

BINARY TOPOLOGY BASED ON SOME NEW SETS

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Abstract

In this chapter, we introduce and some new sets called binary $\mathcal{A}$ -open sets, binary $\mathcal{B}$ -sets, binary $\mathcal{A}_c$ -sets, binary $\mathcal{C}$ -closed sets, binary α^* -sets and binary $\mathcal{B}_{\alpha^*}$ -sets, which are simple forms of binary open sets in binary topological spaces. Also we characterize these sets and study their related properties with binary open sets. The independence of such as binary pre open sets and binary $\mathcal{A}$ -closed sets, binary β -closed sets and binary $\mathcal{A}$ -closed sets, binary α -open sets and binary $\mathcal{B}_{\alpha^*}$ -sets are established. Also decomposition of binary α -open subset into a binary $\mathcal{A}$ -open subset and binary pre open sets is arrived. Finally, the notion of a binary kernel is introduced, and its properties are investigated.

1 Introduction and Preliminaries

Nithyanantha Jothi, et al.[3] introduced topology between two sets and also studied some of their properties. Topology between two sets is the binary structure from X to Y which is defined to be the ordered pairs (A, B) where $A \subseteq X$ and $B \subseteq Y$. In this paper, we introduce and some new sets called binary $\mathcal{A}$ -open sets, binary $\mathcal{B}$ -sets, binary $\mathcal{A}_c$ -sets, binary $\mathcal{C}$ -closed sets, binary α^* -sets and binary $\mathcal{B}_{\alpha^*}$ -sets, which are simple forms of binary open sets in binary

topological spaces. Also we characterize these sets and study their related properties with binary open sets.

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Key words: binary $\mathcal{A}$ -open sets, binary $\mathcal{B}$ -sets, binary $\mathcal{A}_C$ -sets, binary $\mathcal{C}$ -closed sets, binary α^* -sets and binary $\mathcal{B}_{\alpha^*}$ -sets.

Also decomposition of binary α -open subset into a binary $\mathcal{A}$ -open subset and binary pre open sets is arrived. More over, we obtain two different decompositions of a binary $\mathcal{A}_C$ -closed subset and binary $\mathcal{C}$ -closed subset (X, Y) in the binary topological space $(X, Y, \mathcal{M})$.

Let X and Y be any two nonempty sets. A binary topology [3] from X to Y is a binary structure $\mathcal{M} \subseteq \mathbb{P}(X) \times \mathbb{P}(Y)$ that satisfies the axioms namely

1. (ϕ, ϕ) and $(X, Y) \in \mathcal{M}$,
2. $(A_1 \cap A_2, B_1 \cap B_2) \in \mathcal{M}$ whenever $(A_1, B_1) \in \mathcal{M}$ and $(A_2, B_2) \in \mathcal{M}$, and
3. If $\{(A_\alpha, B_\alpha) : \alpha \in \delta\}$ is a family of members of $\mathcal{M}$, then $(\bigcup_{\alpha \in \delta} A_\alpha, \bigcup_{\alpha \in \delta} B_\alpha) \in \mathcal{M}$.

If $\mathcal{M}$ is a binary topology from X to Y then the triplet $(X, Y, \mathcal{M})$ is called a binary topological space and the members of $\mathcal{M}$ are called the binary open subsets of the binary topological space $(X, Y, \mathcal{M})$. The elements of $X \times Y$ are called the binary points of the binary topological space $(X, Y, \mathcal{M})$. If $Y = X$ then $\mathcal{M}$ is called a binary topology on X in which case we write $(X, \mathcal{M})$ as a binary topological space.

Definition 1.1 [3] Let X and Y be any two nonempty sets and let (A, B) and $(C, D) \in \mathbb{P}(X) \times \mathbb{P}(Y)$. We say that $(A, B) \subseteq (C, D)$ if $A \subseteq C$ and $B \subseteq D$.

Definition 1.2 [3] Let $(X, Y, \mathcal{M})$ be a binary topological space and $A \subseteq X$, $B \subseteq Y$. Then (A, B) is called binary closed in $(X, Y, \mathcal{M})$ if $(X \setminus A, Y \setminus B) \in \mathcal{M}$.

Proposition 1.3 [3] Let $(X, Y, \mathcal{M})$ be a binary topological space and $(A, B) \subseteq (X, Y)$.

Let $(A, B)^{1*} = \bigcap \{A_\alpha : (A_\alpha, B_\alpha) \text{ is binary closed and } (A, B) \subseteq (A_\alpha, B_\alpha)\}$ and $(A, B)^{2*} = \bigcap \{B_\alpha : (A_\alpha, B_\alpha) \text{ is binary closed and } (A, B) \subseteq (A_\alpha, B_\alpha)\}$. Then $((A, B)^{1*}, (A, B)^{2*})$ is binary closed and $(A, B) \subseteq ((A, B)^{1*}, (A, B)^{2*})$.

Proposition 1.4 [3] Let $(X, Y, \mathcal{M})$ be a binary topological space and $(A, B) \subseteq (X, Y)$. Let $(A, B)^{1*} = \bigcup \{A_\alpha : (A_\alpha, B_\alpha) \text{ is binary open and } (A_\alpha, B_\alpha) \subseteq (A, B)\}$ and $(A, B)^{2*} = \bigcup \{B_\alpha : (A_\alpha, B_\alpha) \text{ is binary open and } (A_\alpha, B_\alpha) \subseteq (A, B)\}$.

Definition 1.5 [3] The ordered pair $((A, B)^{1*}, (A, B)^{2*})$ is called the binary closure of (A, B) , denoted by $b-cl(A, B)$ in the binary space $(X, Y, \mathcal{M})$ where $(A, B) \subseteq (X, Y)$.

Definition 1.6 [3] The ordered pair $((A, B)^{1*}, (A, B)^{2*})$ defined in proposition 1.4 is called

the binary interior of (A, B) , denoted by $b\text{-int}(A, B)$. Here $((A, B)^{1*}, (A, B)^{2*})$ is binary open and $((A, B)^{1*}, (A, B)^{2*}) \subseteq (A, B)$.

Definition 1.7 [3] Let $(X, Y, \mathcal{M})$ be a binary topological space and let $(x, y) \subseteq (X, Y)$. The binary open set (A, B) is said to be a binary neighbourhood of (x, y) if $x \in A$ and $y \in B$.

Proposition 1.8 [3] Let $(A, B) \subseteq (C, D) \subseteq (X, Y)$ and $(X, Y, \mathcal{M})$ be a binary topological space. Then, the following statements hold:

1. $b\text{-int}(A, B) \subseteq (A, B)$.
2. If (A, B) is binary open, then $b\text{-int}(A, B) = (A, B)$.
3. $b\text{-int}(A, B) \subseteq b\text{-int}(C, D)$.
4. $b\text{-int}(b\text{-int}(A, B)) = b\text{-int}(A, B)$.
5. $(A, B) \subseteq b\text{-cl}(A, B)$.
6. If (A, B) is binary closed, then $b\text{-cl}(A, B) = (A, B)$.
7. $b\text{-cl}(A, B) \subseteq b\text{-cl}(C, D)$.
8. $b\text{-cl}(b\text{-cl}(A, B)) = b\text{-cl}(A, B)$.

Definition 1.9 A subset (A, B) of a binary topological space $(X, Y, \mathcal{M})$ is called

1. a binary α -open set [1] if $(A, B) \subseteq b\text{-int}(b\text{-cl}(b\text{-int}(A, B)))$.
2. a binary semi open set [4] if $(A, B) \subseteq b\text{-cl}(b\text{-int}(A, B))$.
3. a binary pre open set [2] if $(A, B) \subseteq b\text{-int}(b\text{-cl}(A, B))$.
4. a binary regular open set [5] if $(A, B) = b\text{-int}(b\text{-cl}(A, B))$.

The complements of the above mentioned sets are called their respective closed sets.

2 Binary $\mathcal{A}$ -open sets and binary $\mathcal{B}$ -sets

In this section, we will study the concepts of binary $\mathcal{A}$ -open sets and binary $\mathcal{B}$ -sets. We will explore the properties of these sets, their interactions, and how they can be utilized in various mathematical contexts.

Definition 2.1 A subset (O, P) of a binary topological space $(X, Y, \mathcal{M})$ is said binary $\mathcal{A}$ -open if $b\text{-int}(b\text{-cl}(O, P)) \subseteq b\text{-cl}(b\text{-int}(O, P))$. $(O, P)^c = (X, Y) - (O, P)$ is binary $\mathcal{A}$ -closed in $(X, Y, \mathcal{M})$.

Definition 2.2 A subset (O, P) of a binary topological space $(X, Y, \mathcal{M})$ is said a binary $\mathcal{B}$ -set if $b\text{-int}(O, P) = b\text{-int}(b\text{-cl}(O, P))$.

Proposition 2.3 In a binary topological space $(X, Y, \mathcal{M})$, for a subset (O, P) of (X, Y) , the following results hold:

1. (O, P) is a binary α -open set $\Rightarrow (O, P)$ is binary $\mathcal{A}$ -open.

2. (O, P) is a binary $\mathcal{B}$ -set $\Rightarrow (O, P)$ is binary $\mathcal{A}$ -open.

Proof. 1. (O, P) is a binary α -open set in $(X, Y, \mathcal{M})$. $(O, P) \subseteq b\text{-int}(b\text{-cl}(b\text{-int}(O, P))) \subseteq b\text{-cl}(b\text{-int}(O, P))$. Then $b\text{-cl}(O, P) \subseteq b\text{-cl}(b\text{-int}(O, P))$ and $b\text{-int}(b\text{-cl}(O, P)) \subseteq b\text{-cl}(O, P) \subseteq b\text{-cl}(b\text{-int}(O, P))$. This proves that (O, P) is binary $\mathcal{A}$ -open in $(X, Y, \mathcal{M})$.

2. (O, P) is a binary $\mathcal{B}$ -set in $(X, Y, \mathcal{M})$, we have $b\text{-int}(b\text{-cl}(O, P)) = b\text{-int}(O, P) \subseteq (O, P)$. Then $b\text{-int}(b\text{-cl}(O, P)) \subseteq b\text{-int}(O, P) \subseteq b\text{-cl}(b\text{-int}(O, P))$. This proves that (O, P) is binary $\mathcal{B}$ -open in $(X, Y, \mathcal{M})$.

Remark 2.4 The converse of each implication in Proposition 2.3 are need not be true as shown in the following example.

Example 2.5 Let $A = \{a, b\}$, $B = \{1, 2\}$ and $\mathcal{M} = \{(\phi, \phi), (\phi, \{2\}), (\{a\}, \{1\}), (\{a\}, Y), (X, \{1\}), (X, Y)\}$. Then binary $\mathcal{A}$ -open sets are $(\phi, \phi), (\phi, \{2\}), (\{a\}, \{1\}), (\{a\}, Y), (\{b\}, \phi), (\{b\}, \{2\}), (X, \{1\}), (X, Y)$ and binary α -open sets are $(\phi, \phi), (\phi, \{2\}), (\{a\}, \{1\}), (\{a\}, Y), (X, \{1\}), (X, Y)$ and binary $\mathcal{B}$ -sets are $(\phi, \phi), (\phi, \{2\}), (\{b\}, \phi), (\{b\}, \{2\}), (X, \{1\}), (X, Y)$.

1. then the subset $(\{b\}, \{2\})$ is a binary $\mathcal{A}$ -open set but not binary α -open.
2. then the subset $(\{a\}, \{1\})$ is a binary $\mathcal{A}$ -open set but not binary $\mathcal{B}$ -set.

Theorem 2.6 For a subset (O, P) of a binary topological space $(X, Y, \mathcal{M})$, the following statements are equivalent.

1. (O, P) is a binary semi open set.
2. $b\text{-cl}(O, P) = b\text{-cl}(b\text{-int}(O, P))$.

Proof. (1) $\Rightarrow$ (2). Suppose that (O, P) is a binary semi open set in $(X, Y, \mathcal{M})$, then $(O, P) \subseteq b\text{-cl}(b\text{-int}(O, P))$ and $b\text{-cl}(O, P) \subseteq b\text{-cl}(b\text{-int}(O, P))$. But $b\text{-cl}(b\text{-int}(O, P)) \subseteq b\text{-cl}(O, P)$. Thus $b\text{-cl}(O, P) = b\text{-cl}(b\text{-int}(O, P))$.

(2) $\Rightarrow$ (1). let the condition hold. We have $(O, P) \subseteq b\text{-cl}(H)$ and $b\text{-cl}(H) = b\text{-cl}(b\text{-int}(O, P))$. Therefore (O, P) is a binary semi open set in $(X, Y, \mathcal{M})$.

Theorem 2.7 For a subset (O, P) of a binary topological space $(X, Y, \mathcal{M})$, the following statements are equivalent.

1. (O, P) is a binary open set $\Rightarrow (O, P)$ is binary semi open.
2. (O, P) is a binary α -open set $\Rightarrow (O, P)$ is a binary semi open.

Proof. 1. (O, P) is a binary open set $\Rightarrow (O, P) = b\text{-int}(O, P) \subseteq b\text{-cl}(b\text{-int}(O, P))$. This proves that (O, P) is binary semi open in $(X, Y, \mathcal{M})$.

2. (O, P) is a binary α -open set $\Rightarrow (O, P) \subseteq b\text{-int}(b\text{-cl}(b\text{-int}(O, P))) \subseteq b\text{-cl}(b\text{-int}(O, P))$. This proves that (O, P) is semi open in $(X, Y, \mathcal{M})$.

Theorem 2.8 Let (O, P) be a subset of a binary topological space $(X, Y, \mathcal{M})$. Then (O, P) is binary β -closed if and only if $b\text{-int}(b\text{-cl}(b\text{-int}(O, P))) = b\text{-int}(O, P)$.

Proof. Since (O, P) is a binary β -closed set in $(X, Y, \mathcal{M})$, $b\text{-int}(b\text{-cl}(b\text{-int}(O, P))) \subseteq (O, P)$ and then $b\text{-int}(b\text{-cl}(b\text{-int}(O, P))) \subseteq b\text{-int}(O, P)$. But $b\text{-int}(O, P) \subseteq b\text{-int}(b\text{-cl}(b\text{-int}(O, P)))$. We have $b\text{-int}(H) = b\text{-int}(b\text{-cl}(b\text{-int}(O, P)))$.

Conversely, let the condition hold. We have $b\text{-int}(b\text{-cl}(b\text{-int}(O, P))) = b\text{-int}(O, P) \subseteq (O, P)$. Therefore (O, P) is binary β -closed in $(X, Y, \mathcal{M})$.

Theorem 2.9 In a binary topological space $(X, Y, \mathcal{M})$, for a subset (O, P) of (X, Y) , the following results are equivalent:

1. (O, P) is binary semi closed.
2. (O, P) is a binary β -closed set and binary $\mathcal{A}$ -closed.

Proof. (1) $\Rightarrow$ (2): Since (O, P) is a binary semi closed set in $(X, Y, \mathcal{M}) \Rightarrow (O, P)$ is binary β -closed in $(X, Y, \mathcal{M})$. Now, $b\text{-int}(b\text{-cl}(O, P)) \subseteq (O, P)$ and $b\text{-int}(b\text{-cl}(O, P)) \subseteq b\text{-int}(O, P)$. It gives that $b\text{-cl}(b\text{-int}(b\text{-cl}(O, P))) \subseteq b\text{-cl}(b\text{-int}(O, P))$. Thus $b\text{-int}(b\text{-cl}(O, P)) \subseteq b\text{-cl}(b\text{-int}(b\text{-cl}(O, P))) \subseteq b\text{-cl}(b\text{-int}(O, P))$ and hence (O, P) is binary $\mathcal{A}$ -closed in $(X, Y, \mathcal{M})$.

(2) $\Rightarrow$ (1): Since (O, P) is a binary $\mathcal{A}$ -closed set in $(X, Y, \mathcal{M})$, $b\text{-int}(b\text{-cl}(H)) \subseteq b\text{-cl}(b\text{-int}(O, P))$ and $b\text{-int}(b\text{-cl}(O, P)) \subseteq b\text{-int}(b\text{-cl}(b\text{-int}(O, P)))$. Since (O, P) is binary β -closed, $b\text{-int}(b\text{-cl}(b\text{-int}(O, P))) \subseteq (O, P)$. Then $b\text{-int}(b\text{-cl}(O, P)) \subseteq (O, P)$ and hence (O, P) is binary semi closed in $(X, Y, \mathcal{M})$.

Remark 2.10 In a binary topological space $(X, Y, \mathcal{M})$, the notions of binary β -closed sets and the notions of binary $\mathcal{A}$ -closed sets are independent of each other.

Example 2.11 Let $X = \{a, b\}$, $B = \{1, 2\}$ and $\mathcal{M} = \{(\phi, \phi), (\phi, \{1\}), (\{a\}, \phi), (\{a\}, \{1\}), (\{a\}, \{2\}), (\{a\}, Y), (\{b\}, \{1\}), (X, \{1\}), (X, Y)\}$. Then binary $\mathcal{A}$ -closed sets are $(\phi, \phi), (\phi, \{1\}), (\phi, \{2\}), (\phi, Y), (\{a\}, \phi), (\{a\}, \{1\}), (\{a\}, \{2\}), (\{a\}, Y), (\{b\}, \phi), (\{b\}, \{1\}), (X, \phi), (X, \{1\}), (X, \{2\}), (X, Y)$ and binary β -closed sets are $(\phi, \phi), (\phi, \{2\}), (\{a\}, \{2\}), (\{b\}, \phi), (\{b\}, \{1\}), (\{b\}, \{2\}), (\{b\}, Y), (X, \{2\}), (X, Y)$. Then

1. the subset $(\{b\}, \{2\})$ is a binary β -closed set but not binary $\mathcal{A}$ -closed.
2. the subset $(\{a\}, \{1\})$ is a binary $\mathcal{A}$ -closed set but not binary β -closed.

Theorem 2.12 For a subset (O, P) of (X, Y) in the binary topological space $(X, Y, \mathcal{M})$, the following results are true.

1. (O, P) is binary α -open.
2. (O, P) is binary $\mathcal{A}$ -open and binary pre open.

Proof. (1) $\Rightarrow$ (2): Let (O, P) be a binary open set in $(X, Y, \mathcal{M})$. Then $(O, P) \subseteq b\text{-int}(b\text{-cl}(b\text{-int}(O, P)))$ which implies that $b\text{-cl}(O, P) \subseteq b\text{-cl}(b\text{-int}(O, P))$ and $b\text{-int}(b\text{-cl}(O, P)) \subseteq b\text{-int}(b\text{-cl}(b\text{-int}(O, P))) \subseteq b\text{-cl}(b\text{-int}(O, P))$. Hence (O, P) is a binary $\mathcal{A}$ -open set in $(X, Y, \mathcal{M})$. Conversely, since (O, P) is a binary α -open set, (O, P) is a binary pre open set in $(X, Y, \mathcal{M})$.

(2) $\Rightarrow$ (1): Since (O, P) is both binary $\mathcal{A}$ -open and binary pre open in $(X, Y, \mathcal{M})$. Since (O, P) is binary $\mathcal{A}$ -open in $(X, Y, \mathcal{M})$, we have $b\text{-int}(b\text{-cl}(O, P)) \subseteq b\text{-cl}(b\text{-int}(O, P))$ and hence $b\text{-int}(b\text{-cl}(O, P)) \subseteq b\text{-int}(b\text{-cl}(b\text{-int}(O, P)))$. Since (O, P) is binary pre open in $(X, Y, \mathcal{M})$, we have $(O, P) \subseteq b\text{-int}(b\text{-cl}(O, P))$. Therefore $(O, P) \subseteq b\text{-int}(b\text{-cl}(b\text{-int}(O, P)))$. Hence (O, P) is a binary α -open set in $(X, Y, \mathcal{M})$.

Remark 2.13 In a binary topological space $(X, Y, \mathcal{M})$, the notions of binary $\mathcal{A}$ -open sets and the notions of binary pre open sets are independent of each other.

Example 2.14 Let $X = \{a, b\}$, $Y = \{1, 2\}$ and $\mathcal{M} = \{(\phi, \phi), (\phi, \{2\}), (\{a\}, \{1\}), (\{a\}, Y), (X, Y)\}$. Then binary $\mathcal{A}$ -open sets are $(\phi, \phi), (\phi, \{2\}), (\{a\}, \{1\}), (\{a\}, Y), (\{b\}, \phi), (\{b\}, \{2\}), (X, \{1\}), (X, Y)$ and binary pre open sets are $(\phi, \phi), (\phi, \{1\}), (\phi, \{2\}), (\phi, Y), (\{a\}, \phi), (\{a\}, \{1\}), (\{a\}, \{2\}), (\{b\}, Y), (X, \{2\}), (X, Y)$.

1. then the subset $(\{b\}, \{2\})$ is a binary $\mathcal{A}$ -open set but not binary pre open.
2. then the subset $(\phi, \{1\})$ is a binary pre open set but not binary $\mathcal{A}$ -open.

Theorem 2.15 Let (O, P) and (S, K) be the subsets of a binary topological space $(X, Y, \mathcal{M})$. If $(O, P) \subseteq (S, K) \subseteq b\text{-cl}(O, P)$ and (O, P) is binary $\mathcal{A}$ -open then K is binary $\mathcal{A}$ -open.

Proof. Assuming that $(O, P) \subseteq (S, K) \subseteq b\text{-cl}(O, P)$ and (O, P) is binary $\mathcal{A}$ -open in $(X, Y, \mathcal{M})$. Then $b\text{-int}(b\text{-cl}(O, P)) \subseteq b\text{-cl}(b\text{-int}(O, P))$. Since $(O, P) \subseteq (S, K)$, $b\text{-cl}(b\text{-int}(O, P)) \subseteq b\text{-cl}(b\text{-int}(S, K))$ and $b\text{-int}(b\text{-cl}(O, P)) \subseteq b\text{-cl}(b\text{-int}(S, K))$. Since $(S, K) \subseteq b\text{-cl}(O, P)$, we have $b\text{-cl}(S, K) \subseteq b\text{-cl}(b\text{-cl}(O, P)) = b\text{-cl}(O, P)$ and $b\text{-int}(b\text{-cl}(S, K)) \subseteq b\text{-int}(b\text{-cl}(O, P))$. Therefore $b\text{-int}(b\text{-cl}(S, K)) \subseteq b\text{-cl}(b\text{-int}(S, K))$. This proves that (S, K) is a binary $\mathcal{A}$ -open set in $(X, Y, \mathcal{M})$.

Corollary 2.16 For a binary topological space $(X, Y, \mathcal{M})$. If $(O, P) \subseteq (X, Y)$ is binary $\mathcal{A}$ -open and binary dense in $(X, Y, \mathcal{M})$, then each subset of (X, Y) containing (O, P) is binary

$\mathcal{A}$ -open.

It is followed by Theorem 2.15.

Proposition 2.17 Every binary closed set in a binary topological space $(X, Y, \mathcal{M})$ is a binary $\mathcal{B}$ -set.

Proof. Let (O, P) be a binary closed set in $(X, Y, \mathcal{M})$. Then $(O, P) = b-cl(O, P)$ and hence $b-int(b-cl(O, P)) = b-int(O, P)$. Which verifies that (O, P) is a binary $\mathcal{B}$ -set in $(X, Y, \mathcal{M})$.

Remark 2.18 The converse of Proposition 2.17 is need not be true as shown in the following Example.

Example 2.19 Let $X = \{a, b\}$, $Y = \{1\}$ and $\mathcal{M} = \{(\phi, \phi), (\{a\}, \phi), (X, Y)\}$. Then binary closed sets are $(\phi, \phi), (\{b\}, Y), (X, Y)$ and binary $\mathcal{B}$ -sets are $(\phi, \phi), (\phi, Y), (\{b\}, \phi), (\{b\}, Y), (X, Y)$. Then the subset (ϕ, Y) is a binary $\mathcal{B}$ -set but not binary closed.

Theorem 2.20 A subset (O, P) of a binary topological space $(X, Y, \mathcal{M})$ is binary semi closed if and only if (O, P) is a binary $\mathcal{B}$ -set.

Proof. Let (O, P) be a binary semi closed set in $(X, Y, \mathcal{M})$. Then $(X, Y) - (O, P)$ is binary semi-open. By Theorem 2.6, we have $b-cl((X, Y) - (O, P)) = b-cl(b-int((X, Y) - (O, P)))$. Therefore $(X, Y) - b-int(O, P) = b-cl((X, Y) - b-cl(O, P)) = (X, Y) - b-int(b-cl(O, P))$. Thus, $b-int(b-cl(O, P)) = b-int(O, P)$ and hence (O, P) is a binary $\mathcal{B}$ -set in $(X, Y, \mathcal{M})$.

Conversely, let (O, P) be a binary $\mathcal{B}$ -set. Then $b-int(b-cl(O, P)) = b-int(O, P) \subseteq (O, P)$. Which proves that (O, P) is binary semi closed.

3 Binary $\mathcal{A}_C$ -set and binary $\mathcal{C}$ -closed sets

In this section, we introduce and analyze the new kinds of binary $\mathcal{A}_C$ -set and binary $\mathcal{C}$ -closed sets, and their relevant characteristics are investigated.

Definition 3.1 A subset (O, P) of a binary topological space $(X, Y, \mathcal{M})$ is called binary $\mathcal{A}_C$ -set if (O, P) is binary semi open and a binary $\mathcal{B}$ -set.

Theorem 3.2 For a subset (O, P) of (X, Y) in a binary topological space $(X, Y, \mathcal{M})$, the following statements are equivalent:

1. (O, P) is binary $\mathcal{A}_C$ -set.
2. (O, P) is binary β -open and binary semi closed.

Proof. (1) $\Rightarrow$ (2): Since (O, P) is binary $\mathcal{A}_C$ -set in $(X, Y, \mathcal{M})$, then (O, P) is both binary semi open and a binary $\mathcal{B}$ -set. Since every binary semi open set is binary β -open, (O, P) is both binary β -open and a binary $\mathcal{B}$ -set. By Theorem 2.20, we obtain the result.

(2) $\Rightarrow$ (1): let (O, P) be a binary semi closed set and binary β -open in $(X, Y, \mathcal{M})$.

Since (O, P) is a binary semi closed by Theorem 2.20. (O, P) is a binary $\mathcal{B}$ -set. Since (O, P) is binary a β -open set in $(X, Y, \mathcal{M})$. $(O, P) \subseteq b-cl(b-int(b-cl(O, P))) = b-cl(b-int(O, P))$. Therefore (O, P) is binary semi open. Since (O, P) is both binary semi open and a binary $\mathcal{B}$ -set, (O, P) is binary $\mathcal{A}_C$ -set.

Remark 3.3 In a binary topological space $(X, Y, \mathcal{M})$, the notions of binary semi closed and the notions of binary β -open are independent of each other.

Example 3.4 In Example 2.5, then binary semi closed sets are $(\phi, \phi), (\phi, \{2\}), (\{b\}, \phi), (\{b\}, \{2\}), (X, \{1\}), (X, Y)$ and binary β -open sets are $(\phi, \phi), (\phi, \{1\}), (\phi, \{2\}), (\phi, Y), (\{a\}, \phi), (\{a\}, \{1\}), (\{a\}, \{2\}), (\{a\}, Y), (\{b\}, \{1\}), (\{b\}, Y), (X, \phi), (X, \{1\}), (X, \{2\}), (X, Y)$.

1. then the subset $(\{b\}, \{2\})$ is a binary semi closed set but not binary β -open.
2. then the subset $(\{a\}, \{1\})$ is a binary β -open set but not binary semi closed.

Definition 3.5 A subset (O, P) of a binary topological space $(X, Y, \mathcal{M})$ is said binary $\mathcal{C}$ -closed set if $(O, P) = b-cl(b-int(O, P))$. $(O, P)^c = (X, Y) - (O, P)$ is binary $\mathcal{C}$ -open.

Theorem 3.6 In a binary topological space $(X, Y, \mathcal{M})$, for a subset (O, P) of (X, Y) , the following statements are equivalent:

1. $(O, P) \neq (\phi, \phi) = \phi$ is binary $\mathcal{C}$ -closed.
2. There exists a non-empty binary open set $(S, F): (S, F) \subseteq (O, P) = b-cl(S, F)$.
3. There exists a non-empty binary open set $(S, F): (O, P) = (S, F) \cup (b-cl(S, F) - (S, F))$.

Proof. (1) $\Rightarrow$ (2). Suppose $(O, P) \neq (\phi, \phi) = \phi$ is a binary $\mathcal{C}$ -closed set. Then $(O, P) = b-cl(b-int(O, P))$. Let $(S, F) = b-int(O, P)$. (S, F) is the required binary open set. $(S, F) \subseteq (O, P) = b-cl(S, F)$.

(2) $\Rightarrow$ (3). Since $(O, P) = b-cl(S, F) = (S, F) \cup (b-cl(S, F) - (S, F))$ where (S, F) is a nonempty binary open set, (3) follows.

(3) $\Rightarrow$ (1). $(O, P) = (S, F) \cup (b-cl(S, F) - (S, F))$ implies that $(O, P) = b-cl(S, F) = b-cl(b-int(S, F)) \subseteq b-cl(b-int(O, P))$, since (S, F) is binary open and $(S, F) \subseteq (O, P)$. Again $b-int(O, P) \subseteq (O, P)$ implies that $b-cl(b-int(O, P)) \subseteq b-cl(O, P) = b-cl(S, F) = (O, P)$. Therefore $(O, P) = b-cl(b-int(O, P))$. Hence (O, P) is binary $\mathcal{C}$ -closed.

Theorem 3.7 Let (O, P) be a subset of a binary space $(X, Y, \mathcal{M})$. If (O, P) is binary β -open, then $b-cl(O, P)$ is binary $\mathcal{C}$ -closed.

Proof. Suppose (O, P) is a binary β -open set in $(X, Y, \mathcal{M})$. We have $(O, P) \subseteq b-cl(b-int(b-$

$cl(O, P)))$ and $b-cl(O, P) \subseteq b-cl(b-int(b-cl(O, P))) \subseteq b-cl(O, P)$ which implies that $b-cl(O, P) = b-cl(b-int(b-cl(O, P)))$. Hence $b-cl(O, P)$ is binary $\mathcal{C}$ -closed in $(X, Y, \mathcal{M})$.

Theorem 3.8 *In a binary topological space $(X, Y, \mathcal{M})$, for a subset (O, P) of (X, Y) , the following statements are equivalent:*

1. (O, P) is binary $\mathcal{C}$ -closed.
2. (O, P) is binary semi open and binary closed.
3. (O, P) is binary β -open and binary closed.

Proof. (1) $\Rightarrow$ (2): (O, P) is binary $\mathcal{C}$ -closed $\Rightarrow (O, P) = b-cl(b-int(O, P))$ and $b-cl(O, P) = b-cl(b-int(O, P))$. Since $(O, P) \subseteq b-cl(b-int(O, P))$, (O, P) is binary semi open. Also, $(O, P) = b-cl(O, P)$ and hence (O, P) is binary closed in $(X, Y, \mathcal{M})$.

(2) $\Rightarrow$ (3): It follows from the fact that every binary semi open set is a binary β -open.

(3) $\Rightarrow$ (1): If (O, P) is a binary β -open set and binary closed set in $(X, Y, \mathcal{M})$, then $(O, P) \subseteq b-cl(b-int(b-cl(O, P)))$ and $(O, P) = b-cl(O, P)$. Now $b-cl(b-int(O, P)) \subseteq b-cl(O, P) = (O, P)$. Also, $(O, P) \subseteq b-cl(b-int(O, P))$. Therefore $(O, P) = b-cl(b-int(O, P))$. Hence (O, P) is binary $\mathcal{C}$ -closed in $(X, Y, \mathcal{M})$.

Remark 3.9 *In a binary topological space $(X, Y, \mathcal{M})$,*

1. the notions of binary semi open and the notions of binary closed are independent of each other.
2. the notions of binary β -open and the notions of binary closed are independent of each other.

Example 3.10 *In Example 2.5, then binary closed sets are $(\phi, \phi), (\phi, \{2\}), (\{b\}, \phi), (\{b\}, \{2\}), (X, \{1\}), (X, Y)$ and binary semi open sets are $(\phi, \phi), (\phi, \{2\}), (\{a\}, \{1\}), (\{a\}, Y), (X, \{1\}), (X, Y)$ and binary β -open sets are $(\phi, \phi), (\phi, \{1\}), (\phi, \{2\}), (\phi, Y), (\{a\}, \phi), (\{a\}, \{1\}), (\{a\}, \{2\}), (\{a\}, Y), (\{b\}, \{1\}), (\{b\}, Y), (X, \phi), (X, \{1\}), (X, \{2\}), (X, Y)$.*

1. then the subset $(\{a\}, \{1\})$ is a binary semi open set but not binary closed.
2. then the subset $(\{b\}, \phi)$ is a binary closed set but not binary semi open.
3. then the subset $(\{a\}, \{2\})$ is a binary β -open set but not binary closed.
4. then the subset $(\{b\}, \{2\})$ is a binary closed set but not binary β -open.

4 Binary α^* -sets and binary $\mathcal{B}_{\alpha^*}$ -sets

The novel types of binary α^* and binary $\mathcal{B}_{\alpha^*}$ -sets are introduced and analyzed in this section, and their pertinent properties are examined.

Definition 4.1 A subset (O, P) of a binary topological space $(X, Y, \mathcal{M})$ is said to be

1. binary α^* -set if $b-cl(b-int(b-cl(O, P))) = b-int(O, P)$.
2. binary $\mathcal{B}_{\alpha^*}$ -set if $(O, P) = (L, M) \cap (S, F)$, where (L, M) is a binary open set and (S, F) is binary α^* -set.

Proposition 4.2 For a subset (O, P) of a binary topological space $(X, Y, \mathcal{M})$, the following statements are equivalent:

1. $(O, P) \in b-\alpha^*(X, Y, \mathcal{M})$.
2. (O, P) is binary semi pre closed.
3. $b-int(O, P)$ is binary regular open.

Proof. Obvious.

Theorem 4.3 For a subset (O, P) of (X, Y) in a binary topological space $(X, Y, \mathcal{M})$, then the following statements:

1. If (O, P) is a binary $\mathcal{B}$ -set then $(O, P) \in b-\alpha^*(X, Y, \mathcal{M})$.
2. For a subset (O, P) is a binary semi open set and binary $\mathcal{B}$ -set if and only if $(O, P) \in b-\alpha^*(X, Y, \mathcal{M})$.
3. For a subset (O, P) is binary α -open and $(O, P) \in b-\alpha^*(X, Y, \mathcal{M})$ if and only if (O, P) is binary regular open.

Proof. 1. Let (O, P) be a binary $\mathcal{B}$ -set, then $b-int(O, P) = b-int(b-cl(O, P))$ and $b-int(b-cl(b-int(O, P))) = b-int(b-cl(O, P)) = b-int(O, P)$. Therefore, (O, P) is a binary α^* -set.

2. Let (O, P) be binary semi open and $(O, P) \in b-\alpha^*(X, Y, \mathcal{M})$. Since (O, P) is binary semi open, $b-cl(b-int(O, P)) = b-cl(O, P)$ and hence $b-int(b-cl(O, P)) = b-int(b-cl(b-int(O, P))) = b-int(O, P)$. Therefore, (O, P) is a binary $\mathcal{B}$ -set.

3. Let (O, P) be a binary α -open set and $(O, P) \in b-\alpha^*(X, Y, \mathcal{M})$. By Proposition 4.2 and the Definition of binary α -open set, we have $b-int(b-cl(b-int(O, P))) = (O, P)$ and hence $b-int(b-cl(O, P)) = b-int(b-cl(b-int(O, P))) = (O, P)$. The converse is obvious.

Theorem 4.4 In a binary topological space $(X, Y, \mathcal{M})$. Then $b-\alpha^*(X, Y, \mathcal{M}) \subseteq b-\mathcal{B}_{\alpha^*}(X, Y, \mathcal{M})$ and binary open $\subseteq b-\mathcal{B}_{\alpha^*}(X, Y, \mathcal{M})$.

Proof. Since $(X, Y) \in \text{binary open} \cap b-\alpha^*(X, Y, \mathcal{M})$, the inclusions are obvious.

Lemma 4.5 In a binary topological space $(X, Y, \mathcal{M})$. If either $(O, P), (S, K)$ is binary semi open, then $b\text{-int}(b\text{-cl}(O, P) \cap (S, K)) = b\text{-int}(b\text{-cl}(O, P)) \cap b\text{-int}(b\text{-cl}(S, K))$.

Proof. For any subset $(O, P), (S, K) \subseteq (X, Y)$, we generally have $b\text{-int}(b\text{-cl}(O, P) \cap (S, K)) \subseteq b\text{-int}(b\text{-cl}(O, P)) \cap b\text{-int}(b\text{-cl}(S, K))$. Assume that (O, P) is binary semi open. Then $b\text{-cl}(O, P) = b\text{-cl}(b\text{-int}(O, P))$. Therefore $b\text{-int}(b\text{-cl}(O, P)) \cap b\text{-int}(b\text{-cl}(S, K)) = b\text{-int}(b\text{-cl}(b\text{-int}(O, P)) \cap b\text{-int}(b\text{-cl}(S, K))) \subseteq b\text{-int}(b\text{-cl}(b\text{-cl}(O, P) \cap b\text{-int}(b\text{-cl}(S, K)))) = b\text{-int}(b\text{-cl}(b\text{-int}(O, P) \cap b\text{-int}(b\text{-cl}(S, K)))) \subseteq b\text{-int}(b\text{-cl}(b\text{-int}(O, P) \cap (S, K))) \subseteq b\text{-int}(b\text{-cl}((O, P) \cap (S, K)))$.

Theorem 4.6 For a subset (O, P) of (X, Y) in a binary topological space $(X, Y, \mathcal{M})$, the following results are equivalent:

1. (O, P) is a binary open set.
2. (O, P) is a binary α -open and a binary $\mathcal{B}_{\alpha^*}$ -set.

Proof. (1) $\Rightarrow$ (2): It is obvious that every binary open set is a binary α -open set and a binary $\mathcal{B}_{\alpha^*}$ -set.

(2) $\Rightarrow$ (1): let (O, P) be a binary α -open set and a binary $\mathcal{B}_{\alpha^*}$ -set in $(X, Y, \mathcal{M})$. Since (O, P) is a binary $\mathcal{B}_{\alpha^*}$ -set, there exist $(A, B) \in$ binary open and $(C, D) \in b\text{-}\alpha^*(X, Y, \mathcal{M})$ such that $(O, P) = (A, B) \cap (C, D)$. Since (O, P) is a binary α -open set, by using Lemma 4.5, we have $(O, P) \subseteq b\text{-int}(b\text{-cl}(b\text{-int}(O, P))) = b\text{-int}(b\text{-cl}(b\text{-int}((A, B) \cap (C, D)))) = b\text{-int}(b\text{-cl}((A, B))) \cap b\text{-int}(b\text{-cl}(b\text{-int}(C, D))) = b\text{-int}(b\text{-cl}(A, B)) \cap b\text{-int}(C, D)$ and hence $(O, P) = (A, B) \cap (O, P) \subseteq (A, B) \cap (b\text{-int}(b\text{-cl}(A, B)) \cap b\text{-int}(C, D)) = (A, B) \cap b\text{-int}(C, D) \subseteq (O, P)$. Consequently, we obtain $(O, P) = (A, B) \cap b\text{-int}(O, P)$ and (O, P) is binary open.

Remark 4.7 In a binary topological space $(X, Y, \mathcal{M})$, the notions of binary α -open set and the notions of binary $\mathcal{B}_{\alpha^*}$ -set are independent of each other as shown in the following example.

Example 4.8 Let $X = \{a, b\}$, $Y = \{1, 2\}$ and $\mathcal{M} = \{(\phi, \phi), (\phi, \{1\}), (\{a\}, \{1\}), (X, Y)\}$. Then binary α -open sets are $(\phi, \phi), (\phi, \{1\}), (\phi, Y), (\{a\}, \{1\}), (\{a\}, Y), (\{b\}, \{1\}), (\{b\}, Y), (X, \{1\}), (X, Y)$ and binary $\mathcal{B}_{\alpha^*}$ -sets are $(\phi, \phi), (\phi, \{1\}), (\phi, \{2\}), (\{a\}, \phi), (\{a\}, \{1\}), (\{a\}, \{2\}), (\{b\}, \phi), (\{b\}, \{2\}), (X, \phi), (X, \{2\}), (X, Y)$.

1. then the subset $(\{b\}, \{1\})$ is a binary α -open set but not binary $\mathcal{B}_{\alpha^*}$ -set.
2. then the subset $(\{a\}, \{2\})$ is a binary $\mathcal{B}_{\alpha^*}$ -set but not binary α -open.

5 Decompositions of subsets

1. From the Theorem 2.9 and the Remark 2.10 in the binary topological space $(X, Y, \mathcal{M})$ we get a decomposition of binary semi closed subsets into a binary β -closed subset and binary $\mathcal{A}$ -closed.
2. From the Theorem 2.12 and the Remark 2.13 in the binary topological space $(X, Y, \mathcal{M})$ we get a decomposition of binary α -open subsets into a binary $\mathcal{A}$ -open subset and binary pre open.
3. From the Theorem 3.2 and the Remark 3.3 in the binary topological space $(X, Y, \mathcal{M})$ we get a decomposition of binary $\mathcal{A}_c$ -sets into a binary β -open subset and binary semi closed.
4. From the Theorem 3.8 and the Remark 3.2(1) in the binary topological space $(X, Y, \mathcal{M})$ we get a decomposition of binary $\mathcal{C}$ -closed subsets into a binary semi open subset and binary closed.
5. From the Theorem 3.8 and the Remark 3.2(2) in the binary topological space $(X, Y, \mathcal{M})$ we get a decomposition of binary $\mathcal{C}$ -closed subsets into a binary β -open subset and binary closed.
6. From the Theorem 4.6 and the Remark 4.7 in the binary topological space $(X, Y, \mathcal{M})$ we get a decomposition of binary open subsets into a binary α -closed subset and binary $\mathcal{B}_{\alpha^*}$ -set.

6 Binary kernel

In this section, the notion of binary kernel is introduced and its properties are investigated.

Definition 6.1 Let $(X, Y, \mathcal{M})$ be a binary topological space and $(A, B) \subseteq (X, Y)$. The set $b\text{-ker}(A, B) = \cap \{(X, Y) : (A, B) \subseteq (X, Y), (X, Y) \in (X, Y, \mathcal{M})\}$ said to be binary kernel of (A, B) and is denoted by $b\text{-ker}(A, B)$.

Example 6.2 Let $X = \{a, b\}$, $Y = \{1, 2\}$ and $\mathcal{M} = \{(\phi, \phi), (\phi, \{1\}), (\{a\}, \{1\}), (\{b\}, \{1\}), (X, \{1\}), (X, Y)\}$, then the $b\text{-Ker}(\{a\}, \phi) = (\{a\}, \{1\})$, $b\text{-Ker}(\{b\}, \{1\}) = (\{b\}, \{1\})$, $b\text{-Ker}(X, \{2\}) = (X, Y)$.

Theorem 6.3 Let $(X, Y, \mathcal{M})$ be a binary topological space and $(A, B), (C, D) \subseteq (X, Y)$. Then the following properties hold.

1. $(p, q) \in b\text{-ker}(A, B) \Leftarrow (A, B) \cap (F, G) = (\phi, \phi) = \phi$ for any binary closed set containing (p, q) .
2. If $(A, B) \subseteq b\text{-ker}(A, B)$ and then $(A, B) = b\text{-ker}(A, B)$ if (A, B) is binary open in (X, Y) .
3. If $(A, B) \subseteq (C, D)$, then $b\text{-ker}(A, B) \subseteq b\text{-ker}(C, D)$.

Proof. 1. If $(p, q) \in b\text{-ker}(A, B)$ then $(p, q) \in (A, B) \subseteq (F, G)^c$, where $(F, G)^c$ is a binary open set and $(A, B) \subseteq (F, G)^c \Rightarrow (A, B) \cap (F, G)^c = (\phi, \phi) = \phi$. Hence $(A, B) \subseteq (F, G)^c \subseteq (F, G) \Rightarrow (A, B) \cap (F, G) = (\phi, \phi) = \phi$. F is binary closed. Hence $(p, q) \in (A, B) \cap (F, G) = (\phi, \phi) = \phi$. (F, G) is binary closed. Therefore $(A, B) \cap (F, G) = (\phi, \phi) = \phi$ for any binary closed set containing (p, q) .

Conversely, let $(A, B) \cap (F, G) \neq (\phi, \phi) = \phi$. for any binary closed set containing (p, q) . Assume that $(x, y) \notin b\text{-ker}(A, B)$ hence their exist a binary open set $(F, G)^c$ such that $(A, B) \subseteq (F, G)^c$ and $(p, q) \notin (F, G)^c$. Hence $(A, B) \subseteq (F, G)$ and $(p, q) \notin (F, G)$, where (F, G) is a binary closed which is a contradiction. Hence $(p, q) \in b\text{-ker}(A, B)$.

2. Let (A, B) be a binary open set. Since (A, B) is binary open $b\text{-ker}(A, B) \subseteq (A, B)$ holds. Let (C, D) be any binary open set containing (A, B) then we have $(A, B) \subseteq (W, Z)$. Hence $(A, B) \subseteq (W, Z) \cap (A, B) \subseteq (A, B)$ and $(W, Z) \cap (A, B)$ is binary open. Now, $(A, B) \subseteq \cap \{(W, Z): (A, B) \subseteq (W, Z), (W, Z) \in (X, Y, \mathcal{M})\}$. Therefore $(A, B) \subseteq b\text{-ker}(A, B)$ and hence $(A, B) = b\text{-ker}(A, B)$.

3. Let $(A, B) \subseteq (C, D)$, To prove that. $b\text{-ker}(A, B) \subseteq b\text{-ker}(C, D)$. Let $(U, V) \in b\text{-ker}(A, B) \Rightarrow (A, B) \subseteq (U, V)$ and (U, V) is binary open in $(X, Y, \mathcal{M})$. Since $(A, B) \subseteq (C, D)$, so $(A, B) \subseteq (C, D) \subseteq (U, V)$, where (U, V) is a binary open set in $(X, Y, \mathcal{M})$. Therefore $(U, V) \in b\text{-ker}(C, D)$. Hence $(U, V) \in b\text{-ker}(A, B) \Rightarrow (U, V) \in b\text{-ker}(C, D)$ and thus $b\text{-ker}(A, B) \subseteq b\text{-ker}(C, D)$.

7 Conclusion

We introduced and studied some new sets called binary $\mathcal{A}$ -open sets, binary $\mathcal{B}$ -sets, binary $\mathcal{A}_c$ -sets, binary $\mathcal{C}$ -closed sets, binary α^* -sets, and binary $\mathcal{B}_{\alpha^*}$ -sets, which are simple forms of binary open sets in binary topological spaces. We characterized these sets and studied their related properties with binary open sets. The independence of such as binary preopen sets and binary $\mathcal{A}$ -closed sets, binary β -closed sets and binary $\mathcal{A}$ -closed sets, binary α -open sets and binary $\mathcal{B}_{\alpha^*}$ -sets are established. Also, decomposition of a binary α -open subset into a binary $\mathcal{A}$ -open subset and binary preopen sets is arrived at. The notion of a binary kernel is introduced, and its properties are investigated.

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