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BI-TERNARY SEMI NEAR RINGS

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Abstract: In this paper we made an attempt to study the algebraic structure of bi ternary semi near ring, bi ternary sub semi near ring, ideals in bi ternary semi near ring and some of its properties with counter examples. Also, we are exploring on quasi-ideal, bi-ideals, strong ideals are defined and their basic characteristics.

KEY POINTS: Bi ternary semi near ring, right, left, lateral ideals, quasi-ideals, strong irreducible ideals.

INTRODUCTION

Near-semiring is the common generalization of near-ring and semiring. Near-semiring R is an algebraic structure equipped with two binary operations "+" and "." such that (R, +) is a monoid, (R, **.**) is a semigroup, and both structures are linked through a single (left or right) distributive law with 0 as a one-sided absorbing element. In addition, if $0 \in R$ such that $a + 0 = 0 + a = a$, $a.0 = 0.a = 0$, then we call R a zero-symmetric near-semiring (or semi-near ring). Near-semirings established naturally from the mappings on monoid to itself under component-wise addition and composition of mappings. Several algebraists have considered near-semirings in different aspects. Due to having strong relationships of near-semirings with different types of algebras, recently the authors [7] introduced the notion of balanced near-semirings. On the other hand, ideals have their own importance in algebra, particularly when we study rings with two binary operations. In general, ideals in a near-semiring do not match with the standard ideals of rings (resp., near rings, semi-rings) and hence numerous results in rings (resp., near rings, semirings) theory have no similarities in near-semirings using merely ideals. Thus, the classified notion of ideal i.e., S -ideal [2] has been introduced in semi-near ring (zero symmetric near semi-rings) theory. Furthermore, special types of prime ideals of semi-near rings have been explored in [12]. Lehmer. D. N. introduce the theory of ternary algebraic system in1932. Latter Dutta. T. K and Kar. S initiated an idea of ternary semiring.

Bi-groups are especially helpful because they offer answers for a problem that impact all groups. The bi group research was conducted from 1994 to 1996. Maggu [14, 15]was the first one

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developed bi-group theory. This idea was extended by Vasantha Kandasamy and Meiyappan. We are extended this to ternary system.

The study of semi near ring was found by Willey. G. Van Hoorn and Van Rootseloar. B in 1967. Latter different people yield semi near ring and numerous and elegant properties. The ternary semi near ring is the generalization of semi near ring.

Our main purpose of this paper is to introduce the notation of bi ternary semi near ring and derived many examples. We also discussed about bi ternary sub semi near ring, ideal in bi ternary semi near rings and derived their characters.

1. Preliminaries

Definition 1.1: A ternary semi near ring is a non-empty set S together with a binary operation called addition and ternary multiplication denoted by juxtaposition such that

(i) (S, +) is a semigroup

(ii) (S, [ ]) is a ternary semigroup

(iii) $(z + w)xy = zxy + wxy, \forall\, x, y, z, w \in S.$

Definition 1.2: Let N be a non-empty set. We call $(N, +, \cdot)$ is a bi near-ring, if $N = N_1 \cup N_2$ where N_1 and $N2$ are proper subsets of N satisfying the following conditions:

(i) $(N_1, +, \cdot)$ is a near ring

(ii) $(N_2, +, \cdot)$ is a ring.

We say that even if both $(N_i, +, \cdot)$ are right near rings still we call $(N, +, \cdot)$ to be a bi near-ring.

Example 1.3: Let $(N, +, \cdot)$ be a non-empty set. $N = Z \cup Z12$ where Z is a near ring and $Z12$ is a ring. Thus, $(N, +, \cdot)$ is a bi near-ring.

Example 1.4: Let $(N, +, \cdot)$ be a non-empty set. $N = Q \cup Z12$ where Q is a near ring and $Z12$ is a ring. Thus, $(N, +, \cdot)$ is a bi near-ring.

2.Bi-ternary semi near ring

Definition 2.1: Let $\mathfrak{N}$ be a nonempty set. We say $(\mathfrak{N}, +, \cdot)$ is a BTSNR (bi ternary semi near ring), if $\mathfrak{N} = \mathfrak{N}_1 \cup \mathfrak{N}_2$ where $\mathfrak{N}1$ and $\mathfrak{N}2$ are proper subsets of $\mathfrak{N}$ satisfying the following axioms:

(i) $(\mathfrak{N}_1, +, \cdot)$ is a ternary semi near ring

(ii) $(\mathfrak{N}_2, +, \cdot)$ is a ternary semi ring.

We can also say even if both $\mathfrak{N}_1, \mathfrak{N}_2$ are right ternary semi near rings still we call $(\mathfrak{N}, +, \cdot)$ is a BTSNR.

Example 2.2: If $\mathfrak{N} = \mathfrak{N}_1 \cup \mathfrak{N}_2$ where $\mathfrak{N}_1$ is ternary semi near ring.

$(\mathfrak{N}_1, +_6)$ $(\mathfrak{N}_1, .)$

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+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	1	5	3	1	5
2	0	2	4	0	2	4
3	3	3	3	3	3	3
4	0	4	2	0	4	2
5	3	5	1	3	5	1

$\mathrm{N}_2 = \left(Q\sqrt{2}, +, \cdot\right)$ is ternary semi ring.

Example 2.3: If $\mathrm{N} = \mathrm{N}_1 \cup \mathrm{N}_2$ where

$\mathrm{N}_1 = \{0, a, b, c\}$ then $(\mathrm{N}_1, +, \ .)$ is ternary semi near ring

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	C	0	a
c	c	B	A	0

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	0	0
c	0	c	b	c

$\mathrm{N}_2 = (\boldsymbol{Z_0^-}, +, [\quad])$ is ternary semi ring.

Example2.4: If $\mathrm{N} = \mathrm{N}_1 \cup \mathrm{N}_2$ where N_1 is the polynomial functions with real coefficients usual addition and ternary multiplication is circle operation it satisfy the right distributive. $(\mathrm{N}_1, +, .)$ is ternary semi near ring. $\mathrm{N}_2 = \{0,1,2,3,4,5, \ +_6, \ \times_6\}$ is ternary semi ring.

Definition 2.5: Let N be a nonempty set. We say $(\mathrm{N}, +, \cdot\ )$ is a BTSRR (bi ternary semi rare ring). If $\mathrm{N} = \mathrm{N}_1 \cup \mathrm{N}_2$ where $\mathrm{N}1$and $\mathrm{N}2$ are proper subsets of N satisfying the following conditions:

(i) $(\mathrm{N}_1, +, \ \cdot)$ is a ternary semi rare ring(it obey the left distributive property)

(ii) $(\mathrm{N}_2, +, \ \cdot)$ is a ternary semi ring.

We can also say even if both $\mathrm{N}_1, \mathrm{N}_2$ are ternary semi rare rings then the combined structure $(\mathrm{N}, +\ ,\cdot\ )$ is still called a BTSRR.

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Example 2.6: If $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$ where $\mathcal{N}_1 = \{(a,b)/\, a,b \in R\}$ the addition is defined as $(a,b) + (c,d) = (a+c, b+d)$ and ternary multiplication is $(a,b)(c,d)(e,f) = (ace, bce + de + f)$ is ternary semi rare ring. $\mathcal{N}_2 = (\, 3\boldsymbol{Z}\,, +, [\quad])$

Definition 2.7: Let $\mathcal{N}$ be a nonempty set. We say $(\mathcal{N}, +, \cdot\,)$ is a BTSDR (bi ternary semi dual ring). If $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$ where $\mathcal{N}1$and $\mathcal{N}2$ are proper subsets of $\mathcal{N}$ satisfying the following conditions:

(i) $(\mathcal{N}_1, +, \cdot)$ is a ternary semi dual ring(obey the right & left distributive properties)

(ii) $(\mathcal{N}_2, +, \cdot)$ is a ternary semi ring.

We can also say even if both $\mathcal{N}_1, \mathcal{N}_2$ are ternary semi dual rings then the combined structure

$(\mathcal{N}, +, \cdot)$ is still called a BTSDR.

Definition 2.8: A BTSNR $\mathcal{N}$ is said to have an absorbing zero if there exists an element $0 \in \mathcal{N}$ such that $(i)\ x + 0 = 0 + x = x$ $(ii)\ 0xy = x0y = xy0 = 0,\ \forall\, x,\ y \in \mathcal{N}$.

Example 2.9: Let $\mathcal{N}$ be a nonempty set, if $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$ where $\mathcal{N}_1 =$ set of all negative integers with zero and $\mathcal{N}_2 = (2Z, +, [\,])$ then we say $(\mathcal{N}, +, \cdot\,)$ is a BTSNR.

Note 2.10: A BTSNR need not be a BTSR.

Definition 2.11: Let $\mathcal{N}$ be a *BTSNR*. A non-empty subset R of $\mathcal{N}$ is said to be a *BTSSNR* (bi ternary sub semi near ring) *(BTSSRR)*, if $R = R_1 \cup R_2$ itself a BTSNR under same operations of $\mathcal{N}$.

Note 2.12: A non-empty sub set $R = R_1 \cup R_2$ of *BTSNR* $\mathcal{N}$ *(BTSRR)* is a *BTSSNR (BTSSRR)* (bi ternary sub semi near ring) if and only if $R + R \subseteq R$ and $RRR \subseteq R$.

Theorem 2.13: The intersection of two *BTSSNRs (BTSSRR)* of *BTSNR* $\mathcal{N}$ *(BTSRR)* is a *BTSSNR*.

Proof: Let $\mathcal{N}$ be a *BTSNR*, R_1 and R_2 be *BTSSNR* of $\mathcal{N}$.

Since $0 \in R_1$ and $0 \in R_2$ then $0 \in R_1 \cap R_2$ thus $R_1 \cap R_2 \neq \emptyset$.

Let $x,\ y \in R_1 \cap R_2$. Then $x, y \in R_1$ and $x,\ y \in R_2$.

Since R_1 and R_2 are two *BTSSNRs* of $\mathcal{N}$, we have $x + y \in R_1$ and $+y \in R_2 \Longrightarrow x + y \in R_1 \cap R_2$.

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Similarly, if $x,\ y,\ z \in R_1 \cap R_2$, then $x,\ y,\ z \in R_1$ and $x,\ y,\ z \in R_2$.

Since R_1 and R_2 are two *BTSSNRs* of Ɲ, we get $xyz \in R_1$ and $xyz \in R_2 \implies xyz \in R_1 \cap R_2$.

So, $x + y \in R_1 \cap R_2$ and $xyz \in R_1 \cap R_2$.

Hence, $R_1 \cap R_2$ is *(BTSSRR) BTSSNR*.

Theorem 2.14: The intersection of any family of *BTSSNRs(BTSSRR)* of *BTSNR* Ɲ *(BTSRR)* is the *BTSSNR (BTSSRR)* of Ɲ.

Proof: Let $\{R_\alpha\}_{\alpha\in\Delta}$ be a family of *BTSSNR* of Ɲ, and $R = \bigcap_{\alpha\in\Delta} R_\alpha$.

Let $a, b, c \in R \Rightarrow a, b, c \in \bigcap_{\alpha\in\Delta} R_\alpha \Rightarrow a, b, c \in R_\alpha \ \forall\ \alpha \in \Delta$

$a, b, c \in R_\alpha$, R_α is a *BTSSNR 0f* Ɲ $\Rightarrow a + b \in R_\alpha$ and $abc \in R_\alpha$

Now $\Rightarrow a + b \in R_\alpha$ and $abc \in R_\alpha$, $\forall \alpha \in \Delta \Rightarrow a + b \in \bigcap_{\alpha\in\Delta} R_\alpha$ and $abc \in \bigcap_{\alpha\in\Delta} R_\alpha$

Then $a + b \in R$ and $abc \in R$. Therefore R is a *BTSSNR* of Ɲ.

Definition 2.15: Let Ɲ be a *BTSNR (BTSRR)*. A non-empty subset I of Ɲ is said to be a left ideal of *BTSNR (BTSRR)*, if it holds the following conditions

$(i)\ i + j \in I$ $\qquad$ $(ii)\ n_1 n_2 i \in I \ \forall\ i, j \in I$ & $n_1, n_2 \in$ Ɲ .

Similarly, we define the lateral (right) ideal of Ɲ, if it holds the following conditions

$i) i + j \in I$ $ii)$ $n_1 i n_2 \in I$ ($i n_1 n_2 \in I$) $\forall\ i,\ j \in I$ & $n_1,\ n_2 \in$ Ɲ,

Definition 2.16: An element x in a *BTSNR* Ɲ *(BTSRR)* is said to be an idempotent element if $x^3 = x$.

A *BTSNR* Ɲ *(BTSRR)* is said to be idempotent, if each element of Ɲ is an idempotent element. The set of all idempotent elements of Ɲ is denoted by E.

Definition 2.17: Let Ɲ be a *BTSNR (BTSRR)*. Then(Ɲ) $= \{a \in$ Ɲ, $xay = axy = yax\ \forall x$, $y \in$ Ɲ$\}$ is called center of Ɲ.

Definition 2.18: Let Ɲ be a *BTSNR (BTSRR)*. Ɲ is called normal if xyƝ $=$ Ɲxy, for all x, $y \in$ Ɲ.

Theorem 2.19: Let Ɲ be a *BTSNR (BTSRR)*. Ɲ is multiplicatively cancellative (MC) if one of the following criteria holds

(i) Ɲ is (left or right) of MC

(ii) Ɲ is lateral MC

(iii) $uxv = uyv \Longrightarrow x = y$ for all $u, v \in$ Ɲ.

Proof: First $(\boldsymbol{i}) \Longrightarrow (\boldsymbol{ii})$:

Let $uxv = uyv \Rightarrow u(uxv)v = u(uyv)v \Rightarrow uu(xvv) = uu(yvv) \Rightarrow xvv = yvv \Rightarrow x = y$

Obviously $(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{iii})$

(iii) ⟹ (i):

Let $uvx = uvy \Rightarrow (uvx)vu = (uvx)vu \Rightarrow vxv = vyv \Rightarrow x = y$

This show that the left MC holds in Ɲ. Similarly, we show that right MC holds inƝ.

Definition 2.20: An element x in a $BTSNR$ Ɲ (BTSRR) is called a regular element if there exists an element y in Ɲ such that $xyxzx = x$. If $BTSNR$ Ɲ (BTSRR) is regular if all of its elements are regular.

Theorem 2.21: If every lateral ideal of a regular $BTSNR$ Ɲ *(BTSRR)* is a regular.

Proof: Let I be a lateral ideal of BTSNRƝ. Then each $x \in I$, there exists $y \in$ Ɲ

Such that $xyx = x$

Now $x = xyx = xyxyx = x(yxy)x = xzx \in I \Longrightarrow I$ is a regular set.

Theorem 2.22: Let Ɲ be a regular BTSNR *(BTSRR)*. For every x in Ɲ then xy and yx are idempotents.

Proof: Let Ɲ be a regular bi ternary semi near ring.

For every x in Ɲ there exists y in Ɲ such that $x = xyx$

Consider $(xy)^3 = (xy)(xy)(xy) = (xyx)(yxy) = x(yxy) = xy$

$\Longrightarrow (xy)^3 = xy$

Therefore xy is an idempotent.

Similarly,

$(yx)^3 = (yx)(yx)(yx) = (yxy)(xyx) = yx$

So, yx is an idempotent.

Hence, xy and yx are idempotent.

3. Prime-Ideal, Quasi-Ideal, Bi-Ideal of Bi-Ternary semi near ring

Definition 3.1: Let U be an additive sub group of Ɲ. U is the quasi-ideal of S if UƝƝ $+$

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$(\mathrm{N}U\mathrm{N} \cap \mathrm{N}\mathrm{N}U\mathrm{N}\mathrm{N}) + \mathrm{N}\mathrm{N}U \subseteq U$.

Theorem 3.2: The intersection of an arbitrary collection of quasi-ideals is also quasi-ideal in BTSNR(BTSRR) N.

Definition 3.3: A proper quasi-ideal U of N is said to be a prime quasi-ideal of N if $XYZ \subseteq U \Rightarrow X \subseteq U$ or $Y \subseteq U$ or $Z \subseteq U$, for all X, Y, Z are quasi-ideals in$\mathrm{N}$.

Definition 3.4: A proper quasi-ideal U of N is said to be semi prime quasi-ideal of N if $X^3 \subseteq U \Rightarrow X \subseteq U$ where X is a quasi- ideal of N.

Theorem 3.5: Let U be a quasi-ideal of $BTSNR$ N*(BTSRR)*. If U is a prime quasi-ideal, then, U is a left or right or lateral ideal of N.

Proof: Let U be a prime quasi-ideal of $BTSNR$ N.

Consider $(U\mathrm{N}\mathrm{N})(\mathrm{N}U\mathrm{N} + \mathrm{N}\mathrm{N}U\mathrm{N}\mathrm{N})(\mathrm{N}\mathrm{N}U) \subseteq U\mathrm{N}\mathrm{N} \cap (\mathrm{N}U\mathrm{N} + \mathrm{N}\mathrm{N}U\mathrm{N}\mathrm{N}) \cap \mathrm{N}\mathrm{N}U \subseteq U$.

Since, U is a prime quasi-ideal, $U\mathrm{N}\mathrm{N} \subseteq U$ or $\mathrm{N}U\mathrm{N} + \mathrm{N}\mathrm{N}U\mathrm{N}\mathrm{N} \subseteq U$ or $\subseteq U$.

Therefore, U is a right or lateral or left ideal of N.

Theorem 3.6: Let U be quasi-ideal of $BTSNR$ N. U is a prime quasi-ideal $\Leftrightarrow abc \in U$ which implies that $a \in U$ *or* $b \in U$ *or* $c \in U$.

Proof: Suppose U be a prime quasi-ideal of N and $abc \in U$. Then U is an ideal of N(by 3.5). Let $x \in < a >_u < b >_u < c >_u$

Then $\quad x = [(a\mathrm{N}\mathrm{N} \cap \mathrm{N}a\mathrm{N} + \mathrm{N}\mathrm{N}a\mathrm{N}\mathrm{N} \cap \mathrm{N}\mathrm{N}a) + na][(b\mathrm{N}\mathrm{N} \cap \mathrm{N}b\mathrm{N} + \mathrm{N}\mathrm{N}b\mathrm{N}\mathrm{N} \cap \mathrm{N}\mathrm{N}b) + nb][(c\mathrm{N}\mathrm{N} \cap \mathrm{N}c\mathrm{N} + \mathrm{N}\mathrm{N}c\mathrm{N}\mathrm{N} \cap \mathrm{N}\mathrm{N}c) + nc]$

As $abc \in U$ and U is an ideal of N, $x \in \mathrm{N}$ there fore $< a >_u < b >_u < c >_u \subseteq U$

As U is a prime quasi-ideal of N consequently $a \in U$ *or* $b \in U$ *or* $c \in U$.

Converse is true trivially.

Definition 3.7: A BTSSNR B of a BTSNR (BTSRR) N is called a bi –ideal of N is $B\mathrm{N}B\mathrm{N}B \subseteq B$.

Theorem 3.8: Every quasi-ideal of a BTSNR (BTSRR) N is a bi-ideal of N.

Proof: Let Q be the quasi-ideal of N. Then we see that $Q\mathrm{N}Q\mathrm{N}Q \subseteq Q(\mathrm{N}\mathrm{N}\mathrm{N})\mathrm{N} \subseteq Q\mathrm{N}\mathrm{N}$, $Q\mathrm{N}Q\mathrm{N}Q \subseteq \mathrm{N}(\mathrm{N}\mathrm{N}\mathrm{N})Q \subseteq \mathrm{N}\mathrm{N}Q$, and $Q\mathrm{N}Q\mathrm{N}Q \subseteq \mathrm{N}Q\mathrm{N}\mathrm{N}$. So, $Q\mathrm{N}Q\mathrm{N}Q \subseteq \mathrm{N}Q\mathrm{N} +$

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$\mathcal{N}\mathcal{N}Q\mathcal{N}\mathcal{N}$. Consequently, it follows that $Q\mathcal{N}Q\mathcal{N}Q \subseteq Q\mathcal{N}\mathcal{N} \cap (\mathcal{N}Q\mathcal{N} + \mathcal{N}\mathcal{N}Q\mathcal{N}\mathcal{N}) \cap \mathcal{N}\mathcal{N}Q \subseteq Q$. Hence Q is a bi ideal.

Note 3.9: The converse of above theorem is not true.

Proposition 3.10: If L is a bi-ideal of $\mathcal{N}$ and also R is a BTSSNR of $\mathcal{N}$. Then $L \cap R$ is a bi ideal of $\mathcal{N}$.

Proof: Since L is a bi-ideal and R is a BTSSNR.

L and R are additive sub groups of $\mathcal{N}$.

Now $(L \cap R)\mathcal{N}(L \cap R)\mathcal{N}(L \cap R) \subseteq L\mathcal{N}L\mathcal{N}L \subseteq L$, $L$is bi-ideal of $\mathcal{N}$.

Also$(L \cap R)\mathcal{N}(L \cap R)\mathcal{N}(L \cap R) \subseteq [R(RRR)R] \subseteq RRR$, R is BTSSNR of $\mathcal{N}$

Thus $(l \cap R)\mathcal{N}(L \cap R)\mathcal{N}(L \cap R) \subseteq L \cap R$.

Hence $L \cap R$ is a bi-ideal of $\mathcal{N}$.

Theorem 3.11: The intersection of any family of bi ideal of BTSNR (BTSRR) is a bi-ideal.

Definition 3.12: A bi-ideal B of a BTSNR (BTSRR) is called prime bi-ideal if $B_1B_2B_3 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi ideals $B_1,\ B_2,\ B_3$ of $\mathcal{N}$.

Definition 3.13: A bi-ideal B of a BTSNR (BTSRR) is called a strongly prime bi-ideal if $B_1B_2B_3 \cap\ B_2B_3B_1 \cap B_3B_1B_2 \subseteq B\ \Longrightarrow\ B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi ideals $B_1\,, B_2\,, B_3$ of $\mathcal{N}$.

Definition 3.14: A bi-ideal B of a BTSNR (BTSRR) is called a semi prime bi-ideal if $B_1^3 \subseteq B$ implies $B_1 \subseteq B$ for any bi-ideal B_1 of $\mathcal{N}$. 1

Theorem 3.15: Every strongly prime bi ideal of a BTSNR (BTSRR) $\mathcal{N}$ is a prime bi ideal of $\mathcal{N}$.

Proof: Let B be a strongly prime bi ideal of a BTSNR $\mathcal{N}$.

Let $B_1,\ B_2,\ B_3$ be three bi-ideals of $\mathcal{N} \Longrightarrow B_1B_2B_3 \cap\ B_2B_3B_1 \cap B_3B_1B_2 \subseteq B$ then $B_1B_2B_3 \subseteq B$ thus $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence, B is a prime bi ideal of $\mathcal{N}$.

Preposition 3.16: Every prime bi ideal of a BTSNR (BTSRR) $\mathcal{N}$ is a semi prime bi ideal of $\mathcal{N}$.

Theorem 3.17: Let $\mathcal{N}$ be a BTSNR (BTSRR). Let V be a BTSSNR of $\mathcal{N}$. If A is a right ideal, B is a lateral ideal, C is a left ideal of $\mathcal{N}$ such that $ABC\ \subset\ V\ \subseteq\ A\ \cap\ B\ \cap\ C$ then V is a bi ideal

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of ℕ.

Proof: Since V is a BTSSNR ℕ.

Let us consider, $Vℕ Vℕ V \subseteq (A \cap B \cap C)ℕ(A \cap B \cap C)ℕ(A \cap B \cap C)$

$\subseteq A(ℕ Bℕ)C$

$\subseteq ABC \subseteq V$

Therefore, V is a bi ideal of ℕ.

Definition 3.18: Let ℕ be a BTSNR(BTSRR). A nonzero bi ideal B of ℕ is a minimal bi ideal of ℕ if B does not properly contain any nonzero bi-ideal.

Theorem 3.19: Let ℕ be a BTSNR(BTSRR). Let L be a minimal right ideal, M is a minimal lateral ideal, N is a minimal left ideal of ℕ. Then $LMN = 0$ or LMN be a minimal bi ideal of ℕ.

Proof: Let ℕ be a BTSNR, L be a minimal right ideal and M be a minimal lateral ideal. LMN is a BTSSNR of ℕ. Suppose, $LMN \neq 0$ and let, $B = LMN$ then B is a bi ideal of ℕ. Let X be any bi ideal of ℕ and $0 \neq X \subseteq B \subseteq L$. Then $Xℕℕ$ is a right ideal of ℕ and $Xℕℕ \subseteq Bℕℕ = (LMN) \subseteq Lℕℕ \subseteq L$, L is a minimal right ideal of ℕ.

Consequently $Xℕℕ = 0$ *or* $Xℕℕ = L$.

If $Xℕℕ = 0$ then $X = L$ and $LMN = 0$, is a contradiction. Consequently, $Xℕℕ = L$ Similarly, we can show that $ℕXℕ = M$, $ℕℕX = N$.

And $B = LMN = XℕℕℕXℕℕℕX \subseteq XℕXℕX \subseteq X$

$\Longrightarrow B = X$ thus LMN is a minimal bi ideal of ℕ.

Theorem3.20: Any minimal lateral ideal of BTSNR ℕ (BTSRR) is a minimal ideal of ℕ.

Proof: Let M be a minimal lateral ideal of a ℕ. We will show that M is a minimal .

Let $m \in M$, then $ℕ.m.ℕ. + ℕ.ℕ.m.ℕ.ℕ.$ is a lateral ideal of ℕ and $ℕ.m.ℕ. + ℕ.ℕ.m.ℕ.ℕ. \subseteq ℕ.m.ℕ. + ℕ.ℕ.m.ℕ.ℕ. \subseteq M$. Since M is minimal

We have $M = ℕ.m.ℕ. + ℕ.ℕ.m.ℕ.ℕ.$

Now consider, $Mℕ.ℕ. = (ℕmℕ + ℕ.ℕ.m.ℕ.ℕ.)ℕ.ℕ. \subseteq M$

$ℕ.ℕ.ℕ.M = ℕ.ℕ.(ℕ.m.ℕ. + ℕ.ℕ.m.ℕ.ℕ.) = ℕ.ℕ.(ℕ.m.ℕ.) + ℕ.ℕ.(ℕ.ℕ.m.ℕ.ℕ.)$
$\subseteq ℕ.m.ℕ. + ℕ.ℕ.m.ℕ.ℕ. \subseteq M$

Therefore $ℕℕM \subseteq M$ & $Mℕℕ \subseteq M$. This implies that M both left and right ideal of ℕ.

Consequently M is an ideal of ℕ..

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Now it remains to show that M is minimal.

Let Q be an ideal of Ɲ. Such that $Q \subseteq M$ since Q is an ideal of Ɲ, so it is lateral ideal of Ɲ. .Then $M \subseteq Q$

Hence $Q = M.M$ is a minimal of Ɲ.

Theorem3.21: Any minimal quasi ideal if BTSNR Ɲ is contained in a minimal ideal of Ɲ.

Theorem3.22: Every quasi ideal of BTSNR Ɲ is a BI of BTSNR Ɲ.

4. Irreducible, Strongly Irreducible Bi-Ideal of BTSNR(BTSRR)

Definition 4.1: A bi-ideal B of a BTSNR Ɲ (BTSRR) is an irreducible bi-ideal of Ɲ

if $B_1 \cap B_2 \cap B_3 = B \Longrightarrow$ either $B_1 = B$ or $B_2 = B$ or $B_3 = B$ for any bi-ideals B_1, B_2, B_3 of Ɲ.

Definition 4.2: A bi-ideal B of a BTSNR Ɲ (BTSRR) is a strongly irreducible bi-ideal of Ɲ if $B_1 \cap B_2 \cap B_3 \subseteq B$ implies either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi-ideals B_1, B_2, B_3 of Ɲ.

Proposition 4.3: Every strongly irreducible semi prime bi-ideal of Ɲ is a strongly prime bi-ideal of Ɲ.

Proof: Let B be a strongly irreducible semi prime bi-ideal of a bi-ternary semi near ring Ɲ.

Let B_1, B_2, B_3 be three bi-ideals of Ɲ such that

$B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq \boldsymbol{B}$

Then we have to show that either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. As $B_1 \cap B_2 \cap B_3 \subseteq B_1$, $B_1 \cap B_2 \cap B_3 \subseteq B_2, B_1 \cap B_2 \cap B_3 \subseteq B_3 \Longrightarrow (B_1 \cap B_2 \cap B_3)^3 \subseteq B_1B_2B_3$, $(B_1 \cap B_2 \cap B_3)^3 \subseteq B_2B_3B_1$,

$(B_1 \cap B_2 \cap B_3)^3 \subseteq B_3B_1B_2$ thus

$(B_1 \cap B_2 \cap B_3)^3 \subseteq B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq \boldsymbol{B} \Longrightarrow B_1 \cap B_2 \cap B_3 \subseteq B$

Because B is a semi prime bi-ideal of Ɲ. Thus, $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$

Hence B is a strongly prime bi-ideal of Ɲ.

Theorem 4.4: For BTSNR Ɲ (BTSRR) the following assertions are equivalent:

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i) The set of bi-ideals of Ɲ is totally order set by inclusion.

ii) Each bi-ideal of Ɲ is strongly irreducible.

iii) Each bi-ideal of Ɲ is irreducible.

Proof: **i) ⇒ ii)** Let the set of bi ideals of Ɲ is totally ordered by inclusion.

To show that each bi-ideal is strongly irreducible. Let B be an arbitrary bi-ideal of Ɲ and B_1, B_2, B_3 be three bi-ideals of Ɲ such that $B_1 \cap B_2 \cap B_3 \subseteq B$. Since that the set of bi-ideals Ɲ is totally ordered by inclusion.

Then $B_1 \cap B_2 \cap B_3 = B_1$ or B_2 or B_3

Thus $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. So, B is strongly irreducible. Hence, each bi-ideal of Ɲ is strongly irreducible.

ii) ⇒ iii) Let each bi-ideal of Ɲ be strongly irreducible. To show that each bi-ideal of Ɲ is irreducible.

Let B be an arbitrary bi-ideal of Ɲ and B_1, B_2, B_3 be any bi-ideals of Ɲ such that $B_1 \cap B_2 \cap B_3 = B$. Implies $B \subseteq B_1$, $B \subseteq B_2$, $B \subseteq B_3$. On the other hand, B is strongly irreducible.

Thus $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ hence $B_1 = B$ or $B_2 = B$ or $B_3 = B$ So B is irreducible.

Hence each bi-ideal of Ɲ is irreducible.

iii) ⇒ i) Each bi-ideal of Ɲ is irreducible. To show that the set of bi-ideals of Ɲ is totally order set by inclusion. Let B_1, B_2 be any bi-ideals of Ɲ. Then $B_1 \cap B_2$is also bi -ideal of Ɲ. And so is irreducible bi-ideal ofƝ. Since, $B_1 \cap B_2 \cap Ɲ = B_1 \cap B_2 \Longrightarrow B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$ or $B_1 \cap B_2 = Ɲ \Longrightarrow$ either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$ or $B_1 = B_2 = Ɲ$. Hence, the set bi-ideals of Ɲ is totally ordered by inclusion.

5. 3-prime ideals

Corollary 5.1: A proper ideal P of a CBTSNR Ɲ (BTSRR) is prime if and only if $abc \in P$ implies that

$a \in P, b \in P, c \in P \ \forall a, b, c \in$ Ɲ.

Definition 5.2: A Proper Ideal Q of a BTSNR Ɲ (BTSRR) is called a semi prime of Ɲ if $I^3 \subseteq Q$ $\Longrightarrow$ that $I \subseteq Q$ for any ideal I of Ɲ.

Corollary 5.3: A Proper Ideal Q of a CBTSNR Ɲ (CBTSRR) is called a semi prime of Ɲ if $x^3 \subseteq Q$ implies that $x \in Q$ for any element x of Ɲ.

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Definition 5.4: Let Ɲ be a BTSNR(BTSRR) and A be an ideal ofƝ. The radical of A denoted by $Rad(A)$, is defined to be the intersection of all the prime ideals of Ɲ each of which contains A. In a Ɲ, $Rad(A) = \{a \in$ Ɲ$: a^{2n+1} \in A$ for some positive integer $n\}$.

Definition 5.5: A proper ideal P of a CBTSNR Ɲ (CBTSRR) is called primary if for any $a, b, c \in$ Ɲ , $abc \in P \Rightarrow a \in P \; or \; b \in P \; or \; a^{2n+1} \in P$ for some positive integer$n$. An ideal I of a CBTSNR Ɲ (CBTSRR) is called quasi primary if $Rad(I)$ is prime.

Definition 5.6: An ideal I of a BTSNR Ɲ (BTSRR) is called a 3-prime ideal if for any $x, y, z \in$ Ɲ, $xyz \in I$ implies that $x^3 \in I$ or $y^3 \in I$ or $z^3 \in I$.

Theorem 5.7: If an ideal I of a BTSNR Ɲ (BTSRR) is 3-prime as well as semi prime, then I is prime.

Proof: Let $xyz \in I$, for some $x, y, z \in$ Ɲ. Since I is a 3-prime ideal of I, $x^3 \in I$ or $y^3 \in I$ or $z^3 \in I$.

As I is semi prime, we have $x \in I$ or $y \in I$ or $z \in I$. So I is prime.

Proposition 5.8: If I is 3-prime, then $Rad(I)$ is prime.

Proof: Let $xyz \in Rad(I)$ for some $, y, z \in$ Ɲ . Then $(xyz)^{2n+1} \in I$ for some $n \in Z_0^+$. Thus $x^{2n+1}y^{2n+1}z^{2n+1} \in I$, which implies $x^{2n+1} \in I$ or $y^{2n+1} \in I$ or $z^{2n+1} \in I$. So $x \in Rad(I)$ or $y \in Rad(I)$ or $z \in Rad(I)$.

The converse of the above proposition may not be true.

Proposition 5.9: If $Rad(I)$ is prime and $(Rad(I))^3 \subseteq I$.

Proof: Let $Rad(I)$ be a prime and $(Rad(I))^3 \subseteq I$. For $x,\ y, z\ z \in$ Ɲ, Suppose $xyz \in I$. Then $xyz \in Rad(I)$ which implies $x \in Rad(I)$ or $y \in Rad(I)$ or $z \in Rad(I)$. So $x^3 \in (Rad(I))^3 \subseteq I$ or $y^3 \in (Rad(I))^3 \subseteq I$ or $z^3 \in (Rad(I))^3 \subseteq I$. Hence I is 3-prime.

Theorem 5.10: In a regular BTSNR Ɲ(BTSRR) an ideal is prime if and only if it is a 3-prime.

Proof: Clearly if an ideal I is prime then it is 3-prime.

Conversely, let I be a 3-prime ideal and $xyz \in I$. Then $x^3 \in I$ or $y^3 \in I$ or $z^3 \in I$.Suppose $x^3 \in I$.

By regularity, there exist $a, b \in I$ such that $x = xaxbx$, that is, $x = abx^3 \in I$.So, I is prime.

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Proposition 5.11: Let Ɲ be a BTSNR(BTSRR). If an ideal I is a 3-prime ideal of Ɲ, then $(I: a^3: b^3)$ is a 3-prime ideal of Ɲ, where $a, b \in$ Ɲ$/Rad(I)$.

Proof: Let $xyz \in (I: a^3: b^3)$ for some $x, y, z \in$ Ɲ. Then $xyza^3b^3 \in I$. This implies that

$(xab)(yab)(zab) \in I$. Thus, $(xab)^3 = x^3a^3b^3 \in I$ or $(yab)^3 = y^3a^3b^3 \in I$ or $(zab)^3 = z^3a^3b^3 \in I$. Hence

$x^3 \in (I: a^3: b^3)$ or $y^3 \in (I: a^3: b^3)$ or $z^3 \in (I: a^3: b^3)$ and so $(I: a^3: b^3)$ is a 3-prime of Ɲ.

The ternary product of 3-prime ideals may not be 3-prime

Lemma 5.12: Let P be a prime ideal and P', P'' be two ideals with $P \subseteq P'$ and $P \subseteq P''$. Then $PP'P''$ is 3-prime. Moreover, $PP'P''$ is prime if and only if $PP'P'' = P$.

Proof: Let $\in PP'P''$ is 3-prime then $abc \in PP'P'' \subseteq P$ which implies $a \in P$ or $b \in P$ or $c \in P$. So $a^3 \in P^3 \subseteq PP'P''$ or $b^3 \in P^3 \subseteq PP'P''$ or $c^3 \in P^3 \subseteq PP'P''$.

Now, let $PP'P''$ be a prime ideal of Ɲ. Clearly, $PP'P'' \subseteq P$. Consider $a \in P$ it follows that $a^3 \in PP'P''$ As $PP'P''$ is a prime ideal of Ɲ, we have $a \in PP'P''$ and so $P \subseteq PP'P''$. Hence $PP'P'' = P$.

Corollary 5.13: If P is a prime ideal, then, P^3 is a 3-prime ideal.

Proposition 5.14: In a BTSNR (BTSRR) Ɲ, every maximal ideal is 3-prime.

Definition 5.15: A BTSNR (BTSRR) Ɲ is called a 3-P BTSNR (BTSRR) Ɲ if every 3-prime ideal of BTSNR (BTSRR) Ɲ is prime.

Definition 5.16: An ideal I of a BTSNR (BTSRR) Ɲ is called a quasi-3-primary ideal if for any $a, b, c \in$ Ɲ, $abc \in I$ and $a^3 \notin I$, $b^3 \notin I$ implies there exists an integer $n \in Z_0^+$ such that $c^{2n+1} \in I$.

Conclusion:

Here in manuscript, we have systematically initiated and prepared bi structures on ternary semi-rings, sub-,mono-morphism, eip-morphism andiso-morphism. semi rings, ideals, bi ideal, quasi-ideal and interior ideal and various properties. Our results contribute to a deeper understanding of how near semi-ring and how to extend the corresponding properties. This study serves as a stepping stone toward building a richer theory of algebra in ad vanced algebraic systems, and it encourages further investigation into both theoretical development and real-world applicability.

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Future work: The present study opens several promising avenues for future research in the area of algebraic Bi-structures. One immediate direction is the extension of this work to fuzzy ideals and intuitionistic ideals within ternary Gamma semirings in Bi-structures, allowing for the incorporation of degrees of membership and non-membership, which are particularly useful in soft computing and decision-making applications. Another potential area is the exploration of homomorphisms in ternary Gamma semirings, which would contribute to the development of structural theory and classification of ideals under mappings. Future studies could also aim to identify conditions for the existence and uniqueness of certain ideal types and analyze their behavior under algebraic operations such as product and intersection. Moreover, extending the current theoretical findings to computational models, such as automata or logic systems based on ternary Gamma operations, would enhance the applicability of this framework. Finally, interdisciplinary applications in information systems, cryptography, and artificial intelligence should be explored to demonstrate the practical relevance and adaptability of ideals in modeling uncertainty and complex decision processes.

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