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GAUSSIAN SD-DIVISOR LABELING OF GRAPHS

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Abstract

Let $G = (V(G), E(G))$ be a simple, finite and undirected graph of order n. Given a bijection $f : V(G) \rightarrow [\gamma_n]$, we associate two Gaussian integers $S = f(u)+f(v)$ and $D = f(u)-f(v)$ or $f(v)-f(u)$ with every edge uv in $E(G)$. The labeling f induces on edge labeling $f^{r} : E(G) \rightarrow \{0, 1\}$, such that for every edge uv in $E(G)$, $f^{r}(uv) = 1$ if $D \mid S$ and $f^{r}(uv) = 0$ if $D - S$. Let $e_{f^r}(i)$ be the number of edges labeled with $i \in \{0, 1\}$. The labeling f is called an Gaussian SD-divisor labeling if $f^{r}(uv) = 1$ for all $uv \in E(G)$. A graph which admits Gaussian SD-divisor labeling is called Gaussian SD-divisor graph. The labeling f is called an Gaussian SD-divisor cordial labeling if $|e_{f^r}(0) - e_{f^r}(1)| \leq 1$. We say that G is Gaussian SD-divisor cordial if it admits an Gaussian SD-divisor cordial labeling. In this paper, we define SD-divisibility and SD-divisor pair of Gaussian integers and establish some of its properties. Also, we will find out some path and cycle related graphs are Gaussian SD-divisor and some standard graphs such as star, complete, complete bipartite and wheel graphs which are not Gaussian SD-divisor.

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1 Introduction

All graphs in this paper are finite undirected graphs without loops or multiple edges. We follow [5] for definition and information on the Gaussian integers. The Gaussian integers, denoted $\mathbb{Z}[i]$, are the complex numbers of the form $a + bi$, where $a, b \in \mathbb{Z}$ and $i^2 = -1$. The norm of Gaussian integer $a + bi$, denoted by $N(a + bi)$, is given by $a^2 + b^2$. A Gaussian integer is even if it is divisible by $1 + i$ and odd otherwise. A unit in the Gaussian integers is one of $\pm 1, \pm i$. An associate of a Gaussian integer α is $u.\alpha$ where u is a Gaussian unit. A Gaussian integer ρ is prime if its only divisors are $\pm 1, \pm i, \pm \rho$ or $\pm \rho i$. The Gaussian integers α and β are relatively prime if their only common divisors are units in $\mathbb{Z}[i]$.

The Gaussian integers are not totally ordered. So, we use the spiral ordering of the Gaussian integers introduced by Steven Klee in [5].

Definition 1.1. *[5] The spiral ordering of the Gaussian integers is a recursively defined ordering of the Gaussian integers. We denote the nth Gaussian integer in the spiral ordering by γ_n. The ordering is defined beginning with $\gamma_1 = 1$ and continuing as:*

$$\gamma_{n+1} = \begin{cases} \gamma_n + i, & \text{if } Re(\gamma_n) \equiv 1 \pmod 2,\ Re(\gamma_n) > Im(\gamma_n) + 1 \\ \gamma_n - 1, & \text{if } Im(\gamma_n) \equiv 0 \pmod 2,\ Re(\gamma_n) \leq Im(\gamma_n) + 1,\ Re(\gamma_n) > 1 \\ \gamma_n + 1, & \text{if } Im(\gamma_n) \equiv 1 \pmod 2,\ Re(\gamma_n) < Im(\gamma_n) + 1 \\ \gamma_n + i, & \text{if } Im(\gamma_n) \equiv 0 \pmod 2,\ Re(\gamma_n) = 1 \\ \gamma_n - i, & \text{if } Re(\gamma_n) \equiv 0 \pmod 2,\ Re(\gamma_n) \geq Im(\gamma_n) + 1,\ Im(\gamma_n) > 0 \\ \gamma_n + 1, & \text{if } Re(\gamma_n) \equiv 0 \pmod 2,\ Im(\gamma_n) = 0. \end{cases}$$

This is illustrated in Figure 1.

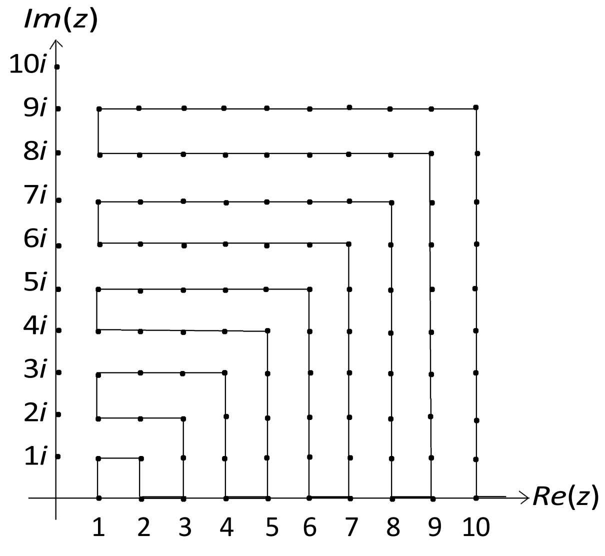

Fig. 1. The spiral ordering of the Gaussian integers.

Under this ordering, the first 10 Gaussian integers are $1, 1 + i, 2 + i, 2, 3, 3 + i, 3 + 2i, 2 + 2i, 1 + 2i, 1 + 3i, \ldots$*, and we write* $[\gamma_n]$ *to denote the set of the*

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first n Gaussian integers in the spiral ordering.

In [5] Steven Klee proved the following properties of Gaussian integers in spiral ordering.

- Let α be a Gaussian integer and u be a unit. Then α and $\alpha+u$ are relatively prime.
- Consecutive Gaussian integers in the spiral ordering are relatively prime.
- Let α be an odd Gaussian integer, let c be a positive integer, and let u be a unit. Then α and $\alpha + u.(1 + i(^{c} are relatively prime.
- Consecutive odd Gaussian integers in the spiral ordering are relatively prime.
- Let α be a Gaussian integer and let ρ be a prime Gaussian integer. Then α and $\alpha + \rho$ are relatively prime if and only if ρ does not divides α.
- Corners of the spiral ordering occur when the spiral turns from north to east or east to north, from south to east or east to south, or from north to west or west to north.
- Branches of the spiral occur when the spiral travels along a straight path going north, south, east, or west. Steps along the real axis and the $Re(z($ = 1 line are not counted as branches.
- $I(a + bi($ to denote the index of $a + bi$ in the spiral ordering.
- Real corners at Gaussian integers on the real axis.
- $Re(z($ = 1 corners at Gaussian integers on the $Re(z($ = 1 line.
- Interior corners at Gaussian integers on the line $Re(z(- Im(z(= 1$.

Gaussian integers at real corners are even when $Re(z($ is even and is odd otherwise. Gaussian integers at $Re(z($ = 1 corners are even when $Im(z($ is odd and are even otherwise.

Lemma 1.2. *[5] Corners in the spiral ordering lie on either the real axis, the* $Re(z($ = 1 *line, or the line* $Im(z(= Re(z(-1$. *Their indices are found as follows:*

$$I(a+bi(= \begin{cases} a^2, \text{ if } b = 0,\ a \equiv 0 \ (mod\ 2(- \text{Even corners on the real axis} \\ (a-1)^2 + 1, \text{ if } b = 0,\ a \equiv 1 \ (mod\ 2(- \text{Odd corners on the real axis} \\ (b+1)^2, \text{ if } a = 1,\ b \equiv 0 \ (mod\ 2(- \text{Odd corners on the } Re(z(= 1 \text{ line} \\ b^2 + 1, \text{ if } a = 1,\ b \equiv 1 \ (mod\ 2(- \text{Even corners on the } Re(z(= 1 \text{ line} \\ b^2 + b + 1, \text{ if } b = a - 1 - \text{Interior corners.} \end{cases}$$

Theorem 1.3. *[5] Let $a + bi$ be a Gaussian integer with $a > 0$ and $b \geq 0$. Then its index in the spiral ordering, $I(a + bi($, is given by the following formula:*

$$I(a+bi(= \begin{cases} (a-1)^2 + 1 + b, \text{ if } a \equiv 1 \pmod{2(}, \ a \geq (b + 1(\text{—Up-oriented branches} \\ (b+1)^2 - a + 1, \text{ if } b \equiv 0 \pmod{2(}, \ a \leq (b + 1(\text{—Left-oriented branches} \\ b^2 + a, \text{ if } b \equiv 1 \pmod{2(}, \ a \leq (b + 1(\text{—Right-oriented branches} \\ a^2 + b, \text{ if } a \equiv 0 \pmod{2(}, \ a \geq (b + 1(\text{—Down-oriented branches} \end{cases}$$

Definition 1.4. *[5] Let G be a graph on n vertices. A Gaussian prime labeling of G is a bijection $l : V(G(\to [\gamma_n]$ such that if $uv \in E(G($, then $l(u($ and $l(v($ are relatively prime; that is, neighbouring vertices have relatively prime labels.*

Definition 1.5. *[8] Let $\alpha, \beta \in \mathbb{Z}[i]$. Then α divides β if there exists $\gamma \in \mathbb{Z}[i]$ such that $\beta = \alpha\gamma$.*

Example 1.6. $2 + i$ *divides* $1 + 3i$. *Since* $1 + 3i = (2 + i((1 + i($.

Theorem 1.7. *[8] A Gaussian integer $\alpha = a + bi$ is divisible by an ordinary integer c if and only if $c \mid a$ and $c \mid b$ in $\mathbb{Z}$.*

Theorem 1.8. *[8] For α, β in $\mathbb{Z}[i]$, if $\beta \mid \alpha$ in $\mathbb{Z}[i]$ then $N(\beta(\mid N(\alpha($ in $\mathbb{Z}$.*

Corollary 1.9. *[8] A Gaussian integer has even norm if and only if it is a multiple of $1 + i$.*

Theorem 1.10. *[8] (Division Theorem). For $\alpha, \beta \in \mathbb{Z}[i]$ with $\beta \neq 0$, there are $\gamma, \rho \in \mathbb{Z}[i]$ such that $\alpha = \beta\gamma + \rho$ and $N(\rho(< N(\beta($. In fact, we can choose ρ so $N(\rho(\leq (1/2(N(\beta($.*

There is a close relationship between graph theory [1, 4] and number theory [2, 8]. Motivated by the various concepts of labeling [3, 6, 7, 9], we introduce a new concepts Gaussian SD-divisor labeling and Gaussian SD-divisor cordial labeling. In this paper, we discuss SD-divisibility and SD-divisor pair of Gaussian integers and establish some of its properties. Also, we will find out some path and cycle related graphs are Gaussian SD-divisor and some standard graphs such as star, complete, complete bipartite and wheel graphs which are not Gaussian SD-divisor.

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2 Gaussian SD-Divisibility and its Properties

First we define SD-divisibility of two Gaussian integers.

Definition 2.1. *Let γ_i and γ_j be the two distinct Gaussian integers, we say that γ_i SD-divides γ_j if $(\gamma_i - \gamma_j(\mid (\gamma_i + \gamma_j($ or $(\gamma_j - \gamma_i(\mid (\gamma_i + \gamma_j($. It is denoted by $\gamma_i \mid_{SD} \gamma_j$. If γ_i does not SD-divide γ_j, then it is denoted by $\gamma_i -_{SD} \gamma_j$.*

Note that $\gamma_3 \mid_{SD} \gamma_7$ and $\gamma_2 -_{SD} \gamma_5$.

Remark 2.2.

1. *Gaussian divisibility and Gaussian SD-divisibility are different concepts. For example*
   - *(a)* $\gamma_4 \mid \gamma_{22}$ *and* $\gamma_4 -_{SD} \gamma_{22}$.
   - *(b)* $\gamma_6 - \gamma_{13}$ *and* $\gamma_6 \mid_{SD} \gamma_{13}$.
2. *By the definition 2.1, Gaussian SD-divisibility is not reflexive.*
3. *By the definition 2.1,* $\gamma_i \mid_{SD} \gamma_j \Rightarrow (\gamma_i - \gamma_j(\mid (\gamma_i + \gamma_j($
   $\Rightarrow (\gamma_j - \gamma_i(\mid (\gamma_j + \gamma_i($
   $\Rightarrow \gamma_j \mid_{SD} \gamma_i$.

   Thus, Gaussian SD-divisibility is symmetric.
4. *Gaussian SD-divisibility is not transitive. For example*
   - *(a)* $\gamma_2 \mid_{SD} \gamma_3$ *and* $\gamma_3 \mid_{SD} \gamma_5$ *but* $\gamma_2 -_{SD} \gamma_5$.
   - *(b)* $\gamma_3 \mid_{SD} \gamma_7$ *and* $\gamma_7 \mid_{SD} \gamma_{12}$ *but* $\gamma_3 -_{SD} \gamma_{12}$.

Therefore, Gaussian SD-divisibility is not an equivalence relation.

Lemma 2.3. *Two consecutive Gaussian integers in the spiral ordering are SD-divisor.*

Proof. Let α and β be the consecutive Gaussian integers.
Let $\gamma = \alpha + \beta$ and $\lambda = \alpha - \beta$ or $\beta - \alpha$. Now to prove $\lambda \mid \gamma$.
Since the difference of two consecutive Gaussian integers in the spiral ordering is one of the unit of $\mathbb{Z}[i]$, λ must be a unit. Thus $\alpha \mid_{SD} \beta$. □

Lemma 2.4. *Two consecutive odd Gaussian integers in the spiral ordering are SD-divisor.*

Proof. Let α and β be the consecutive odd Gaussian integers.
Let $\gamma = \alpha + \beta$ and $\lambda = \alpha - \beta$ or $\beta - \alpha$. Now to prove $\lambda \mid \gamma$.
Since the differences of two consecutive odd Gaussian integers in the spiral ordering are $1 + i$, 2, or one of their associates. Therefore λ are $1 + i$, 2, or one of their associates. Since sum of two odd Gaussian integers is even Gaussian integer, γ must be a even Gaussian integer. Thus $\alpha \mid_{SD} \beta$. □

Lemma 2.5. *Two consecutive even Gaussian integers in the spiral ordering are SD-divisor.*

Proof. Let α and β be the consecutive even Gaussian integers.
Let $\gamma = \alpha + \beta$ and $\lambda = \alpha - \beta$ or $\beta - \alpha$. Now to prove $\lambda \mid \gamma$.
Since the differences of two consecutive even Gaussian integers in the spiral ordering are $1 + i$, 2, or one of their associates. Therefore λ are $1 + i$, 2, or one of their associates. Since sum of two even Gaussian integers is even Gaussian integer, γ must be a even Gaussian integer. Thus $\alpha \mid_{SD} \beta$. □

Result 2.6. *γ_1 SD-divides only to the Gaussian integers γ_2, γ_3, γ_4, γ_5 and γ_9.*

Proof. Let γ_1 and $\gamma_i > \gamma_1$ be the any Gaussian integer.
If $\gamma_1 \mid_{SD} \gamma_i$, then $(\gamma_i - \gamma_1(\mid (\gamma_i + \gamma_1($.
This is possible only if $\gamma_i = \gamma_2, \gamma_3, \gamma_4, \gamma_5$ and γ_9. □

Result 2.7. *γ_2 SD-divides only to the Gaussian integers γ_1, γ_3, γ_4, γ_6, γ_8, γ_9, γ_{10} and γ_{12}.*

Proof. Let γ_2 and γ_i be the any Gaussian integer.
If $\gamma_2 \mid_{SD} \gamma_i$, then $(\gamma_i - \gamma_2(\mid (\gamma_i + \gamma_2($. This is possible only if $\gamma_i = \gamma_1, \gamma_3, \gamma_4, \gamma_6, \gamma_8, \gamma_9, \gamma_{10}$ and γ_{12}. □

Result 2.8. *γ_3 SD-divides only to the Gaussian integers γ_1, γ_2, γ_4, γ_5, γ_6, γ_7, γ_8, γ_9, γ_{10}, γ_{11}, γ_{14}, γ_{15}, γ_{17}, γ_{23}, and γ_{33}.*

Proof. Let γ_3 and γ_i be the any Gaussian integer.
If $\gamma_3 \mid_{SD} \gamma_i$, then $(\gamma_i - \gamma_3(\mid (\gamma_i + \gamma_3($. This is possible only if $\gamma_i = \gamma_1, \gamma_2, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}, \gamma_{14}, \gamma_{15}, \gamma_{17}, \gamma_{23}$, and γ_{33}. □

Result 2.9. *γ_4 SD-divides only to the Gaussian integers γ_1, γ_2, γ_3, γ_5, γ_6, γ_8, γ_{14}, γ_{16}, γ_{24}, and γ_{36}.*

Proof. Let γ_4 and γ_i be the any Gaussian integer.
If $\gamma_4 \mid_{SD} \gamma_i$, then $(\gamma_i - \gamma_4(\mid (\gamma_i + \gamma_4($. This is possible only if $\gamma_i = \gamma_1, \gamma_2, \gamma_3, \gamma_5, \gamma_6, \gamma_8, \gamma_{14}, \gamma_{16}, \gamma_{24}$, and γ_{36}. □

Observation 2.10. *Let γ_k with $k \geq 3$ be any Gaussian integer. Then γ_{k-2}, γ_{k-1}, γ_{k+1}, γ_{k+2} are SD-divisible by γ_k.*

3 Gaussian SD-Divisor Pair

In this section, we define SD-divisor pair of Gaussian integers and establish some results.

Definition 3.1. *Let α and β be the two distinct Gaussian integers. If $\alpha \mid_{SD} \beta$, then we say that $(\alpha, \beta($ is called Gaussian SD-divisor pair.*

Example 3.2. *For $k \geq 1$, $(\gamma_k, \gamma_{k+1}($ is Gaussian SD-divisor pair.*

Note: If $n \geq 1$ is any positive integer, then $(n\gamma_k, n\gamma_{k+1}($ is Gaussian SD-divisor pair.

Result 3.3. *If the pair $(\alpha, \beta($ is Gaussian SD-divisor, then $(n\alpha, n\beta($ is Gaussian SD-divisor pair for $n \geq 1$.*

Proof. Let α and β be the Gaussian SD-divisor pair.
Without loss of generality, we take $\alpha > \beta$.
Then $\alpha \mid_{SD} \beta \Rightarrow (\alpha - \beta(\mid (\alpha + \beta($
$\Rightarrow n(\alpha - \beta(\mid n(\alpha + \beta($ for $n \geq 1$
$\Rightarrow (n\alpha - n\beta(\mid (n\alpha + n\beta($
$\Rightarrow n\alpha \mid_{SD} n\beta$. □

Result 3.4. *The pair $(\gamma_{k-1}, \gamma_{k+1}($ for $k \geq 2$ is Gaussian SD-divisor.*

Proof. Let $\alpha = \gamma_{k-1}$ and $\beta = \gamma_{k+1}$ for $k \geq 2$.
From the lemma 2.3 and lemma 2.4, it follows that $\alpha \mid_{SD} \beta$. □

4 Gaussian SD-Divisor Labeling of Graphs

Definition 4.1. *Let $G = (V, E($ be a simple, finite and undirected graph of order n. Given a bijection $f : V(G(\rightarrow [\gamma_n]$, we associate two Gaussian integers $S = f(u(+ f(v($ and $D = f(u(- f(v($ or $f(v(- f(u($ with every edge uv in $E(G($. The labeling f induces on edge labeling $f^{r} : E(G(\rightarrow \{0, 1\}$, such that for every edge uv in $E(G($, $f^{r}(uv(= 1$ if $D \mid S$ and $f^{r}(uv(= 0$ if D - S. The labeling f is called an Gaussian SD-divisor labeling if $f^{r}(uv(= 1$ for all $uv \in E(G($. A graph which admits Gaussian SD-divisor labeling is called Gaussian SD-divisor graph.*

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Example 4.2. *The following Figure 2. is Gaussian SD-divisor graph.*

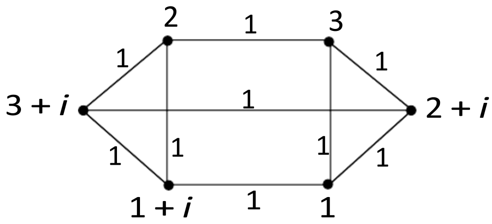

Fig. 2. Gaussian SD-divisor graph

Definition 4.3. *Let $G = (V, E($ be a simple, finite and undirected graph of order n. Given a bijection $f : V(G(\rightarrow [\gamma_n]$, we associate two Gaussian integers $S = f(u(+ f(v($ and $D = f(u(- f(v($ or $f(v(- f(u($ with every edge uv in $E(G($. The labeling f induces on edge labeling $f' : E(G(\rightarrow \{0, 1\}$, such that for every edge uv in $E(G($, $f'(uv(= 1$ if $D \mid S$ and $f'(uv(= 0$ if $D \nmid S$. The labeling f is called an Gaussian SD-divisor cordial labeling if $|e_{f'}(0(- e_{f'}(1(| \leq 1$. We say that G is Gaussian SD-divisor cordial if it admits an Gaussian SD-divisor cordial labeling.*

Example 4.4. *The following Figure 3. is Gaussian SD-divisor cordial graph.*

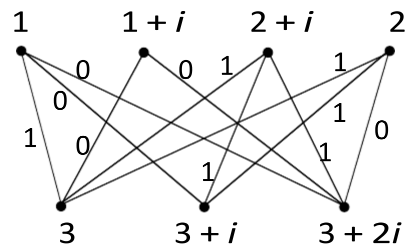

Fig. 3. Gaussian SD-divisor cordial graph

Theorem 4.5. *The path P_n is Gaussian SD-divisor.*

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of path P_n and $E(P_n(= \{v_iv_{i+1} : 1 \leq i \leq n - 1\}$. Therefore, P_n is of order n and size $n - 1$.
Define $f : V(P_n(\rightarrow \{\gamma_1, \gamma_2, \gamma_3, ..., \gamma_n\}$ as follows:

$$f(v_i(= \gamma_i,\ 1 \leq i \leq n.$$

From the above labeling pattern we get, $e_{f'}(1(= n - 1$.
Hence, P_n is Gaussian SD-divisor. □

Theorem 4.6. *The comb $P_n \odot K_1$ is Gaussian SD-divisor.*

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of path P_n. Let $V(P_n \odot K_1(= \{v_i, u_i : 1 \leq i \leq n\}$ and $E(P_n \odot K_1(= \{v_iv_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_iu_i : 1 \leq i \leq n\}$.

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Therefore, $P_n \odot K_1$ is of order $2n$ and size $2n-1$.
Define $f : V(P_n \odot K_1) \to \{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{2n}\}$ is follows:

$$f(v_i) = \gamma_{2i-1},\ 1 \le i \le n;$$
$$f(u_i) = \gamma_{2i},\ 1 \le i \le n.$$

From the above labeling pattern we get, $e_{f^*}(1) = 2n-1$.
Hence, $P_n \odot K_1$ is Gaussian SD-divisor. □

Theorem 4.7. *The ladder L_n is Gaussian SD-divisor.*

Proof. Let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of L_n. Let $V(L_n) = \{v_i, u_i : 1 \le i \le n\}$ and $E(L_n) = \{v_iv_{i+1}, u_iu_{i+1} : 1 \le i \le n-1\} \cup \{v_iu_i : 1 \le i \le n\}$. Therefore, L_n is of order $2n$ and size $3n-2$.
Define $f : V(L_n) \to \{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{2n}\}$ as follows:

$$f(v_i) = \gamma_{2i-1},\ 1 \le i \le n;$$
$$f(u_i) = \gamma_{2i},\ 1 \le i \le n.$$

From the above labeling pattern we get, $e_{f^*}(1) = 3n-2$.
Hence, L_n is Gaussian SD-divisor. □

Theorem 4.8. *The triangular ladder TL_n is Gaussian SD-divisor.*

Proof. Let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of L_n. Let $V(TL_n) = \{v_i, u_i : 1 \le i \le n\}$ and $E(TL_n) = \{v_iv_{i+1}, u_iu_{i+1}, u_iv_{i+1} : 1 \le i \le n-1\} \cup \{v_iu_i : 1 \le i \le n\}$. Therefore, TL_n is of order $2n$ and size $4n-3$.
Define $f : V(TL_n) \to \{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{2n}\}$ as follows:

$$f(v_i) = \gamma_{2i-1},\ 1 \le i \le n;$$
$$f(u_i) = \gamma_{2i},\ 1 \le i \le n.$$

From the above labeling pattern we get, $e_{f^*}(1) = 4n-3$.
Hence, TL_n is Gaussian SD-divisor. □

Theorem 4.9. *The graph P_n^2 is Gaussian SD-divisor.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n. Let $V(P_n^2) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n^2) = \{v_iv_{i+1} : 1 \le i \le n-1\} \cup \{v_iv_{i+2} : 1 \le i \le n-2\}$. Therefore, P_n^2 is of order n and size $2n-3$. We define $f : V(P_n^2) \to \{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n\}$ by $f(v_i) = \gamma_i$ for $1 \le i \le n$. We observe that $e_{f^*}(1) = 2n-3$. Hence, P_n^2 is Gaussian SD-divisor. □

Remark 4.10. *Note that $T(P_n) = P_{2n-1}^2$. Therefore, $T(P_n)$ is also Gaussian SD-divisor.*

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Theorem 4.11. *The triangular snake T_n is Gaussian SD-divisor.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of path P_n. Let $V(T_n) = V(P_n) \bigcup \{u_i : 1 \leq i \leq n-1\}$ and $E(T_n) = E(P_n) \bigcup \{v_i u_i, v_{i+1} u_i : 1 \leq i \leq n-1\}$. Therefore, T_n is of order $2n-1$ and size $3n-3$. Define $f : V(T_n) \to \{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{2n-1}\}$ as follows:

$$f(v_i) = \gamma_{2i-1}, \ 1 \leq i \leq n;$$
$$f(u_i) = \gamma_{2i}, \ 1 \leq i \leq n-1.$$

From the above labeling pattern we get, $e_{f^*}(1) = 3n-3$.
Hence, T_n is Gaussian SD-divisor. □

Theorem 4.12. *The alternate triangular snake $A(T_n)$ is Gaussian SD-divisor.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of path P_n. Let $V(A(T_n)) = V(P_n) \bigcup \{u_i : 1 \leq i \leq \lceil \frac{n}{2} \rceil\}$ and $E(A(T_n)) = E(P_n) \bigcup \{v_{2i-1} u_i, u_i v_{2i} : 1 \leq i \leq \lceil \frac{n}{2} \rceil\}$. Also,

$$|V(A(T_n))| = \begin{cases} \frac{3n-1}{2} & \text{if } n \text{ is odd} \\ \frac{3n}{2} & \text{if } n \text{ is even} \end{cases}$$

$$|E(A(T_n))| = \begin{cases} 2n-2 & \text{if } n \text{ is odd} \\ 2n-1 & \text{if } n \text{ is even} \end{cases}$$

Define $f : V(A(T_n)) \to \{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{|V(A(T_n))|}\}$ as follows:

$$f(v_{2i-1}) = \gamma_{3i-2}, \ 1 \leq i \leq \lceil \tfrac{n}{2} \rceil;$$
$$f(v_{2i}) = \gamma_{3i}, \ 1 \leq i \leq \lceil \tfrac{n}{2} \rceil;$$
$$f(u_i) = \gamma_{3i-1}, \ 1 \leq i \leq \lceil \tfrac{n}{2} \rceil.$$

From the above labeling pattern we get, $e_{f^*}(1) = |E(A(T_n))|$.
Hence, $A(T_n)$ is Gaussian SD-divisor. □

Theorem 4.13. *The alternate quadrilateral snake $A(Q_n)$ is Gaussian SD-divisor.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of path P_n. Let, $V(A(Q_n)) = V(P_n) \bigcup \{u_i, w_i : 1 \leq i \leq \lceil \frac{n}{2} \rceil\}$ and $E(A(Q_n)) = E(P_n) \bigcup \{v_{2i-1} u_i, u_i w_i, w_i v_{2i} : 1 \leq i \leq \lceil \frac{n}{2} \rceil\}$. Also,

$$|V(A(Q_n))| = \begin{cases} 2n-1 & \text{if } n \text{ is odd} \\ 2n & \text{if } n \text{ is even} \end{cases}$$

$$|E(A(Q_n))| = \begin{cases} \frac{5(n-1)}{2} & \text{if } n \text{ is odd} \\ \frac{5n}{2} - 1 & \text{if } n \text{ is even} \end{cases}$$

Define $f : V(A(Q_n)) \to \{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{|V(A(Q_n))|}\}$ as follows:

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$$f(u_i) = \gamma_{4i-2},\ 1 \le i \le \lceil \tfrac{n}{2} \rceil;$$
$$f(w_i) = \gamma_{4i},\ 1 \le i \le \lceil \tfrac{n}{2} \rceil.$$

From the above labeling pattern we get, $e_{f^*}(1) = |E(A(Q_n))|$.
Hence, $A(Q_n)$ is Gaussian SD-divisor. □

Theorem 4.14. *The cycle C_n is Gaussian SD-divisor.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle C_n and $E(C_n) = \{v_iv_{i+1} : 1 \le i \le n-1\} \cup \{v_nv_1\}$. Therefore, C_n is of order n and size n. Define $f : V(C_n) \to \{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n\}$ as follows:

$$f(v_i) = \gamma_{2i-1},\ 1 \le i \le \lceil \tfrac{n}{2} \rceil;$$
$$f(v_{n+1-i}) = \gamma_{2i},\ 1 \le i \le \lceil \tfrac{n}{2} \rceil.$$

From the above labeling pattern we get, $e_{f^*}(1) = n$.
Hence, C_n is Gaussian SD-divisor. □

Theorem 4.15. *For $n \ge 4$, every cycle C_n with zigzag chords is Gaussian SD-divisor.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n and G be a graph obtained from the cycle $C_n (n \ge 4)$ by adding the chords $v_1v_{n-1}, v_{n-1}v_2, v_2v_{n-2}, \ldots, v_\alpha v_\beta$, where $\alpha = \frac{n-2}{2}$ and $\beta = \frac{n+2}{2}$ if n is even, $\alpha = \frac{n+3}{2}$ and $\beta = \frac{n-1}{2}$ if n is odd. Then $V(G) = \{v_i : 1 \le i \le n\}$ and $E(G) = E(C_n) \cup \{v_1v_{n-1}, v_{n-1}v_2, v_2v_{n-2}, \ldots, v_\alpha v_\beta\}$.

Therefore, G is of order n and size $2n-3$. Define $f : V(G) \to \{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n\}$ as follows:

$$f(v_i) = \gamma_{2i},\ 1 \le i \le \lceil \tfrac{n}{2} \rceil;$$
$$f(v_{n+1-i}) = \gamma_{2i-1},\ 1 \le i \le \lceil \tfrac{n}{2} \rceil.$$

From the above labeling pattern we get, $e_{f^*}(1) = 2n-3$.
Hence, G is Gaussian SD-divisor. □

Theorem 4.16. *The (m, n)-kite graph is Gaussian SD-divisor.*

Proof. Let $v_1, v_2, \ldots, v_m$ be the vertices of path P_m and $u_1, u_2, \ldots, u_n$ be the vertices of cycle C_n. Let $G = (m, n)$ be a kite graph obtained from the vertex v_1 of the path P_m is attached to the vertex u_1 of the cycle C_n, i.e., $u_1 = v_1$. Then $V(G) = V(C_n) \cup \{v_i : 1 \le i \le m-1\}$ and $E(G) = E(P_m) \cup E(C_n)$. Therefore, G is of order $m+n-1$ and size $m+n-1$. Define $f : V(G) \to \{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{m+n-1}\}$ as follows:
Case(1): m is odd and n is odd.

$$f(v_{m+1-i}) = \gamma_i,\ 1 \le i \le m-1;$$
$$f(u_{i-\lceil \frac{m}{2} \rceil+1}) = \gamma_{2i+1},\ \lceil \tfrac{m}{2} \rceil \le i \le \tfrac{m+n}{2} - 1;$$

$$f(u_{n+\lceil \frac{m}{2}\rceil - i}) = \gamma_{2i+2}, \ \lceil \tfrac{m}{2} \rceil \leq i \leq \tfrac{m+n}{2} - 2.$$

Case(2): m is odd and n is even.

$$f(v_{m+1-i}) = \gamma_i, \ 1 \leq i \leq m-1;$$
$$f(u_{i-\lceil \frac{m}{2}\rceil + 1}) = \gamma_{2i+1}, \ \lceil \tfrac{m}{2} \rceil \leq i \leq \lceil \tfrac{m+n}{2} \rceil - 1;$$
$$f(u_{n+\lceil \frac{m}{2}\rceil - i}) = \gamma_{2i+2}, \ \lceil \tfrac{m}{2} \rceil \leq i \leq \lceil \tfrac{m+n}{2} \rceil - 1.$$

Case(3): m is even and n is odd.

$$f(v_{m+1-i}) = \gamma_i, \ 1 \leq i \leq m-1;$$
$$f(u_{i-\frac{m}{2}+1}) = \gamma_{2i}, \ \tfrac{m}{2} \leq i \leq \lceil \tfrac{m+n}{2} \rceil;$$
$$f(u_{n+\frac{m}{2}-i}) = \gamma_{2i+1}, \ \tfrac{m}{2} \leq i \leq \lceil \tfrac{m+n}{2} \rceil - 1.$$

Case(4): m is even and n is even.

$$f(v_{m+1-i}) = \gamma_i, \ 1 \leq i \leq m-1;$$
$$f(u_{i-\frac{m}{2}+1}) = \gamma_{2i}, \ \tfrac{m}{2} \leq i \leq \tfrac{m+n}{2} - 1;$$
$$f(u_{n+\frac{m}{2}-i}) = \gamma_{2i+1}, \ \tfrac{m}{2} \leq i \leq \tfrac{m+n}{2} - 1.$$

From the above labeling pattern we get, $e_{f^*}(1) = m + n - 1$.
Hence, G is Gaussian SD-divisor. □

5 Graphs which are not Gaussian SD-Divisor

In this section, we prove some standard graphs such as star, complete, complete bipartite and wheel graphs which are not Gaussian SD-divisor.

Theorem 5.1. *If $\delta(G) \geq 6$, then G is not Gaussian SD-divisor.*

Proof. Suppose G is Gaussian SD-divisor.

Let v be the vertex of degree $\delta(G) \geq 6$, which is labeled with γ_1.

Then, any one of the δ adjacent vertices of v must have the labels other than γ_2, γ_3, γ_4, γ_5 and γ_9, say u.

From the result 2.6, it follows that γ_1 does not Gaussian SD-divide the label of u.

This is contradiction to G is Gaussian SD-divisor. □

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Next, we will investigate whether the star graph $K_{1,n}$ is Gaussian SD-divisor or not. Clearly $K_{1,1}$ and $K_{1,2}$ are Gaussian SD-divisor. In the following Figure 4. labeling pattern shows, $K_{1,n}(n \leq 10($ are Gaussian SD-divisor.

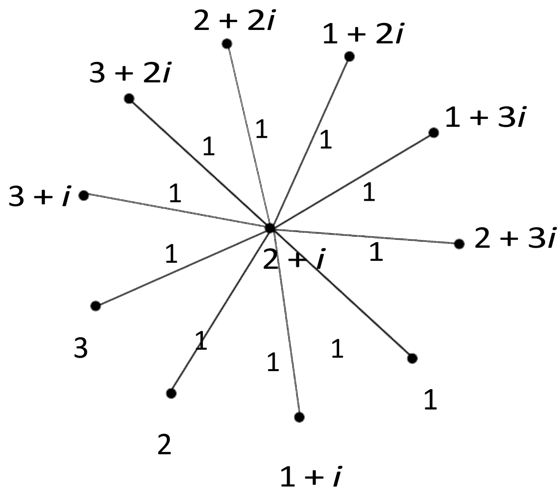

Fig. 4. Star graph $K_{1,10}$

Next, we will prove $K_{1,n}$ is not Gaussian SD-divisor for $n \geq 11$.

Theorem 5.2. *For $n \geq 11$, the star graph $K_{1,n}$ is not Gaussian SD-divisor.*

Proof. Consider the star graph $K_{1,n}$ with the vertex set $\{v, v_1, v_2, v_3, ..., v_n\}$, $n \geq 11$.

Let v be the central vertex of $K_{1,n}(n \geq 11($.

If we label γ_1 to v and other Gaussian integers to the end vertices of $K_{1,n}$, then it follows from the result 2.6, γ_1 does not Gaussian SD-divide γ_6, γ_7, γ_8, γ_{10}, γ_{11},...,γ_n.

If we label γ_2 to v and other Gaussian integers to the end vertices of $K_{1,n}$, then it follows from the result 2.7, γ_2 does not Gaussian SD-divide γ_5, γ_7, γ_{11}, γ_{13}, γ_{14},...,γ_n.

If we label γ_3 to v and other Gaussian integers to the end vertices of $K_{1,n}$, then it follows from the result 2.8, γ_3 does not Gaussian SD-divide γ_{12}, γ_{13}, γ_{16}, γ_{18},...,γ_n.

If we label γ_4 to v and other Gaussian integers to the end vertices of $K_{1,n}$, then it follows from the result 2.9, γ_4 does not Gaussian SD-divide γ_7, γ_9, γ_{10}, γ_{11}, γ_{12},...,γ_n.

If we label γ_5 to v and other Gaussian integers to the end vertices of $K_{1,n}$, then it follows from the result 2.7 and remark 2.2, γ_5 does not Gaussian SD-divide γ_2.

Suppose, we label $\gamma_n(n \geq 6($ to v except γ_9. Since any one of the end vertex has the label γ_1, then it follows from the result 2.6, γ_1 does not Gaussian SD-divide to the label of v.

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If we label γ_9 to v and other Gaussian integers to the end vertices of $K_{1,n}$, then it follows from the result 2.8 and remark 2.2, γ_9 does not Gaussian SD-divide γ_4.

Thus, $K_{1,n}$ is not Gaussian SD-divisor for $n \geq 11$. □

Next, we discuss the Gaussian SD-divisibility of complete graphs. Clearly, the complete graphs K_1, K_2, K_3 and K_4 are Gaussian SD-divisor. Now, we will prove K_n is not Gaussian SD-divisor for $n \geq 5$.

Corollary 5.3. *For $n \geq 5$, the complete graph K_n is not Gaussian SD-divisor.*

Proof. If $n = 5$ and $n = 6$ in the complete graph K_n, label vertex v_n with γ_n. Then it follows from the result 2.7, γ_2 does not Gaussian SD-divide γ_5.

Since $\delta(K_n(\geq 6$ for $n \geq 7$, the result follows from theorem 5.1. □

Next, we discuss the Gaussian SD-divisibility of complete bipartite graphs. Clearly, $K_{1,1}$, $K_{1,2}$ and $K_{2,2}$ are Gaussian SD-divisor. The following Figure 5. labeling patterns show that $K_{2,3}$, $K_{2,4}$ and $K_{3,3}$ are Gaussian SD-divisor.

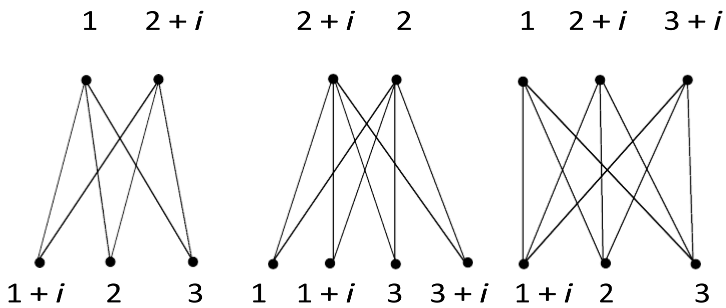

Fig. 5. $K_{2,3}$ $K_{2,4}$ $K_{3,3}$

Similarly, $K_{3,2}$ and $K_{4,2}$ are Gaussian SD-divisor.

Corollary 5.4. *For $m, n \geq 2$ and $m + n \geq 7$, the complete bipartite graph $K_{m,n}$ is not Gaussian SD-divisor.*

Corollary 5.5. *The wheel graph $W_{n+1}(n \geq 11($ is not Gaussian SD-divisor.*

6 Conclusion

In this paper, the concepts of Gaussian SD-divisibility and Gaussian SD-divisor labeling have been introduced. Then, Gaussian SD-divisor labeling of graphs has been obtained. In our further research, we will investigate

1. Gaussian SD-divisor labeling and Gaussian SD-divisor cordial labeling of some special graphs.
2. Other inherent properties of Gaussian SD-divisibility.
3. We will search the suitable application for Gaussian SD-divisor labeling of graphs.

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