is even, take $T \cong B_{2m,m}$. In both the cases $diam\, T = m$. In T, let $S = \{v_{\left(\frac{m}{2}\right)+1}\}$, $B_R(S) = \{v_{m+1}, \dots, v_{2m+1} \quad if\ m\ is\ odd\ v_1, v_{m+1}, \dots, v_{2m} \quad if\ m\ is\ even$. Hence $\partial_R(S) = m$. No other subset S of T exists with ∂_R value greater than m. Therefore, $\partial_R(T) = m$. ■

Note that when m is odd, $B_{2m+1,m}$ proves the existence of family of graphs with diameter = max radial number $=\Delta - 2$.

Theorem 3.5 For any tree with radius $r \geq 2$ and maximum degree $\Delta \geq 3$, $\partial_R(T) \leq \frac{\Delta(\Delta-1)^r-2}{\Delta-2} - 4(r-1)$ and the inequality is sharp.

Proof We first prove the existence of a tree with $\partial_R(T) = \frac{\Delta(\Delta-1)^r-2}{\Delta-2} - 4(r-1)$ and then we prove that it is the maximum possible with given Δ and r.

Let us construct T as follows.

(i) Take $(\Delta - 1)$-ary tree T' of height r with rooted vertex v.

(ii) Take another $(\Delta - 1)$-ary tree T'' of height $r - 1$ with rooted vertex u.

(iii) Construct T from T' and T'' as follows.

$V(T) = V(T') \cup V(T'')$ and $E(T) = E(T') \cup E(T'') \cup \{uv\}$.

Now T is of maximum degree Δ and radius r. It is easy to note that T is of order $\frac{\Delta(\Delta-1)^r-2}{\Delta-2}$.

Fix a diametral path P_{2r+1} in T and let the vertex on P_{2r+1} be $v_1, v_2, \dots, v_{2r+1}$ with $v_{r+1} = v$. Let $S = \{v_2, v_3, \dots, v_{2r-1}\}$. Then $B_R(S) = V(T) - S$. Therefore, $\partial_R(S) = \frac{\Delta(\Delta-1)^r-2}{\Delta-2} - 4(r-1)$.

Note that no other S' of T could have ∂_R-value greater than this. Therefore, $\partial_R(T) = \frac{\Delta(\Delta-1)^r-2}{\Delta-2} - 4(r-1)$. Since T is constructed with every vertex other than pendant vertices, to be a maximum degree vertex, T yields maximum ∂_R-value with given Δ and radius r.

Any other subtree T_0 of T cannot exceed ∂_R-value of T. Hence $\frac{\Delta(\Delta-1)^r-2}{\Delta-2} - 4(r-1)$ serves as an upper bound for ∂_R-value of T. ■

For given $\Delta = 4$ and $r = 3$, the constructed T is shown in Figure 2.6.

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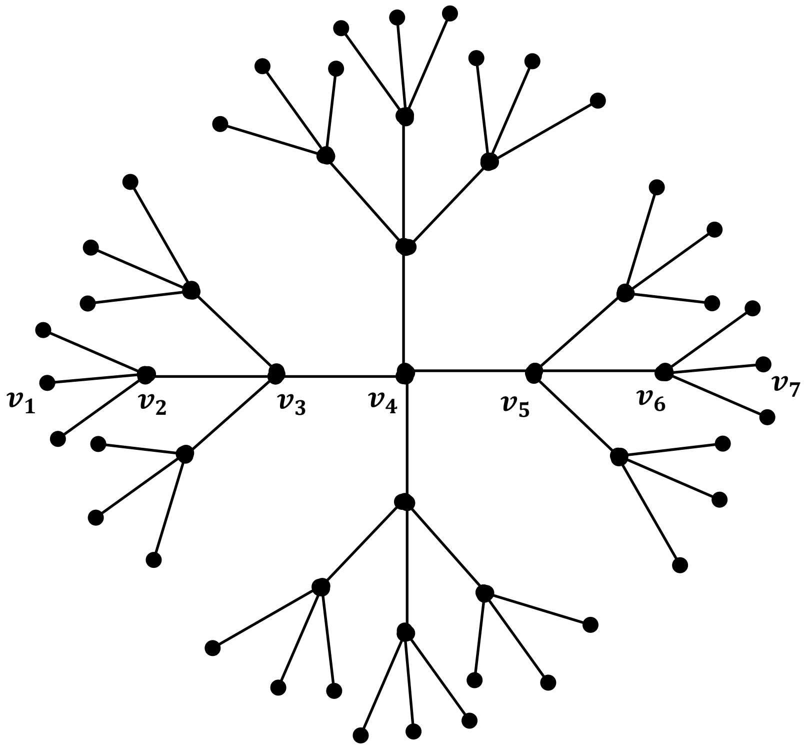

Figure 2.6

Here $\partial_R(T) = 45$ and $\frac{\Delta(\Delta-1)^r-2}{\Delta-2} - 4(r-1) = 45$.

Theorem 3.6 For any given positive integer k such that $\Delta - 2 \leq k \leq \frac{\Delta(\Delta-1)^2-2}{\Delta-2} - 4$, there exists a graph of maximum degree $\Delta \geq 3$ and the max-radial number k.

Proof Let us construct G as follows.

$V(G) = \{v_0, v_1, \ldots, v_n\}$ where $n = k + 3$ and let $n + 1 = r\Delta + s$, $E(G) = \{v_0 v_{j\Delta+1} \,/\, 0 \leq j \leq r\} \cup \{v_{j\Delta+1} v_{j\Delta+i} \,/\, 1 \leq i \leq \Delta, 0 \leq j \leq r-1\} \cup \{v_{r\Delta+1} v_{r\Delta+i} \,/\, 1 \leq i \leq s-1\}$.

Let $S = \begin{cases} \{v_0\} & if\ r < 2 \\ \{v_0, v_1\} & if\ r \geq 2 \end{cases}$. Then $B_R(S) = \begin{cases} V(G) - \{v_1, v_{\Delta+1}\} - S & if\ r < 2 \\ V(G) - S & if\ r \geq 2 \end{cases}$.

In both the cases, $\partial_R(S) = \begin{cases} k + 4 - 3 - 1 \\ k + 4 - 2 - 2 \end{cases} = k$. And since no other set S exists with ∂_R-number greater than this, we have $\partial_R(G) = k$. ■

For example, when $\Delta = 4$ and $k = 8$, we have the constructed graph G as follows.

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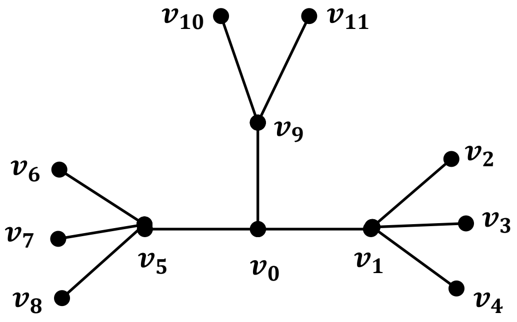

Figure 2.5

Since the max radial number is the difference in cardinalities of a set and its radial set, there is no clear way to conclude the number of vertices chosen in the set and their radial vertices individually.

For example, consider the following trees with max radial number 3 but the radial set of each tree has different cardinality.

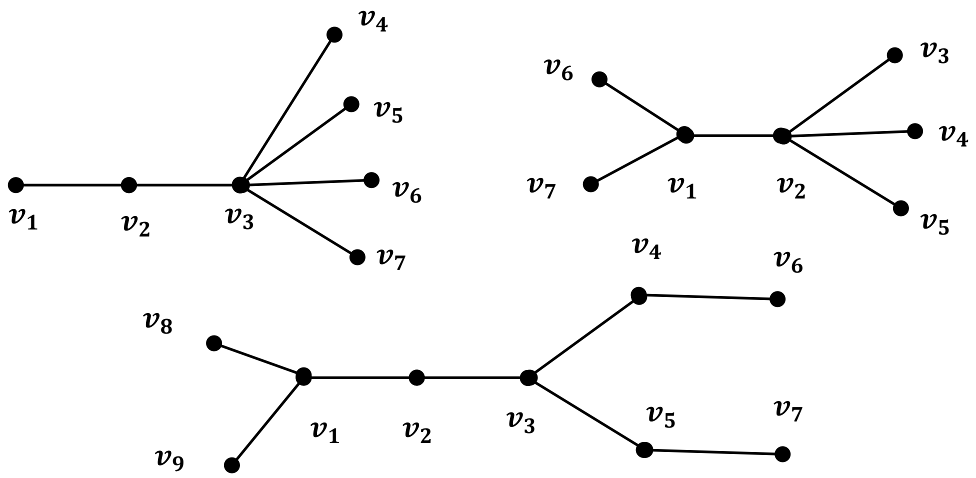

Figure 2.7

The following theorem takes control over the cardinalities of the radial set.

Theorem 3.7 For any two given integers $k \geq 2$, $m \geq 1$, we can find a family of trees with radius k and max radial number mk. In particular, their radial set contains k vertices.

Proof Construct a tree as follows.

$$V(T) = \{v_{ij} \, / \, 1 \leq i \leq m+2 \,\&\, 1 \leq j \leq k\}$$

$E(T) = \{v_{ij}, v_{i(j+1)} \, / \, 1 \leq i \leq m+2 \,\&\, 1 \leq j \leq k-1\} \cup \{v_{11}v_{j1} \, / \, 2 \leq j \leq m+2 \,\&\, i \neq j\}$. T is of order $(m+2)k$. The radius of tree is obviously k. And let $S = \{v_{11}, v_{12}, \dots v_{1k}\}$. Then $B_R(S) = V(T) \backslash S$. Which implies $\partial_R(T) = mk$.

Example 3.8 For $k = 4, m = 3$, the constructed tree T is as follows:

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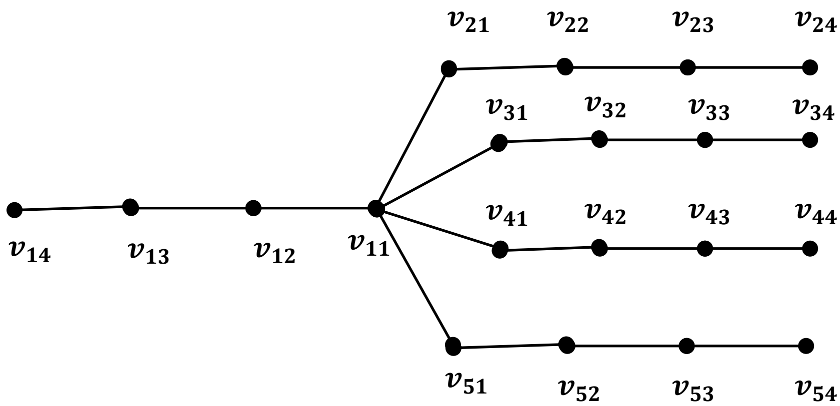

Figure 2.8

Here $S = \{v_{11}, v_{12}, v_{13}, v_{14}\}$ and $B_R(S) = \{v_{ij} \,/\, 2 \leq i \leq 5, 1 \leq j \leq 4\}$. Therefore, $\partial_R(T) = 12$.

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We note that, no tree exists with max-radial number $n-5$. Because, there is no ∂_R-set in T with $\partial_R(X) = n-5$. Let us now characterize trees with radial number $n-6$.

A double broom $B(l, s, t)$ is a tree obtained from a path P_l by attaching s pendent edges to one end vertex of P_l and t pendent edges to the other end vertex of P_l where $l, s, t \geq 2$. For reference, the double broom $B(7,5,4)$ is shown in Figure 2.8.

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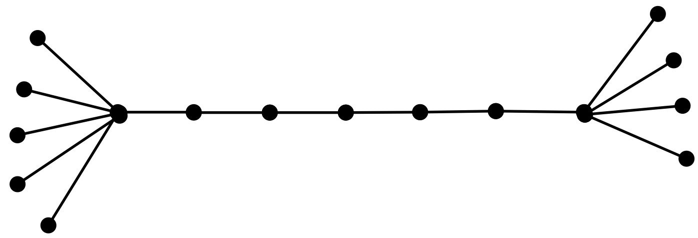

Figure 2.8

Let P_5^* denote the family of trees obtained from P_5 by attaching n_1, n_2, n_3 pendent vertices to at least one pendent vertex and remaining one or two other vertices respectively where $n_1 \geq 0, n_2 \geq 0, n_3 \geq 0$ $(not\ both\ n_1\ \&\ n_3 = 0)$. Some trees in the P_5^*-family are shown in Figure 2.9.

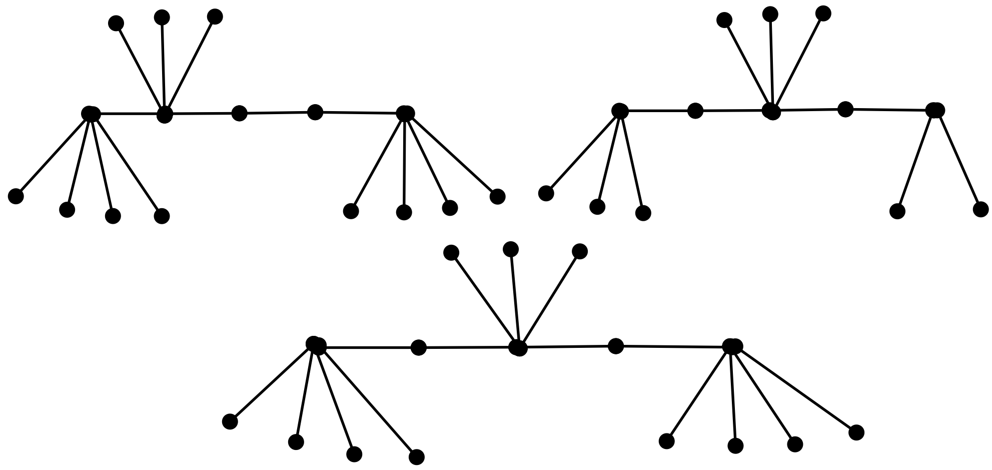

Figure 2.9

We note that, no tree exists with max-radial number $n-5$. Because, there is no ∂_R-set in T such that $\partial_R(X) = n-5$. Let us now characterize trees with radial number $n-6$.

Theorem For any tree T, $\partial_R(T) = n-6$ if and only if $T \cong B(l,s,t)$ $(l = 4, 5)$ or $B_{n,5}$ $(n \geq 6)$ or $T \cong T'$, where $T' \in P_5^*$.

Proof Let T be any with $\partial_R(T) = n-6$. We know that for any subset $S \subseteq V(T), |B_R(S)| \neq n-1$, which is possible only for $T \cong K_{1,m}$. Therefore, $|B_R(S)| = n-2$ or $n-3$ or $n-4$ for any $S \subseteq V(T)$. This force that $|S| = 3$or 2 or 1 respectively. Let P denote a diametral path in T.

If possible, let length of $P \geq 7$. Then no matter if the vertices of S (one or both) are in P or not, $B_R(S)$ contains atmost six vertices of P. Implies, $|B_R(S)| < n-5$. Hence $\partial_R(S) < n-6$ for any $S \subseteq V(T)$, which is a contradiction. Therefore, we conclude that P is of length atmost 6. Also, since $\partial_R(T) = n-6$, $len(P) > 4$. Then P is a length 5 or 6. Thus, any tree of diameter 5 or 6 must be a $B(l,s,t)(l = 4, 5)$ or $B_{n,5}$ $(n \geq 6)$ or a member of P_5^*-family. Hence the theorem.

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Corollary For any tree T with $\partial_R(T) = n - 6$, then $diam\ (T) = 5\ or\ 6$. Converse need not true. For example, any caterpillar T from a path of length 3 or 4. we have $\partial_R(T) = n - 8$ or $n - 9$.

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