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NEW ALGEBRAIC CHARACTERIZATIONS OF MIDDLE BOL LOOP

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Abstract

The concept of isostrophy of a loop is a generalization of parastrophy since an isostrophy of a loop is a map (transfor-

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mation) that combines an isotopy (bijections) with paras-trophy. This paper examines the properties of the middle Bol loop (MBL) under the isostrophy of a loop to further introduce new algebraic characteristics of MBL. We establish a necessary and sufficient condition for some loops to be a MBL under isostrophy. It is revealed that a MBL under isostrophy of a loop has an alternative property. The necessary and sufficient conditions for a MBL under the isostrophy of a loop to be an inverse property loop were shown. We show that a middle Bol loop under isostrophy is a Steiner loop. In addition, it is demonstrated that a commutative loop under the isostrophy of MBL is Moufang. It is further obtained that commutative inverse property loops under the isostrophy of MBL are commutative Moufang loops.

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Key Words and Phrases: Isostrophy, middle Bol loop, Moufang loop

1 Introduction

A pair $(Q, \odot)$ is known as a groupoid if Q is a non-empty set and" $\odot$ " is a binary operation in Q, such that $s \odot y \in Q$ for all $s, y \in Q$. If each of the equations:

$$a \odot y = b \quad \text{and} \quad s \odot a = b$$

has unique solutions in G for y and s respectively, then $(Q, \odot)$ is called a quasigroup. If $(Q, \odot)$ is a quasigroup and there is a unique element e in Q called the identity element, which has the property that for any s in Q, $s \odot e = e \odot s = s$, then $(Q, \odot)$ is called to as a loop.

We sometimes write sy instead of $s \odot y$ and declare that juxtaposition of factors to be multiplied has higher precedence than $\odot$. If $(Q, \odot)$ is a groupoid and "a" is a fixed element in Q, then the left L_a and right R_a translations are respectively defined by $sL_a = a \odot s$

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and $sR_a = s \odot a$. Also, the map $E_s: Q \to Q$ defined by $y \backslash s = yE_s$ and $s/y = yE_s^{-1}$ are called middle translations.

Every quasigroup $(Q, \odot)$ belongs to a set of six quasigroup, called adjugates by (Fisher, Yates [7] 1934) and parastrophes by (Belousov [4], 1967) and conjugates by (Stein [35], 1957).

Definition 1. (Shcherbacov [37, 36], 2008): Given a groupoid $(Q, \odot)$ with a binary operation" $\odot$ " such that the equality $(a_1 \odot a_2) = a_3$ shown the result for any two elements of a_1, a_2, a_3. It follows that for any quasigroup $(Q, \odot)$, have six of its substructures called parastrophes, that is $(Q, \odot)$: $(a_1 \odot a_2) = a_3 \iff (a_2 \odot^{(12)} a_1) = a_3 \iff (a_3 \odot^{(13)} a_2) = a_1 \iff (a_1 \odot^{(23)} a_2) = a_3 \iff (a_2 \odot^{(123)} a_3) = a_1 \iff (a_3 \odot^{(132)} a_1) = a_2$.

Definition 2. Let $(Q, \odot)$ be a quasigroup, then the isostrophy of $(Q, \odot)$ is a map (transformation) that is a combination of isotopy (bijections) with parastrophy. That is, the isostrophic images of a quasigroup $(Q, \odot)$ are parastrophic images of its isotopic images.

Definition 3. (Shcherbacov [37, 36], 2008): Let $(Q, \odot)$ be a quasigroup, $(Q, \circ)$ is an isostrophic image of $(Q, \odot)$ if there ex-ists a collection of permutations $(\sigma, (\alpha_1, \alpha_2, \alpha_3)) = (\sigma, T)$, where $\sigma \in S_3$, $T = (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha_1, \alpha_2, \alpha_3$ are permutation of the set Q such that $(a_1 \circ a_2) = (a_1 \odot a_2)(\sigma, T) = (\odot^\sigma(a_1, a_2))T = \alpha^{-1}(\alpha_1 a_{\sigma^{-1}1}, \alpha_2 a_{\sigma^{-1}2})$ for all $a_1, a_2 \in Q$. A collection of permutation $(\sigma, (\alpha_1, \alpha_2, \alpha_3)) = (\sigma, T)$ is called isostrophy of a quasigroup $(Q, \odot)$.

Definition 4. Let Q be a non-empty set, the set of all permutations on Q forms a group $SYM(Q)$ called the symmetric group of Q. Let $(Q, \odot)$ be a loop and let $A, B, C \in SYM(Q)$. If

$$sA \odot yB = (s \odot y)C \ \forall \ s, y \in G,$$

then the triple (A, B, C) is called an autotopism and such triples form a group $AUT(G, \odot)$ called the autotopism group of $(Q, \odot)$. If $A = B = C$, then A is called an isomorphism of $(Q, \odot)$ which form a group $AUM(Q, \odot)$ called the isomorphism group of $(Q, \odot)$.

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Definition 5. (Pflugfelder [33], 1990) Let $(Q,\cdot)$ and $(Q,\odot)$ be two quasigroup, then $(Q,\odot)$ is said to be isotope to $(Q,\cdot)$ if there exist three bijections $\alpha,\beta,\gamma : Q^{\cdot} \rightarrow Q^{\odot}$ such that $s\alpha \cdot y\beta = (s \odot y)\gamma \ \forall\, s, y \in Q$.

For more on quasigroups and loops, see (Jaiyéọlá [19], 2009), (Shcherbacov [44], 2017), (Pflugfelder [33], 1990) and Osoba et al. [25, 26, 27]

Definition 6. Let $(Q,\odot)$ be a "certain" loop where "certain" is an isomorphic invariant property. $(Q,\odot)$ is a universal "certain" loop if and only if every f, g-principal isotope $(Q,\star)$ of $(Q,\odot)$ has the "certain" loop property.

The invariant properties under the isotopy of a loop are called universal properties. It is well known that the RIP, LIP, and AAIP of a loop are respectively universal in a loop if and only if the loops are right Bol, left Bol, and middle Bol. Also, in a loop $(Q,\odot)$, both right and left inverse properties are universal if and only if the loop $(Q,\odot)$ is Moufang.

Definition 7. A loop $(G,\odot)$ is called a MBL if

$$(a/b)(c\backslash a) = (a/(cb))a \ or \ (a/b)(c\backslash a) = a((cb)\backslash a) \qquad (1)$$

for all $a, b, c \in Q$.

Initially, MBL were explored by V. D. Belousov [4], who established identities (1) delineating loops that satisfy the universal anti-automorphic inverse property. Building upon Belousov's foundational work, Gvaramiya [11] unveiled the isostrophic relationship between the right (left) Bol loop and a MBL structure in 1971. Following these insightful findings by Gvaramiya [11], research work on MBLs remained scarce until 1994 and 1996, when Syrbu [38, 39] conducted a study on the universality of the flexible law. In 2003, Kuznetsov investigated gyrogroups, which are a subclass of Bol loops, and elucidated various algebraic traits of the MBL. Additionally, he demonstrated a method for constructing a MBL from a gyrogroup. Furthermore, a new identity of MBL was discovered in 2012 by Drapal and Shcherbacov [6]. In 2010, Syrbu [40] conducted research on the isostrophic relationships between the structure and characterization of MBLs and their corresponding left Bol loops. In 2012, Grecu and Syrbu [10] unveiled that two MBLs are only isotopic if and only if the corresponding right (left) Bol loops are also isotopic. Expanding on their findings, in 2013, Syrbu and Grecu [41] extended their results to the quotient structure of a MBL.

In 2014, Grecu [42] established that the right multiplication group of a MBL coincides with the left multiplication group of the corresponding right Bol loop. Additionally, in the same year, Grecu and Syrbu [9] investigated the commutant loop of a MBL, demonstrating that a commutant loop of an MBL is an AIP-subloop. Furthermore, in 2018, the multiplicative group of MBL related to the right Bol loop was examined in [22], while the characterization of isostrophic connections between quasigroups and loops was studied in [23].

The characterizations of Bryant-Schneider group under the struc-ture of Smarandache loop and Osborn loop, along with their universality, were examined by Jaiyéolá [15] in 2008, and by Jaiyéolá et al. [17, 18] in 2011 and 2013. In 2017, Jaiyéolá et al. [12] examined the holomorphic structure of the MBL. Generalized Bol loops were the subject of study by Adeniran et al. [1] in 2014 and by Jaiyéolá and Popoola [16] in 2015. Further exploration into the structure of MBLs was conducted by Jaiyéolá et al. [14] in 2018, where new algebraic identities were obtained. Recently, a studied on a characterization of invariant loops under isostrophy was presented by Syrbu and Grecu [43].

In 2021, Jaiyéọlá et al. [13] introduced additional algebraic properties of the MBL, while the parastrophy of MBL were explored in [31]. Additionally, Osoba et al. [28], in 2025 investigated the algebraic characterizations of the generalized MBL. It was shown that (12)−parastrophy of GMBL is a GMBL. The authors further revealed the conditions for (13)− and (123)−parastrophes of a GMBL to be GMBL and further revealed the holomorphic properties of

GMBL which is the generalization of the results of Jaiyéolá ẹt al [12].

In 2022, Osoba and Jaiyéọlá [24] established algebraic connections between the MBL and RBL, along with their cores. They provided a necessary and sufficient condition for the core of a right Bol loop to exhibit elastic property and right idempotent law. Building upon the work of the author in [15], Osoba [32] and Osoba et al. [30] further studied some properties of MBL. In 2023, Jaiyéolá et al. [20] conducted a study on the Bryant-Schneider group of a MBL. Most recently, Osoba et al. [26] unveiled Bryant-Schneider group Characterizations of GMBL and revealed the condition for the two identities of generalized middle Bol loop to be equivalent.

Definition 8. A quasigroup $(Q, \odot)$ is said to have

1. LIP if there exists a mapping $J_\lambda : s \to s^\lambda$ such that $s^\lambda \odot st = t$ for all $s,t \in Q$.
2. RIP if there exists a mapping $J_\rho : s \to s^\rho$ such that $ts \odot s^\rho = t$ for all $s,t \in Q$.
3. a RAP if $t \odot ss = ts \odot s$ for all $s.t \in Q$.
4. a LAP if $s \odot st = ss \odot t$ for all $s,t \in Q$.
5. flexible or elastic if $st \odot s = s \odot ts$ holds for all $s,t \in Q$.
6. a power associative loop if $< s >$ is a subgroup for all $s \in Q$ and a diassociative loop if $< s,t >$ is a subgroup for all $s,t \in Q$.

Definition 9. A loop $(Q, \odot)$ is said to be:

1. a commutative loop if $R_s = L_s$ for all $s \in G$.
2. an anti- automorphic inverse property loop (AAIPL) if $(s \odot t)^\rho = t^\rho \odot s^\rho$ or = or $(s \odot t)^{-1} = t^{-1} \odot s^{-1}$ for all $s,t \in G$.

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Definition 10. A loop $(Q, \odot)$ is called a:

1. right Bol loop if $(st \odot z)t = s(tz \odot t)$ for all $s, t, z \in Q$.
2. Moufang loop if $(st \odot z)t = s(t \odot zt), tz \odot st = t(zs \odot t)$ and $(tz \odot t)s = t(z \odot ts)$ for all $s, t, z \in Q$.

Definition 11. A groupoid (quasigroup) $(Q, \odot)$ is

1. right symmetric if $ts \odot s = t$ for all $s, t \in Q$
2. left symmetric if $s \odot st = t$ for all $s, t \in Q$
3. middle symmetric if $s \odot ts = t$ or $st \odot s = t$ for all $s, t \in Q$.

Theorem 12. [33] A quasigroup $(Q, \odot)$ is totally symmetric if and only if $st = ts$ and $s(s \odot t) = t$ for all $s, t \in Q$.

Theorem 13. [33] A loop $(Q, \odot)$ is totally symmetric if and only if $(Q, \odot)$ is a commutative I.P. loop of exponent two.

Corollary 14. [33] Every T.S. quasigroup is a commutative I.P. quasigroup.

Definition 15. [33] If a totally symmetric quasigroup $(Q, \odot)$

is a loop, then it is called Steiner loop.

In Definition 1 shows that every parastrophy of a quasigroup has six sub quasigroups, which were used in Lemma 16 to present the isotopic characterization of a quasigroup which is further characterized as loop in Lemma 17. We used the parastrophes in Lemma 17 on Definition 7 to present the necessary and sufficient condition for some certain loops to be a middle Bol loop under the isostrophe. It is well known that a middle Bol loop is not an inverse property loop, therefore it cannot be a Moufang loop. In continuation of previous research, this study is aimed to investigate the loops that are MBL under the isostrophic characterization of Lemma 17. A necessary and sufficient condition for the invariant MBL under the

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isostrophy of the loop is found. It was reviewed that the MBL under the isostrophy of a loop is a left (right) alternative property (LAP) RAP, Steiner loop, and a left (right) inverse property loop LIP (RIP). In addition, commutative inverse property loops with universal MBL under the isostophes are Moufang loops.

2 Main Results

Lemma 16

Let (Q,⊗) be isotopic to a quasigroup (Q,⊙,\,/) with isotopy (ϕ,β,γ). For any maps ϕ,β,γ : Q(⊗) → Q(⊙) and for all s,y ∈ Q, the following equations hold:

1. $s \otimes y = (s\beta \odot y\gamma)\, \phi^{-1}$
2. $s \otimes y = (y\gamma \odot (12)\, s\beta)\, \phi^{-1}$
3. $s \otimes y = (s\beta \,/_{(13)}\, y\gamma)\, \phi^{-1}$
4. $s \otimes y = (s\beta \,\backslash_{(23)}\, y\gamma)\, \phi^{-1}$
5. $s \otimes y = (y\gamma \,\backslash_{(123)}\, s\beta)\, \phi^{-1}$
6. $s \otimes y = (y\gamma \,/_{(132)}\, s\beta)\, \phi^{-1}$

Proof

1. By definition of the isotopy (ϕ,β,γ), we have $s \otimes y = \phi^{-1}(s\beta \odot y\gamma)$.

2. Applying the (12)-permutation to the factors in the ⊙-product yields $s \otimes y = \phi^{-1}(y\gamma \odot (12)\, s\beta)$.

3. Let $s\beta \odot y\gamma = z\phi$. Solving by right division gives $z\phi = s\beta / y\gamma$, hence $s \otimes y = \phi^{-1}(s\beta \,/_{(13)}\, y\gamma)$.

□

4. Let $s\beta \odot y\gamma = z\phi$. Solving by left division gives $z\phi = s\beta \setminus y\gamma$, hence

$s \otimes y = \phi^{-1}(s\beta \setminus_{(23)} y\gamma)$.

5. From $s\beta \odot y\gamma = z\phi$ and $y\gamma \odot z\phi = s\beta$, we get $z\phi = y\gamma \setminus s\beta$, so

$$s \otimes y = \phi^{-1}(y\gamma \setminus_{(123)} s\beta).$$

6. From $s\beta \odot y\gamma = z\phi$ and $z\phi \odot s\beta = y\gamma$, we get $z\phi = y\gamma / s\beta$, so $s \otimes y = \phi^{-1}(y\gamma \,/_{(132)}\, s\beta)$.

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Lemma 17

Let (Q, ⊗) be a loop. The isostrophy given to any loop (Q, ⊙) such that the mappings $\phi, \beta, \gamma : Q \rightarrow Q$ defined on the loop operation (Q, ⊗) have the following six forms:

1. If $s \otimes y = (y\gamma \setminus s\beta)\phi^{-1}$, then $\otimes = (\setminus)(\phi E_g^{-1}, \phi L_h, \phi)$. That is, (Q, ⊗) ≅ (Q, ∗), where $s * y = (yE_g^{-1} \setminus sL_h)$.

2. If $s \otimes y = (s\beta \setminus y\gamma)\,\phi^{-1}$, then $\otimes = (\setminus)(\phi E_h^{-1}, \phi L_g, \phi)$. That is, (Q, ⊗) ≅ (Q, ∗), where $s * y = (sE_h^{-1} \setminus yL_g)$.

3. If s⊗y = (sβ / yγ) ϕ^{-1}, then $\otimes = (/)(\phi R_h, \phi E_g, \phi)$. That is, $(Q,\otimes) \cong (Q,*)$, where $s * y = (sR_h \,/\, yE_g)$.

4. If $s \otimes y = (s\beta \odot y\gamma)\,\phi^{-1}$, then $\otimes = (\odot)(\phi R_h^{-1}, \phi L_g^{-1}, \phi)$. That is, $(Q,\otimes) \cong (Q,*)$, where

$s * y = (sR_h^{-1} \odot yL_g^{-1})$.

5. If $s \otimes y = (y\gamma \,/\, s\beta)\,\phi^{-1}$, then $\otimes = (/)(\phi R_g, \phi E_h, \phi)$. That is, $(Q,\otimes) \cong (Q,*)$, where $s * y = (yR_g \,/\, sE_h)$.

6. If $s \otimes y = (y\gamma \circ s\beta)\phi^{-1}$, then $\otimes = (\circ)(\phi R_g^{-1}, \phi L_h^{-1}, \phi)$. That is, $(Q,\otimes) \cong (Q,*)$, where $s * y = (yR_g^{-1} \circ sL_h^{-1})$.

Proof

1. Considering the isostrophic loop $s \otimes y = (y\gamma \setminus s\beta)\phi^{-1}$, let $s = e$ the identity element in Q and setting $e\beta = g$, we have $y\phi = y\gamma \setminus g = y\gamma E_g\ \forall\, y \in Q. \Rightarrow \gamma = \phi E_g{}^{-1}$. (2)

Also, let $y = e$ the identity element in Q, setting $e\gamma = h$, we have

$s\phi = h \setminus s\beta = s\beta L_h{}^{-1}\ \forall\, s \in Q. So\ \beta = \phi L_h$. ` (3)

Using equations (2) and (3), the isostrophy in Lemma 17(1) implies

$\otimes = (\setminus)(\phi E_g^{-1}, \phi L_h, \phi)$Thus

$(Q,\otimes) \cong (Q,*)$, where $s * y = (yE_g{}^{-1} \setminus sL_h)$. (4)

2. Considering the isostrophic loop $s \otimes y = (s\beta \setminus y\gamma)\phi^{-1}$, setting $s = e$ the identity element in Q and denoting $e\beta = g$, we have y$\phi = g \setminus y\gamma = y\gamma L_g{}^{-1}\ \forall\, y \in Q$. Thus $\gamma = \phi L_g$. (5)

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Also, setting $y = e$, the identity element in Q, let $e\gamma = h$. Then $s\phi = s\beta \setminus h = s\beta E_h \, \forall \, s \in Q. So \, \beta = \phi {E_h}^{-1}$. (6)

Using (5) and (6), Lemma 17(2) implies $\otimes = (\setminus)(\phi {E_h}^{-1}, \phi L_g, \phi)$.. Thus $(Q,\otimes) \cong (Q,*)$, where $s * y = (s{E_h}^{-1} \setminus yL_g)$. (7)

3. Considering the isostrophic loop $s \otimes y = (s\beta \odot y\gamma)\phi^{-1}$, let $s = e$ and set $e\beta = g$. Then $y\phi = g \odot y\gamma = y\gamma L_g \, \forall \, y \in Q$. So $\gamma = \phi {L_g}^{-1}$. (8)

Also, setting $y = e$ and $e\gamma = h$, we have $s\phi = s\beta \odot h =$

$s\beta Rh \, \forall \, s \in Q$. So $\beta = \phi {R_h}^{-1}$. (9)

Using (8) and (9), Lemma 17(3) implies $\otimes = (\odot)(\phi {R_h}^{-1}, \phi {L_g}^{-1}, \phi)$. Thus $(Q ,\otimes) \cong (Q,*)$, where $s * y = (s{R_h}^{-1} \odot y{L_g}^{-1})$. (10)

The results for (4), (5), and (6) of Lemma 17 follow by similar steps as in cases (1), (2), and (3)

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Theorem 18. Let $(Q,\odot,\backslash,/)$ be a loop. Any loop isostrophic to $(Q,\odot,\backslash,/)$ is a MBL if and only if

$$(gy/s)\backslash((h/z)\odot s) = (h/s)\backslash[h/(h/z)\backslash gy \odot s] \quad (11)$$

$$(s \odot z\backslash h)/(s\backslash yg) = [s \odot yg/(z\backslash h)\backslash h]/(s\backslash h) \quad (12)$$

$$s/(h\backslash z) \odot (y/g)\backslash s = s/[h\backslash (y/g) \odot (h\backslash z)] \odot h\backslash s \quad (13)$$

for all $s, y, z, g, h, \in Q$.

Proof. 1. Using Lemma 17(2) on MBL,

$$(s/^{*} y) * (z\backslash^{*} s) = s * ((z * y)\backslash^{*} s) \quad (14)$$

where: $s * y = (sE_h^{-1}\backslash yL_g)$.

Considering the LHS of the identity (14), it is not difficult to see that

$$s/^{*}y = (yL_g/s)E_h \quad \text{and} \quad z\backslash^{*} s = (zE_h^{-1} \odot s)L_g^{-1}.$$

Then, $(s/^{*} y) * (z * s) = (yL_g/s)E_hE_h^{-1}\backslash (zE_h^{-1} \odot s)L_g^{-1} L_g = (yL_g/s)\backslash(zE_h^{-1} \odot s)$.

Thus,

$$(s/^{*}y) * (z\backslash^{*}s) = (yL_g/s)\backslash(zE_h^{-1} \odot s). \quad (15)$$

Considering the RHS of identity (14), we have

$$s * ((z * y)\backslash^{*}s) = sE_h^{-1}\backslash[(z * y)\backslash^{*}s]L_g = sE_h^{-1}\backslash(zE_h^{-1}\backslash yL_g)\backslash^{*}sL_g =$$
$$sE_h^{-1}\backslash(zE_h^{-1}\backslash yL_g)E_h^{-1} \odot sL_g^{-1} L_g$$
$$= sE_h^{-1}\backslash(zE_h^{-1}\backslash yL_g)E_h^{-1} \odot s.$$

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Thu, $s * ((z * y)\backslash^* s) = sE_h^{-1}\backslash(zE_h^{-1}\backslash yL_g)E_h^{-1} \odot s$ (16)

Using equalities (15) and (16), we have the RHS = LHS ⇔

$$(yL_g/s)\backslash(zE_h^{-1} \odot s) = sE_h^{-1}\backslash\ (zE_h^{-1}\backslash yL_g)E_h^{-1} \odot s$$

$$\Leftrightarrow (gy/s)\backslash((h/z) \odot s) = (h/s)\backslash\ h/[(h/z)\backslash gy] \odot s\ . \quad (17)$$

2. Using $s * y = (yR_g/sE_h)$ of Lemma 17(5) on the identity (14) to obtains;

$$z\backslash^* s = (s \odot zE_h)R_g^{-1} \quad \text{and}$$

$$s/^* y = (s\backslash yR_g)\ E_h^{-1} \quad (18)$$

And considering the LHS of (14), we have

$$(s/^* y) * (z\backslash^* s) = (s \odot zE_h)R_g^{-1}]R_g/\ (s\backslash yR_g)E_h^{-1}\ E_h = (s \odot zE_h)/(s\backslash yR_g).$$

Using the RHS of identity (14) to obtain

$$s * ((z * y)\backslash^* s) = [(z * y)\backslash^* s]R_g/sE_h = [(yR_g/zE_h)\backslash^* s]R_g/sE_h = s \odot (yR_g/zE_h)E_hR_g^{-1}\ R_g/sE_h.$$

Then, RHS=LHS

$$\Leftrightarrow (s \odot zE_h)/(s\backslash yR_g) = s \odot (yR_g/zE_h)E_h\ R_g^{-1}\ R_g/sE_h$$

$$\Leftrightarrow (s \odot zE_h)/(s\backslash yR_g) = s \odot (yR_g/zE_h)E_h/sE_h \Leftrightarrow (s \odot z\backslash h)/(s\backslash yg) = s \odot\ yg/(z\backslash h)\backslash h\ /(s\backslash h). \quad (19)$$

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3. Using $s * y = (yR_g^{-1} \odot yL_h^{-1})$ on the LHS of identity (14), we have

$$s/^{*}y = (yR_g^{-1}\backslash s)L_h \quad \text{and} \quad z\backslash^{*}s = (s/zL_h^{-1})R_g. \quad (20)$$

Using (20) on the LHS of (14) to get

$$(s/^{*}y) * (z\backslash^{*}s) = (yR_g^{-1}\backslash s)L_h * (s/zL_h^{-1})R_g =$$
$$(s/zL_h^{-1})R_g\, R_g^{-1} \odot (yR_g^{-1}\backslash s)L_h\, L_h^{-1} =$$
$$(s/zL_h^{-1}) \odot (yR_g^{-1}\backslash s).$$

Thus,

$$(s/^{*}y) * (z\backslash^{*}s) = (s/zL_h^{-1}) \odot (yR_g^{-1})\backslash s. \quad (21)$$

Considering the RHS, we have

$$s * (z * y\backslash^{*}s) = (z * y\backslash^{*}s)R_g^{-1} \odot sL_h^{-1}$$
$$= (yR_g^{-1} \odot zL_h^{-1})\backslash^{*}s\, R_g^{-1} \odot sL_h^{-1}$$
$$= s/\, (yR_g^{-1} \odot zL_h^{-1}L_h^{-1}\, R_g\, R_g^{-1} \odot sL_h^{-1}.$$

Hence, RHS=LHS

$$\Leftrightarrow (s/zL_h^{-1}) \odot (yR_g^{-1}\backslash s) =$$
$$s/\, (yR_g^{-1} \odot zL_h^{-1})L_h^{-1}\, R_g\, R_g^{-1} \odot sL_h^{-1} \Leftrightarrow$$
$$(s/zL_h^{-1}) \odot (yR_g^{-1}\backslash s) =$$
$$s/\, (yR_g^{-1} \odot zL_h^{-1})L_h^{-1} \odot sL_h^{-1}$$
$$\Leftrightarrow s/(h\backslash z) \odot (y/g)\backslash s = s/\, h\backslash(y/g) \odot (h\backslash z)\, \odot h\backslash s. \quad (22)$$

□

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Therefore, we used the isostrophes 2, 5, and 6 of Lemma 17, on the identity (14) to obtain identities (11), (12) and (13) respectively. Analogously, using the isostrophes 1, 3 and 4 of Lemma 17 on identity (14), we shall obtain identities that are identically equivalent to the identities (11), (12) and (13) respectively. We further this study by considering the algebraic characterizations of the identities (11) and (12) above. Identities (13) is equivalent to identity obtained in Theorem 2.1 [13] when considered the principal isotope of the MBL.

Remark 1. The qualities (11), (12) and (13) are not equivalent in the study of properties of loop. Hence, the Example 2.1 below satisfies the equality in (11), but does not satisfy the identities in equations (12) and (13). Although, identity (13) is a middle Bol loop.

Example

$(\cdot)$	a	b	c	d	e	f	g	h
a	a	b	c	d	e	f	g	h
b	b	a	d	c	f	e	h	g
c	c	d	a	b	h	g	e	f
d	d	c	b	a	g	h	f	e
e	e	f	g	h	a	b	c	d
f	f	e	h	g	b	a	d	c
g	g	h	e	f	d	c	a	b
h	h	g	f	e	d	c	b	a

Theorem 19. Every loop $(Q, \odot, /, \backslash)$ isostrophic to MBL satisfies the following;

1. $(y/s)\backslash zs = (h/s)\backslash[h/(z\backslash y) \odot s]$.
2. $z(y\backslash z) = z/(z\backslash y)$ or $E_z L_z = L_z E^{-1}_z$.
3. $(y/s)\backslash s = (s/y) \odot s$ or $R^{-1}_s E_s = E^{-1}_s R_s$.
4. $(z \odot t) \odot t^{-1} = z$ (RIP) $\forall$ $t, z, \in Q$.
5. $s = s^{-1}$.

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6. $(h \odot s) \odot s = h \ \forall h, s, \in Q$.
7. $t \odot h^2 = (t \odot h) \odot h \ \forall t, h, \in Q$.

Proof. 1. Set $z \to h\backslash z$ and $y \to h\backslash y$ in (1), we have

$$(y/s)\backslash zs = h/s\backslash h/(z\backslash y) \odot s \tag{23}$$

2. Set $s = e$ and $h \to z$ in (23), we have

$$y\backslash z = z\backslash(z/(z\backslash y)) \Rightarrow z \odot (y\backslash z) = z/(z\backslash y) \Rightarrow E_z L_z = L_z E_z^{-1}$$

3. Let $z = e$ and $h \to s$ in (23), we have $(y/s)\backslash s = (s/y)s$
4. Put $h = s = e$ in (23), give

$$y\backslash z = e/(z\backslash y) \Rightarrow y\backslash z = (z\backslash y)^{-1} \Rightarrow z = y \odot (z\backslash y)^{-1}$$

Let $z\backslash y = t \Rightarrow z \odot t = y$ for any $t \in Q$, then $(z \odot t) \odot t^{-1} = z \ \forall z \in Q$

5. Apply RIP into (1), we have

$$(ys^{-1})\backslash z \odot s = hs^{-1}\backslash h[(z\backslash y)^{-1}] \odot s$$

Put $z = y$, and $y = e$ get $s = s^{-1}$ for all $s \in Q$

6. Put $y = z = e$ in (1), we have

$$s^{-1}\backslash s = (h/s)\backslash(hs) \Rightarrow h/s = hs$$

By applying $s = s^{-1}$, we have $(h \odot s) \odot s = h$ for all $s, h \in Q$.

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7. Let s = h in (23), we have

$$(y/h)\backslash z \odot g = [h/(z\backslash y) \odot h].$$

Then, let y → z, we have $(z/h)\backslash z \odot h = h^2 \Leftrightarrow (z/h) \odot h^2 = z \odot h$. Let $z/h = t \Rightarrow z = t \odot h$ for all $t \in Q, \Rightarrow t \odot h^2 = (t \odot h) \odot h.$ □

Theorem 20. Every loop $(Q, \odot, /, \backslash)$ isostrophic to MBL satisfies the following;

1. $(s \odot z)/(s\backslash y) = s \odot (y/z)\backslash h/(s\backslash h)$.
2. $h \odot ht = h^2 \odot t \ \forall \ h, t \in Q$.
3. $h/(h\backslash y) = h \odot (y\backslash h) \ \forall \ h, y \in Q$.
4. $h/h^{-1} = h^2 \ \forall h \in Q$.
5. $z = t^{-1} \odot tz \ \forall \ t, z \in Q$.
6. $s = s^{-1} \ \forall s \in Q$.
7. $s \odot z = s \odot (z^{-1})^{-1} \ \forall \ s, z \in Q$.

Proof. 1. Considering the equality (12). Doing the following steps: Let y → y/g and z → z/h, we have

$$(s \odot z)/(s\backslash y) = s \odot (y/z)\backslash h/(s\backslash h). \tag{24}$$

2. Let s → h and y → z in (24), we have

$$hz/(h\backslash z) = h^2 \Rightarrow hz = h^2(h\backslash z)$$

Let $h\backslash z = t \Rightarrow h \odot t = z$ for any $t \in Q$, so, $h \odot ht = h^2 \odot t \ \forall \ h, t \in Q$.

3. Let s → h and z = e, we have

$$h/(h\backslash y) = h \odot (y\backslash h)$$

for all $h, y, \in Q$.

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4. Let $y = e$ in $h/(h\backslash y) = h \odot (y\backslash h)$, give $h/h^{-1} = h^2$ for all $h \in Q$.

5. Let $s \to h = e$ in (24), we have

$$z/y = (y/z)^{-1} \Rightarrow z = (y/z)^{-1} \odot y$$

. Let $y/z = t \Rightarrow y = tz$. Then, $z = t^{-1} \odot tz$ for all $t, z \in Q$.

6. Let $z \to y$ and $s \to h$ in (24), we have $s = s^{-1}$ for all $s \in Q$.

7. Apply LIP and put $h \to y = e$ in (24), get $s \odot z = s \odot (e/z)^{-1} = s \odot (z^{-1})^{-1}$.

□

Corollary 21. Let $(Q, \odot)$ be a loop under the isostrophy of MBL. Then,

$$(E_z L_z)^n = \begin{cases} L_z^n E_z^{-1}, & if\ n\ is\ odd \\ L_z^n, & if\ n\ is\ even \end{cases}$$

For all natural numbern.

Proof.Using Theorem 19(2) □

.

Corollary 22. Let $(Q, \odot)$ be a loop under the isostrophy of MBL. Then, $(E_x^{-1} R_x)^2 = e$.

Proof. Follows from Theorem 19(3). □

Corollary 23. Let $(Q, e, \odot)$ be a loop under the isostrophy of MBL. Then, $(Q, e, \odot)$ is a left inverse property loop if and only if it satisfies the identity

$$sz/(h^{-1}y) = s \odot (y/z)^{-1} h/(s^{-1}h).$$

Proof. Suppose that Q is a left inverse property, then from identity (12), we have

$$(s \odot z)/(s^{-1}y) = s \odot (y/z)^{-1} h/(s^{-1}h)$$

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for all $s, y, z, h \in Q$.
Conversely, suppose that $(s \odot z)/(s^{-1}y) = s \odot (y/z)^{-1}h\ /(s^{-1}h)$, then put $h \to y$ we have

$$(s \odot z)/(s^{-1}y) = s \odot (y/z)^{-1}y/(s^{-1}y) \Leftrightarrow$$
$$sz = s \odot (y/z)^{-1}y.$$

Let $y/z \to t \Leftrightarrow y = tz$. Then, $s \odot z = s \odot (t^{-1} \odot (t \odot z)) \Leftrightarrow z = t^{-1} \odot (t \odot z)$. □

Corollary 24. Let $(Q, e, \odot)$ be a loop under the isostrophy of MBL. Then, $(Q, e, \odot)$ is a right inverse property loop if and only if it satisfies

$$(y \odot s^{-1})\backslash zs = hs^{-1}\backslash[h/(z\backslash y)^{-1} \odot s]$$

Proof. Follow the steps in Corollary 23. □

Corollary 25. Every commutative loop $(Q, \odot)$ under the isostrophy of middle Bol loop is an inverse property loop.

Proof. It follows from Corollaries 23 and 24. □

Corollary 26. Every commutative loop $(Q, \odot)$ under isostrophy to MBL has an alternative property.

Proof. Apply the Theorem 19 and 20 couple with the fact that $(Q, \odot)$ is commutative, then $(Q, \odot)$ is an alternative property loop.

□

Corollary 27. Every commutative loop $(Q, \odot)$ under isostrophy of MBL is power associative.

Proof. Consequence of Theorem 19 and Theorem 20 couple with the fact that $(Q, \odot)$ is commutative. □

Corollary 28. Every commutative loop $(Q, \odot)$ under the isostrophy of MBL is Steiner loop.

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Proof. Apply the right symmetric property in Theorem 19 couple with the fact that $(Q, \odot)$ is commutative, then $(Q, \odot)$ is symmetric. Thus, $(Q, \odot)$ is a Steiner loop □

Remark 2. It is observed that identity (11) is a mirror of the identity (12) under the commutative property.

Theorem 29. Every loop isostrophic to a commutative loop $(Q, \odot)$ is MBL if and only if it satisfies the eqaulity

$$(gy \odot s^{-1})^{-1}\ (fz^{-1}) \odot s = (hs^{-1})^{-1}\ \ h(\ hz^{-1})^{-1} \odot gy^{-1} \odot s\ \ .$$

Proof. Suppose $(Q, \odot)$ is a commutative loop which is MBL invariant under isostrophy of loops, apply the inverse property in Theorem 19 on Theorem 18, we obtain

$$(gy \odot s^{-1})^{-1}\ (hz^{-1}) \odot s = (hs^{-1})^{-1} h(\ hz^{-1})^{-1} \odot gy^{-1} \odot s. \quad (25)$$

Conversely, let $(Q, \odot)$ be a loop with the equality (25).
Let $z \to h$ in (25) get

$$(gy \odot s^{-1})^{-1} \odot s = (h \odot s^{-1})^{-1}\ h(gy)^{-1} \odot s \quad (26)$$

. Let $s = e$ in (26) the identity element in loop $(Q, \odot)$, we have

$$(gy)^{-1} = h^{-1}(h(gy)^{-1}) \quad (27)$$

Put $(g \odot y)^{-1} \to z$ for any $z \in Q$, give

$$z = h^{-1} \odot (h \odot z)\ \forall\ h, z \in Q$$

Since $(Q, \odot)$ is a commutative loop, we get that it is right (left) inverse property loop, that is, $(Q, \odot)$ is a loop with invariant MBL under the isostrophy of loops. □

Theorem 30. A commutative loop $(Q, \odot)$ with invariant MBL under the isostrophy of a loop is a Moufang.

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Proof. Suppose that $(Q, \odot)$ is a commutative loop with invariant MBL under the isostrophy of a loops then, we show that $(Q, \odot)$ is Moufang loop. Using Theorems 18, Theorem 19 and corollary 25 then, $(Q, \odot)$ is an inverse property loop and satisfies the identity (25). Doing the following steps: Let $y \to g^{-1}y$ and $z^{-1} \to h^{-1}z$ in (25), we have

$$(ys^{-1})^{-1} \odot zs = (hs^{-1})^{-1}\ h(z^{-1}y)^{-1} \odot s$$
$$\Rightarrow (y^{-1}s) \odot zs = (hs^{-1})^{-1}\ h(zy^{-1}) \odot s$$

Since, $(Q, \odot)$ is commutative, we have

$$(sy^{-1}) \odot (z \odot s) = (hs^{-1})^{-1}\ h(y^{-1}z) \odot s. \tag{28}$$

Setting $s \to h$ in (28), we have

$$(hy^{-1}) \odot zh = (h(y^{-1}z)) \odot h$$
$$\Rightarrow hy^{-1} \odot hz = h \odot (h(y^{-1}z)).$$

Using Corollary 26 to get

$$hy^{-1} \odot hz = h^2(y^{-1}z).$$

Replace y^{-1} with s, we have

$$hs \odot hz = h^2(s \odot z)\ \forall h, s, z \in Q.$$

Thus, $(Q, \odot)$ is a commutative Moufang loop.

□

3 Conclusion

This research provides valuable insights into the algebraic properties of a middle Bol loop. It is well established that a middle Bol loop (MBL) does not exhibit the alternative property, is not an inverse property loop, and cannot be classified as a Moufang

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loop. However, this study presents the novel finding that under the isostrophic characterization, MBL does indeed satisfy the aforementioned properties of a loop. Specifically, we have demonstrated that the middle Bol loop, when subjected to the isostrophy of a loop, qualifies as an inverse property loop. Consequently, it is also identified as a commutative Moufang loop.

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