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DETERMINING THE OPTIMAL REPLACEMENT POLICY IN A TWO-COMPONENT SERIES SYSTEM UNDER PARTIAL PRODUCT PROCESS REPAIR ASSUMPTION

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Abstract

In this article, a series repairable system consisting of two non-identical components and a single repairer is analyzed. It is assumed that after each repair, a component does not return to an "as good as new" condition. Based on the number of failures of components 1 and 2, an (M, N) replacement policy is proposed under the framework of a partial product process repair model. Finding the best replacement strategy (M^*, N^*) to minimize the long-term expected cost per unit of time is the challenge. Analytical or numerical methods can be used to find the best replacement policy, and the explicit formula for the long-run expected cost per unit time is generated. Lastly, a suitable numerical example is provided to demonstrate a few theoretical findings, such as the uniqueness of the optimal replacement policy (M^*, N^*) and the sensitivity analysis components 1 and 2.
Keywords: Series system, Partial product process, Renewal process, Replacement policy (M, N).

Introduction

In the earliest study of the maintenance problem, the repair–replacement models mainly concentrate on the study of perfect repair models in which the system after repair is “as good as new”. In practice, most systems deteriorate due to ageing effects and accumulated wear. In other words, a system after repair cannot be “as good as new”.

It might be that a repair does not change the age of the system. Under this assumption, a minimal repair model was presented by Barlow and Hunter [3]. Thereafter an imperfect repair model in which a repair with probability p is a perfect repair, and with probability $q = 1 - p$ is a minimal repair, was first introduced by Brown and Proschan [4]. Many research works have been done by Park [15], Phelps [16], Block et al. [5], Kijima [7] and others along this direction. The geometric process repair model—was first proposed by Lam [8,9]. Using this model, he studied two kinds of replacement policies, one based on the working age T of the system and the other based on the failure number N of the system. The object is to choose optimal replacement policies T^* and N^* such that the long-run average cost per unit time is minimized. The explicit expressions for the

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long-run average cost per unit time for these two kinds of policies are derived, and the corresponding optimal replacement policies T^* and N^* are found analytically or numerically. Because the geometric process is a special monotone process, Stadje and Zuckerman [17] introduced a general monotone process repair model to generalize Lam's work. Zhang [25] generalized Lam's work with a bivariate replacement policy (T, N) under which the system is replaced at the working age T or at the time of the Nth failure, whichever occurs first. Many research works have been done by Lam [10], Stadje and Zuckerman [18], Finkelstein [6], Stanley [19], Lam and Zhang [11], Zhang et al. [24], Zhang [26,27], Wang and Zhang [22,23] and others along this direction.

In practical applications, some complex repairable systems such as series systems, standby systems, parallel systems and $k - out - of - n$: F $(or\ G)$ systems and others are often installed. By applying the geometric process repair model, Zhang and Wu [28] first reported some reliability indices of a two-component series repairable system when the operating time of component 1 follows the exponential distribution while that of component 2 and the repair times of both components follow general distributions. Lam and Zhang [12] provided a more in depth analysis of the series system studied by Zhang and Wu [28] under the assumptions that the operating times and repair times of both components all follow the exponential distribution. Many research works for some reliability indices and replacement policies for some complex repairable systems have been done by Lam and Zhang [13,14], Zhang [29], Zhang et al. [30], Zhang et al. [31], Zhang and Wang [32] etc. Even now, little attention has been paid to the study of a replacement policy for a series repairable system.

However, a more reasonable repair model the partial product process repair model—was first proposed by Babu [1,2]. Using this model, he studied two kinds of replacement policies, one based on the working age T of the system and the other based on the failure number N of the system. The object is to choose optimal replacement policies T^* and N^* such that the long-run average cost per unit time is minimized. The explicit expressions for the long-run average cost per unit time for these two kinds of policies are derived, and the corresponding optimal replacement policies T^* and N^* are found analytically or numerically. In this article a two-component series system's optimal replacement policy under the assumption of partial product process repair is studied.The purpose of this article is to apply the geometric process repair model to a two non-identical components series repairable system with one repairer, because a two-component series repairable system not only is one of the fundamental models in reliability theory but also is the usually used model in practice. For example, a computer system may be treated as a series system consisting of hardware and software; an auto-control system is a series system consisting of control component and operating component; an electronic system may also be regarded as a series system consisting of power supply unit and function unit etc.

In this article, a series repairable system consisting of two non-identical components and one repairer is studied. It is assumed that the successive survival times after repair constitute a decreasing geometric process and the consecutive repair times form an increasing geometric process for each component. A replacement policy (M, N) is considered based on the numbers of failures of components 1 and 2. The problem is to determine an optimal replacement policy (M^*, N^*) such that the long-run expected cost per unit time is minimized. The explicit expression for the long-run expected cost per unit time is derived and the corresponding optimal replacement policy can be determined analytically or numerically.

For convenience, the definitions of stochastic order and geometric process are stated first.

Definition 1. [20] Given two random variables X and Y, X is said to be stochastically greater than Y, or Y is stochastically less than X, if

$$P(X > \alpha) \geq P(Y > \alpha) \text{ for all real } \alpha.$$

This is denoted by $X \geq_{st} Y$ or $Y \leq_{st} X$ Furthermore, a stochastic process $\{X_n, n = 1, 2, \dots\}$ is stochastically decreasing (increasing) if $X_n \geq_{st} (\leq_{st})\ X_{n+1}$ for all $n = 1, 2, \dots$.

Definition 2.[21] A stochastic process $\{X_n, n = 1, 2, \dots\}$ is a geometric process if there exists a real $a > 0$ such that $\{a^{n-1}X_n, n = 1, 2, \dots\}$ forms a renewal process. The real a is called the ratio of the geometric process.

Obviously, if $a > 1$, then $\{X_n, n = 1, 2, \dots\}$ is stochastically decreasing, i.e.

$$X_n \geq_{st} X_{n+1}, n = 1, 2, \dots.$$

If $0 < a < 1$, then $\{X_n, n = 1, 2, \dots\}$ is stochastically increasing, i.e.

$$X_n \leq_{st} X_{n+1}, n = 1, 2, \dots.$$

If $a = 1$, then the geometric process becomes a renewal process.

Definition 3[1]. Let $\{X_n,\ n = 1,\ 2,\ 3,\ \dots\}$ be a sequence of non-negative independent random variables and let $G(x)$ be the distribution function of X_n. Then $\{X_n,\ n = 1,\ 2,\ 3,\ \dots\}$ is called a partial product process, if the distribution function of X_{i+1} is $F\,(a_i x)$ $(i = 1,\ 2,\ 3,\ \dots)$ where $a_i > 0$ are constants and $a_i = a_0 a_1 a_2 \cdots a_{i-1}$.

By induction, it is clear that for real a_i $(i = 1, 2, \dots)$, $a_i = {a_0}^{2^{i-1}}$. Thus, the distribution function of X_{i+1} is $F\left({a_0}^{2^{i-1}}\right)$ for $= 1, 2, \dots$. If $a_0 = 1$, then the partial product process becomes a renewal process.

Theorem:[1] Let, $\{X_n\}$ be a partial product process and $E(X_1) = \mu$. Then for n = 2,3,4 ...

$$E(X_n) = \frac{\mu}{a^{2^{n-1}}}, \text{where } a \text{ is a constant}$$

Model

A two non-identical components series repairable system with one repairer is studied by making the following assumptions.

Assumption 1. At the beginning, the two components in the system are both new. Whenever one component fails, the system breaks down, and the failed component will be immediately repaired. It is assumed that two components shut each other off, and each component after repair is not "as good as new", but follows a partial product process repair.

Assumption 2. For $i = 1, 2$, $X_n^{(i)}$ is the operating time for component i after the $(n-1)$th repair, and $Y_n^{(i)}$ is the repair time after the nth failure. $X_n^{(i)}$ and $Y_n^{(i)}$ have the distributions $F\left(\left(a_0^{2^{i-1}}\right)^{n-1} x\right)$ and $G\left(\left(b_0^{2^{i-1}}\right)^{n-1} y\right)$. $(n = 1, 2, \ldots;\ a_0 > 1, 0 < b_0 < 1)$, respectively. The expectations of $EX_1^{(i)}$ and $EY_1^{(i)}$ are assumed to be $\lambda_i > 0$ and $\mu_i > 0$ $(i = 1, 2)$, respectively.

Assumption 3. Component 1 and component 2 will be replaced by new and identical ones at the times of the Mth and Nth failures, and both the replacement times are negligible.

Assumption 4. $X_n^{(i)}\ and\ Y_n^{(i)}, i = 1, 2;\ n = 1, 2, \ldots$, are mutually independent.

Assumption 5. The working reward per unit time of the system is c_0, the repair cost per unit time of component i for every failure is c_i, and the replacement cost of component i each time is $d_i, i = 1, 2$.

Assumption 6. The replacement policy (M, N) based on the number of failures of component 1 and component 2 is used.

Long-run expected cost under policy (M, N)

According to the assumptions above, the explicit expression for the long-run expected cost per unit time for the series system can be derived under the replacement policy (M, N).

Let $T_k^{(i)}$ and $S_k^{(i)}$ be respectively the operating time and the repair time for component i between the $(k-1)$ th replacement and k th replacement, $k = 1, 2, \ldots; i = 1, 2$. Let $T(t)$ be the total operating time of the system before time t, and let $S_k^{(i)}(t)$ be the total repair time of the component i before time $t, i = 1, 2$; then

$$T(t) = T_1^{(1)} + T_2^{(1)} + \cdots + T_m^{(1)} + \varphi_1(t)$$
$$= T_1^{(2)} + T_2^{(2)} + \cdots + T_m^{(2)} + \varphi_2(t) \qquad (1)$$
$$S^{(1)}(t) = S_1^{(1)} + S_2^{(1)} + \cdots + S_m^{(1)} + \psi_1(t) \qquad (2)$$
$$S^{(2)}(t) = S_1^{(2)} + S_2^{(2)} + \cdots + S_n^{(2)} + \psi_2(t) \qquad (3)$$

where m is the replacement number of component 1 before time t, and $\varphi_1(t)$ is the operating time of the system between the mth replacement and time t, while n is the replacement number of component 2 before time t, and $\varphi_2(t)$ is the operating time of the system between the nth replacement and time t. And $\psi_1(t)$ is the repair time of component 1 between the mth replacement and time t, while $\psi_2(t)$ is the repair time of component 2 between the nth replacement and time t.

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Let $\tau_i^{(i)}$ be the first replacement time of the component i under policy (M, N). Let $\tau_j^{(i)}$ $(j \geq 2)$ be the time between the $(j-1)$th replacement and the jth replacement of the component i under policy (M, N). Obviously, $\{\tau_1^{(i)}, \tau_2^{(i)}, \dots\}$ forms a renewal process, while the inter-arrival time between two consecutive replacements is called a renewal cycle. We also know that $T_1^{(i)}, T_2^{(i)}, \dots$ are respectively the regeneration points of component i. Thus, $\{T_1^{(i)}, T_2^{(i)}, \dots\}$ forms a renewal process. Similarly, $\{R_1^{(i)}, R_2^{(i)}, \dots\}$ is also a renewal process, and they have the same renewal cycle. Let $D(t)$ be the cost function of the system at time t. And the time t can be expressed as

$$\begin{aligned} t = {} & T_1^{(1)} + T_2^{(1)} + \cdots + T_m^{(1)} + \varphi_1(t) \\ & + S_1^{(1)} + S_2^{(1)} + \cdots + S_m^{(1)} + \psi_1(t) + S_1^{(2)} + S_2^{(2)} + \cdots + S_m^{(2)} + \psi_2(t) \\ = {} & T_1^{(2)} + T_2^{(2)} + \cdots + T_m^{(2)} + \varphi_2(t) \\ & + S_1^{(1)} + S_2^{(1)} + \cdots + S_m^{(1)} + \psi_1(t) + S_1^{(2)} + S_2^{(2)} + \cdots + S_m^{(2)} + \psi_2(t) \\ = {} & T(t) + S^{(1)}(t) + S^{(2)}(t) \end{aligned}$$

Thus, $D(t) = c_1 S^{(1)}(t) + c_2 S^{(2)}(t) + d_1 m(t) + d_2 n(t) - c_0 T(t)$ (4)

Let $C(M, N)$ be the long-run expected cost per unit time of the system under the replacement policy (M, N); we have

$$\begin{aligned} C(M,N) &= \lim_{t\to\infty} \frac{E[D(t)]}{t} \\ &= \lim_{t\to\infty} \frac{E[c_1 S^{(1)}(t) + c_2 S^{(2)}(t) + d_1 m(t) + d_2 n(t) - c_0 T(t)]}{E[T(t) + S^{(1)}(t) + S^{(2)}(t)]} \\ &= \lim_{t\to\infty} \frac{c_1 \frac{E[S^{(1)}(t)]}{E[T(t)]} + c_2 \frac{E[S^{(2)}(t)]}{E[T(t)]} + d_1 \frac{E[m(t)]}{E[T(t)]} + d_2 \frac{E[n(t)]}{E[T(t)]} - c_0}{1 + \frac{E[S^{(1)}(t)]}{E[T(t)]} + \frac{E[S^{(2)}(t)]}{E[T(t)]}}. \end{aligned} \tag{5}$$

$\{T_1^{(1)}, T_2^{(1)}, \dots\}$ and $\{S_1^{(1)}, S_2^{(1)}, \dots\}$ are renewal processes; the time interval between two consecutive replacements is a renewal cycle. Let W_i be the length of a renewal cycle of component i under policy (M, N). Then according to the renewal reward theorem (see, for example, Ross [29]), we have

$$\lim_{n\to\infty} \frac{E[S^{(1)}(t)]}{E[T(t)]} = \frac{\lim_{t\to\infty} \frac{E\left[S^{(1)}(t)\right]}{E[T(t)]}}{\lim_{t\to\infty} \frac{E[T(t)]}{t}} = \frac{\frac{ES^{(i)}}{EW_i}}{\frac{ET^{(i)}}{EW_i}} = \frac{ES^{(1)}}{ET^{(i)}}, \qquad i = 1,2$$

Similarly

$$\lim_{n\to\infty} \frac{E[m(t)]}{E[T(t)]} = \frac{1}{ET^{(1)}}, \qquad \lim_{n\to\infty} \frac{E[n(t)]}{E[T(t)]} = \frac{1}{ET^{(2)}},$$

Where

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$$E\left(T^{(1)}\right) = E\left(\sum_{m=1}^{M} X_m^{(1)}\right) = \sum_{m=1}^{M} \frac{\lambda_1}{a_1^{2^{m-1}}}$$

$$E\left(T^{(2)}\right) = E\left(\sum_{m=1}^{M} X_n^{(2)}\right) = \sum_{m=1}^{M} \frac{\lambda_1}{a_1^{2^{n-1}}}$$

$$E\left(S^{(1)}\right) = E\left(\sum_{m=1}^{M-1} Y_m^{(1)}\right) = \sum_{m=1}^{M-1} \frac{\mu_1}{b_1^{2^{m-1}}}$$

$$E\left(S^{(2)}\right) = E\left(\sum_{m=1}^{N-1} Y_n^{(2)}\right) = \sum_{m=1}^{N-1} \frac{\mu_2}{b_2^{2^{m-1}}}$$

Substituting the results above in Eq. (5), we have

$$C(M,N) = \frac{c_1 \frac{ES^{(1)}}{ET^{(1)}} + c_2 \frac{ES^{(2)}}{ET^{(2)}} + d_1 \frac{1}{ET^{(1)}} + d_2 \frac{1}{ET^{(2)}} - c_0}{1 + \frac{ES^{(1)}}{ET^{(1)}} + \frac{ES^{(2)}}{ET^{(2)}}}$$

$$= \frac{c_1 \frac{\sum_{m=1}^{M-1} \frac{\mu_1}{b_1^{2^{m-1}}}}{\sum_{m=1}^{M} \frac{\lambda_1}{a_1^{2^{m-1}}}} + c_2 \frac{\sum_{m=1}^{M-1} \frac{\mu_2}{b_2^{2^{m-1}}}}{E\sum_{m=1}^{M} \frac{\lambda_2}{a_2^{2^{m-1}}}} + d_1 \frac{1}{\sum_{m=1}^{M} \frac{\lambda_1}{a_1^{2^{m-1}}}} + d_2 \frac{1}{\sum_{m=1}^{M} \frac{\lambda_2}{a_2^{2^{m-1}}}} - c_0}{1 + \frac{\sum_{m=1}^{M-1} \frac{\mu_1}{b_1^{2^{m-1}}}}{\sum_{m=1}^{M} \frac{\lambda_1}{a_1^{2^{m-1}}}} + \frac{\sum_{m=1}^{M-1} \frac{\mu_2}{b_2^{2^{m-1}}}}{\sum_{m=1}^{M} \frac{\lambda_2}{a_2^{2^{n-1}}}}} \quad (6)$$

Our problem is finding an optimal replacement policy (M^*, N^*) for minimizing $C(M,N)$.

Optimal replacement policy $(\boldsymbol{M}^*, \boldsymbol{N}^*)$

In this section, we shall determine an optimal replacement policy (M^*, N^*) for minimizing $C(M,N)$ explicitly. To do this, we shall consider two cases.

4.1. Case I: Series system of two identical components

When the two components are identical, i.e. $a_1 = a_2 \triangleq a, b_1 = b_2 \triangleq b, \lambda_1 = \lambda_2 \triangleq \lambda, \mu_1 = \mu_2 \triangleq \mu$, there are also $c_1 = c_2 \triangleq c$ and $d_1 = d_2 \triangleq d$ accordingly. Obviously, the replacement numbers of the two components should be the same, and $C(M,N)$ can be replaced by $C(N,N) \triangleq C_1(N)$. To prove the existence and uniqueness of the optimal replacement policy for the system, we introduce a lemma as follows.

Lemma 1. If $a \geq 1, 0 < b \leq 1$, let $f(N) = \frac{1}{a^{2^N}} \sum_{k=1}^{N-1} a^{2^k} = \frac{1}{a^{2^N}} \sum_{k=1}^{N-1} b^{2^k}$.

Then $f\ (N)$ is nondecreasing in N.

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Proof.

$$f(N+1) - f(N) = \left(\frac{1}{a}\right)^{2^N} - \left(\frac{1}{a}\right)^{2^{N+1}} \left(b + b^2 + \cdots + b^{2^N}\right) + \left(\frac{1}{a}\right)^{2^N} \left(b + b^2 + \cdots + b^{2^{N-1}}\right)$$

$$= \left(\frac{1}{a}\right)^{2^N} + \left(\left(\frac{1}{a}\right)^{2^N} - \left(\frac{1}{a}\right)^{2^{N+1}}\right)\left(b + b^2 + \cdots + b^{2^{N-1}}\right) - \left(\frac{1}{a}\right)^{2^{N+1}} b^{2^N}$$

$$= \left(\frac{1}{a}\right)^{2^N} \left(1 - \frac{b^{2^N}}{a}\right) + \left(\left(\frac{1}{a}\right)^{2^N} - \left(\frac{1}{a}\right)^{2^{N+1}}\right)\left(b + b^2 + \cdots + b^{2^{N-1}}\right)$$

≥ 0

This completes the proof of Lemma 1.

Now, the long-run expected cost per unit time of the system under the replacement policy (N, N) is given by

$$C_1(N) = C(N,N) = \frac{-c_0 E\,T(N) + 2cE\,S(N) + 2d}{ET(N) + 2ES(N)}$$

$$= c + \frac{-(c_0 + c)E\,T\,(N) + 2d}{E\,T\,(N) + 2E\,S(N)}$$

where $T(N)$ and $S(N)$ respectively denote the operating time and the repair time of the component under policy (N, N); the expressions for $ET(N)$ and $ES(N)$ are, respectively,

$$ET(N) = E\left(\sum_{n=1}^{N} X_n\right) = \sum_{n=1}^{N} \frac{\lambda}{a^{2^{n-1}}}, \qquad N \geq 1$$

$$ES(N) = E\left(\sum_{n=1}^{N-1} Y_n\right) = \sum_{n=1}^{N} \frac{\mu}{b^{2^{n-1}}}. \qquad N \geq 2$$

$$ES(1) = 0$$

When the replacement number $N \to +\infty$, then the expectations of the operating time, the repair time and the long-run expected cost per unit time of the system are, respectively,

$$\lim_{N\to\infty} ET(N) = \lim_{N\to\infty} E\left(\sum_{n=1}^{N} X_n\right) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\lambda}{a^{2^{n-1}}} = \lambda\left[\frac{a^2 + a - 1}{a(a^2 - 1)}\right]$$

$$\lim_{N\to\infty} ES(N) = \lim_{N\to\infty} E\left(\sum_{n=1}^{N-1} Y_n\right) = \lim_{N\to\infty} \sum_{n=1}^{N-1} \frac{\mu}{b^{2^{n-1}}} = \mu\left[\frac{b^2 + b - 1}{b(b^2 - 1)}\right]$$

$$\lim_{N\to\infty} C_1(N) = c + \lim_{N\to\infty} \frac{-(c_0 + c)E\,T\,(N) + 2d}{E\,T\,(N) + 2E\,S(N)} = c.$$

Now, we can obtain the following result.

Theorem 1. (1) If $\frac{d}{(c_0 + c)\mu\lambda} \leq \frac{1}{\frac{\lambda}{a^2} + 2\mu}$, then $N = 1$ is the optimal replacement policy,

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i.e. $N^* = 1$.

(2) If $\frac{d}{(c_0+c)\mu\lambda} \geq \frac{a^2+a-1}{a(a^2-1)} \frac{1}{2\mu}$, i.e. $\frac{d}{(c_0+c)\lambda} \geq \frac{1}{2} \frac{a^2+a-1}{a(a^2-1)}$, then $C_1(N)$ is nonincreasing in N; the minimum of $C_1(N)$ can be obtained when $N \to \infty, i.e. N^* = \infty$.

(3) If $\frac{\lambda}{a^2} + 2\mu \leq \frac{d}{(c_0+c)\mu\lambda} \leq \frac{a^2+a-1}{a(a^2-1)} \frac{1}{2\mu}$, then the optimal replacement policy exists for finite N, and the optimal policy is unique.

Proof.

$$C_1(N+1) - C_1(N) = \frac{-(c_0+c)E\,T(N+1)\ +2d}{ET(N+1)\ +\ 2ES(N+1)} - \frac{-(c_0+c)E\,T(N)\ +2d}{ET(N)\ +\ 2ES(N)}$$

$$= \frac{2d(ET(N) - ET(N+1) + 4d(ES(N) - ES(N+1))}{(ET(N+1)\ +\ 2ES(N+1))\,(ET(N)\ +\ 2ES(N))}$$

$$+ \frac{2(c_0+\ c)\big(ES(N+1)ET(N) - ES(N)\,ET(N+1)\big)}{(ET(N+1)\ +\ 2ES(N+1))\,(ET(N)\ +\ 2ES(N))}$$

$$= \frac{2(c_0+\ c)\mu\lambda\big(\sum_{n=1}^{N} a^{2^n} - \sum_{n=1}^{N-1} b^{2^n}\big) - 2d\big(\lambda b^{2^{N-1}} + 2\mu a^{2^N}\big)}{(ET(N+1)\ +\ 2ES(N+1))\,(ET(N)\ +\ 2ES(N))\, a^{2^N}\, b^{2^{N-1}}}.$$

According to the numerator of $C_1(N+1) - C_1(N)$, let

$$g(N) =\ 2(c_0+\ c)\ \mu\lambda\ \big(\sum_{n=1}^{N} a^{2^n} - \sum_{n=1}^{N-1} b^{2^n}\big) - 2d\big(\lambda b^{2^{N-1}} + 2\mu a^{2^N}\big). \qquad (7)$$

Now, an auxiliary function is considered through formula (7):

$$h(N) = \frac{\sum_{n=1}^{N} a^{2^n} - \sum_{n=1}^{N-1} b^{2^n}}{\lambda b^{2^{N-1}} + 2\mu a^{2^N}} = \frac{\sum_{n=0}^{N-1} \left(\frac{1}{a}\right)^{2^n} - \frac{1}{a^{2^N}} \sum_{n=1}^{N-1} b^{2^n}}{\frac{\lambda b^{2^{N-1}}}{a^{2^N}} + 2\mu}$$

Because the denominator of $C_1(N+1) - C_1(N)$ is always positive, obviously, the sign of $C_1(N+1) - C_1(N)$ is the same as the sign of its numerator. Thus, the following result is straightforward.

$$C_1(N+1) \gtreqless C_1(N) \Leftrightarrow g(N) \gtreqless 0 \Leftrightarrow h(N) \gtreqless \frac{d}{(c_0+c)\mu\lambda} \qquad (8)$$

By virtue of Lemma 1, $h(N)$ is the nondecreasing in N. Therefore

$$h(1) \leq\ h(N) \leq\ \lim_{N\to\infty} h(N), \qquad N\ \geq\ 1.$$

Furthermore

$$h(1) = \frac{1}{\frac{\lambda}{a^2} + 2\mu}, \qquad \lim_{N\to\infty} h(N) = \frac{a^2+a-1}{a(a^2-1)}\,\frac{1}{2\mu}.$$

If $\frac{d}{(c_0+c)\mu\lambda} \leq h(1) = \frac{1}{\frac{\lambda}{a^2}+2\mu}$

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with the help of formula (8), then $C_1(N)$ is nondecreasing for all $N \geq 1$. Thus, the optimal replacement policy will be $N^* = 1$. This means that an optimal replacement policy is to replace the system immediately without any repair as soon as it fails.
If

$$\frac{d}{(c_0 + c)\mu\lambda} \geq \frac{a^2 + a - 1}{a(a^2 - 1)} \frac{1}{2\mu} \quad or \quad \frac{d}{(c_0 + c)\lambda} \geq \frac{1}{2} \frac{a^2 + a - 1}{a(a^2 - 1)},$$

with the help of (8), then $C_1(N)$ is non increasing for all $N \geq 1$; the optimal replacement policy will be $N^* = +\infty$. This means that the optimal policy is to continually repair the system (i.e. two identical components) as it ages without ever replacing it.
If

$$\frac{1}{\frac{\lambda}{a^2} + 2\mu} \leq \frac{d_1}{(c_0 + c)\mu\lambda} \leq \frac{a^2 + a - 1}{a(a^2 - 1)} \frac{1}{2\mu},$$

then there exists finite $N^* = min_{1 \leq N < \infty}\{N \mid g(N) \geq 0\}$ satisfying

$$C_1(N^*) = \min_{1 \leq N < \infty} C_1(N).$$

Thus, (N^*, N^*) is the unique optimal replacement policy of the system. In this way, we complete the proof of Theorem 1.

4.2. Case II: Series system of two non-identical components

By formula (6), we have

$$C(M, N) = \frac{c_1 \frac{ES^{(1)}}{ET^{(1)}} + c_2 \frac{ES^{(2)}}{ET^{(2)}} + d_1 \frac{1}{ET^{(1)}} + d_2 \frac{1}{ET^{(2)}} - c_0}{1 + \frac{ES^{(1)}}{ET^{(1)}} + \frac{ES^{(2)}}{ET^{(2)}}}$$

When M is fixed, $C(M, N)$ is a function of N. If $M = m$, then $C(M, N)$ can be written as

$$\begin{aligned} C_m(N) &= \frac{\alpha + c_2 \frac{ES^{(2)}}{ET^{(2)}} + d_2 \frac{1}{ET^{(2)}}}{\beta + \frac{ES^{(2)}}{ET^{(2)}}} \\ &= c_2 + \frac{-\gamma ET^{(2)} + d_2}{\beta ET^{(2)} + ES^{(2)}} \\ &\triangleq c_2 + \frac{-\gamma ET^{(2)} N + d_2}{\beta ET^{(2)} N + ES^{(2)} N} \end{aligned}$$

where

$$\alpha = -c_0 + c_1 \frac{ES^{(1)}}{ET^{(1)}} + d_1 \frac{1}{ET^{(1)}},$$

$$\beta = 1 + \frac{ES^{(1)}}{ET^{(1)}}, \quad \gamma = c_2 \beta - \alpha$$

Now, the difference of $C_M(N + 1)$ and $C_M(N)$ can be calculated:

$$C_m(N + 1) - C_m(N) = \frac{-\gamma ET^{(2)}(N + 1) + d_2}{\beta ET^{(2)}(N + 1) + ES^{(2)}(N + 1)} - \frac{-\gamma ET^{(2)}(N) + d_2}{\beta ET^{(2)}(N) + ES^{(2)}(N)}$$

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$$= \frac{\gamma(ET^{(2)}(N+1) + ET^{(2)}(N) - ES^{(2)}(N)\ \ ET^{(2)}(N+1))}{\left(\beta ET^{(2)}(N+1) + ES^{(2)}(N+1)\right)(\beta ET^{(2)}(N) + ES^{(2)}(N))}$$
$$+ \frac{d_2\beta(ET^{(2)}(N) - ET^{(2)}(N+1)) + d_2(ES^{(2)}(N) - ES^{(2)}(N+1))}{\left(\beta ET^{(2)}(N+1) + ES^{(2)}(N+1)\right)\ (\beta ET^{(2)}(N) + ES^{(2)}(N))}$$
$$= \frac{\delta(N)}{\left(\beta ET^{(2)}(N+1) + ES^{(2)}(N+1)\right)(\beta ET^{(2)}(N) + ES^{(2)}(N))}$$

where

$$\delta(N) = \left(\frac{1}{b_2}\right)^{2^{N-1}} \left(\frac{1}{a_2}\right)^{2^{N-1}} \left\{\gamma\mu_2\lambda_2\left[\sum_{n=1}^{N} a_2^{2^n} - \sum_{n=1}^{N-1} b_2^{2^n}\right] - d_2(\beta\lambda_2 b_2^{2^{N-1}} + \mu_2 a_2^{2^N})\right\}.$$

Similarly, let

$$B(N) = \frac{\gamma\lambda_2\mu_2\, I(N)}{d_2(\beta\lambda_2 b_2^{2^{N-1}} + \mu_2 a_2^{2^N})}$$

where

$$I(N) = \sum_{n=1}^{N} a_2^{2^n} - \sum_{n=1}^{N-1} b_2^{2^n}$$

Because the denominator of $C_m(N+1) - C_m(N)$ is always positive, the sign of $C_m(N+1) - C_m(N)$ is the same as the sign of its numerator. Therefore, the following result is clear:

$$C_m(N+1) \gtreqless C_m(N) \iff \delta(N) \gtreqless 0 \iff B(N) \gtreqless 1 \qquad (9)$$

It shows that the monotonicity of $C_m(N)$ is determined by the value of $B(N)$. Similarly, the difference between $B(N+1)\ and\ B(N)$ is

$$B(N+1) - B(N) = \frac{-\gamma\lambda_2\mu_2 I(N+1)}{d_2\left(\beta\lambda_2\, b_2^{2^N} + \mu_2 a_2^{2^{N+1}}\right)} - \frac{\gamma\lambda_2\mu_2 I(N)}{d_2\left(\beta\lambda_2\, b_2^{2^{N-1}} + \mu_2 a_2^{2^N}\right)}$$
$$= \frac{\gamma\lambda_2\mu_2[I(N+1)\left(\beta\lambda_2\, b_2^{2^{N-1}} + \mu_2 a_2^{2^N}\right) - I(N)(\beta\lambda_2\, b_2^{2^N} + \mu_2 a_2^{2^{N+1}})}{d_2(\beta\lambda_2\, b_2^{2^N} + \mu_2 a_2^{2^{N+1}})\ (\beta\lambda_2\, b_2^{2^{N-1}} + \mu_2 a_2^{2^N})}$$
$$= \frac{\gamma\lambda_2\mu_2\ [\beta\lambda_2\, b_2^{2^{N-1}}\left(I(N+1) - b_2\, I(N)\right) + \mu_2 a_2^{2^N}\left(I(N+1) - a_2 I(N)\right)]}{d_2(\beta\lambda_2\, b_2^{2^N} + \mu_2 a_2^{2^{N+1}})\ d_2(\beta\lambda_2\, b_2^{2^{N-1}} + \mu_2 a_2^{2^N})}$$
$$\geq 0$$

The holding of the last inequality results because

$$I(N+1) - b_2 I(N) = \left(\sum_{n=1}^{N+1} a_2^{2^n} - \sum_{n=1}^{N} b_2^{2^n}\right) - b_2\left(\sum_{n=1}^{N} a_2^{2^n} - \sum_{n=1}^{N-1} b_2^{2^n}\right)$$
$$= (1 - b_2)\left(\sum_{n=1}^{N} a_2^{2^n} - \sum_{n=1}^{N-1} b_2^{2^n}\right) - \left(a_2^{2^{N+1}} - b_2^{2^N}\right) \geq 0.$$

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$$I(N+1) - a_2 I(N) = \left(\sum_{n=1}^{N+1} a_2^{2^n} - \sum_{n=1}^{N} b_2^{2^n}\right) - a_2\left(\sum_{n=1}^{N} a_2^{2^n} - \sum_{n=1}^{N-1} b_2^{2^n}\right)$$
$$= (a_2 - 1)\sum_{n=1}^{N-1} b_2^{2^n} + (a_2 - b_2^{2^N}) \geq 0$$

Thus, the following lemma can be obtained.

Lemma 2. $B(N)$ is nondecreasing in N.

According to the formula (9) and Lemma 2, an analytic expression for an optimal policy is obtained through the study of $B(N)$. Thus, we have the following theorem.

Theorem 2. For fixed $M = m$, the optimal replacement policy $N^* \triangleq N_m^*$ can be determined by
$$N_m^* = \min\{N | B(N) \geq 1\}. \quad (10)$$
Furthermore, if $B(N_m^*) > 1$, then the optimal policy N_m^* is unique.

Proof. Because $B(N)$ is nondecreasing in N, there exists an integer N_m^* such that
$$B(N) \geq 1 \Leftrightarrow N \geq N_m^*$$
and $B(N) < 1 \Leftrightarrow N < N_m^*$.

Then, N_m^* is the minimum integer satisfying (10), and the policy N_m^* is an optimal replacement policy. Thus, if $B(N_m^*) > 1$, then the optimal replacement policy also uniquely exists.

In other words, for fixed m, N_m^* can be found such that $C_m(N_m^*)$ is minimized; namely when $M = 1, 2, \ldots, m, \ldots$, $N_1^*, N_2^*, \ldots, N_m^*, \ldots$ can be found respectively such that the corresponding $C_1(N_1^*) = C(1, N_1^*) = C_2(N_2^*) = C(2, N_2^*), \ldots, C_m(N_m^*) = C(m, N_m^*), \ldots$
are minimized.

Similarly, for every fixed $N = n$, M_n^* can be found such that $C_n(M_n^*)$ is minimized; Likewise when $N = 1, 2, \ldots, n, \ldots$, $M_1^*, M_2^*, \ldots, M_n^*, \ldots$ can be found respectively such that the corresponding $C_1(M_1^*) = C(M_1^*, 1) = C_2(M_2^*) = C(M_2^*, 2), \ldots, C_m(M_m^*) = C(M_m^*, n), \ldots$
are minimized.

Because the total lifetime of the series repairable system is limited, the minimum of $C(M, N)$ exists. Thus, an optimal replacement policy (M^*, N^*) can be determined such that $C(M^*, N^*)$ is minimized based on $C(m, M_m^*)$ and $C(M_m^*, n)$, $m, n = 1, 2, \ldots$.

A numerical example

In this section, we provide a numerical example for the system to illustrate some theoretical results. Let $a_1 = 1.05, b_1 = 0.95, \lambda_1 = 100, \mu_1 = 20, c_1 = 20$, $d_1 = 2000$, $a_2 = 1.08$, $b_2 = 0.92$, $\lambda_2 = 150, \mu_2 = 30$, $c_2 = 25$, $d_2 = 2400, c_0 = 30$. Substituting the above values into Eq. (6), the minimum of the long-run expected cost per unit time of the system can be obtained, i.e. $C(M, N) = -6.62365$, and the corresponding optimal replacement policy is (M^*, N^*) = (4, 3). Obviously, we can also see from Fig. 1 that $(M^*, N^*) = (4, 3)$ is the unique optimal policy, and $C(M^*, N^*) = C(4, 3) = -6.62365$ is the unique minimum of $C(M, N)$. More calculation results can be found in Table 1.

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According to Definition 2, a_i and b_i $(i = 1, 2)$ are respectively the ratios of the stochastically decreasing partial product process $\{X_n^{(i)}, n = 1, 2, \ldots; i = 1, 2\}$ and the stochastically increasing partial product process $\{Y_n^{(i)}, n = 1, 2, \ldots; i = 1, 2\}$. To study the influence of these ratios of partial product processes on the optimal policy, we tabulate the optimal replacement policy (M^*, N^*) and the minimum $C(M^*, N^*)$ of the long-run expected cost per unit time of the system for different values of $a_i > 1$ and $0 < b_i < 1$ $(i = 1, 2)$ in Table 1.

Table 1

M/N	1	2	3	4	5	6	7
1	8.28	2.691817	1.573867	1.792068	3.10974	6.041543	12.68718
2	0.269972	-4.01537	-4.73338	-4.25625	-2.64302	0.825623	8.916494
3	-1.75767	-5.69038	-6.31001	-5.78228	-4.12335	-0.56766	7.828709
4	-2.27226	-6.03684	-6.62365	-6.09785	-4.4632	-0.94878	7.438882
5	-1.87024	-5.52081	-6.10617	-5.62019	-4.07042	-0.70898	7.428771
6	-0.46723	-3.98781	-4.5934	-4.19408	-2.81596	0.229675	7.78552
7	2.682842	-0.55105	-1.18404	-0.94861	0.092563	2.497689	8.787972
8	10.45688	8.304447	7.772399	7.741311	8.107985	9.111658	12.20042
9	19.26407	19.06337	19.00351	18.98337	18.98648	19.01675	19.14422
10	19.99888	19.99857	19.99847	19.99844	19.99844	19.99848	19.99865

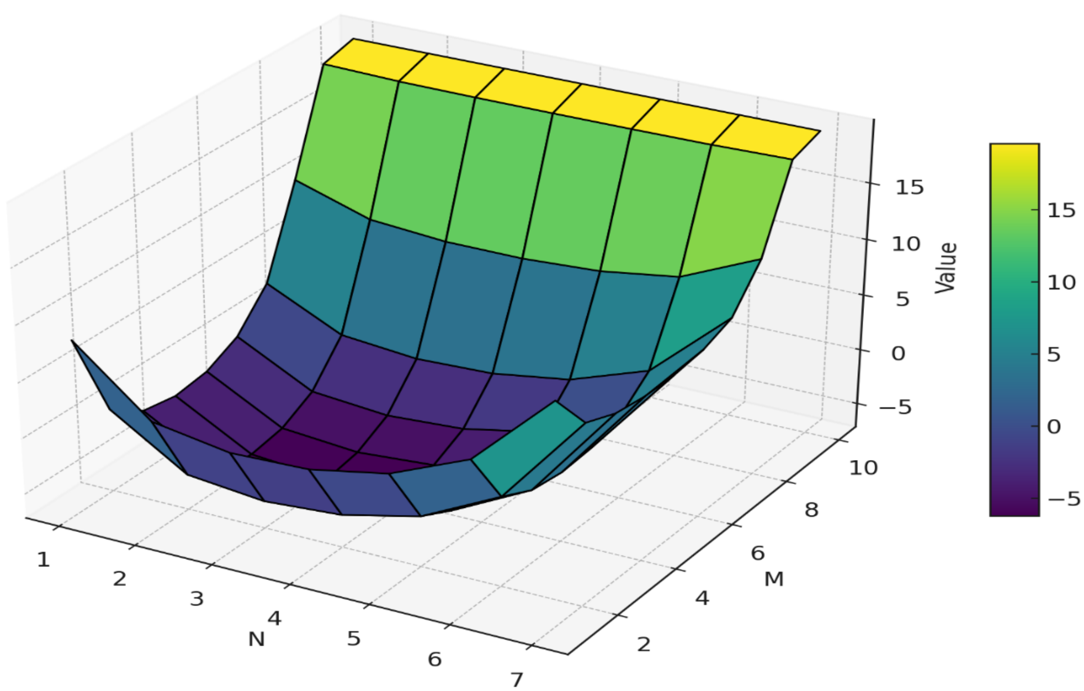

Figure 1

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