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NUMERICAL APPROACH FOR FUZZY FRACTIONAL RICCATI DIFFERENTIAL EQUATIONS USING RESIDUAL POWER SERIES METHOD

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Abstract - The power series solutions to fuzzy fractional Riccati differential equations (FFRDEs) along with a suitable fuzzy constant through an interactive derivative, more specifically, the strongly generalized differentiability (SGD) is analyzed by using the Residual power series (RPS) method. To analyze the accuracy of our proposed algorithm, examples were examined numerically and compared the results with the exact solution.

Keywords - Keywords: Fuzzy-valued function, Strongly generalized differentiability, Fractional riccati differential equation, Residual power series method.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. Many physical events may be described with the help of fractional calculus, which is a very useful and significant technique. It is used for many purposes in several fields, such as chemistry, physics, dynamics systems, engineering and mathematical biology [1]. Fuzzy sets are a powerful mathematical tool that are used to model uncertainty-related problems. It has been successfully used to describe a wide variety of natural events that have been restated in fuzzy language.

Fuzzy differential equations (FDEs) are commonly used in science and engineering to simulate problems with fuzzy initial or boundary conditions. Since it is frequently difficult or impossible to obtain exact solutions, approximate solutions have become more important. Understanding the notion of a fuzzy derivative, which was first presented by Chang and Zadeh [2] and then expanded upon by Dubois and Prade [3], who used the extension principle in their methodology, is crucial before delving into numerical techniques for FDEs. Puri and Ralescu [4] and Goetschel and Vaxman [5] created fuzzy derivatives as expansions of the Hukuhara derivative for multivalued functions and provided additional definitions.

Fuzzy fractional differential equations (FFDEs) are essential in many scientific fields, such as population models [6], military systems evaluation [7], civil engineering [8], and electro-hydraulic systems modeling [9]. The first formal study in FFDEs was conducted in 2010 by Agarwal et al. [10]. Indeed, they defined the differentiability of Riemann-Liouville in the context of the differentiability of Hukuhara and applied this novel approach to solve FFDEs.

Riccati differential equations, studied in the 18th century by Count Jacopo Riccati, are nonlinear first-order ordinary differential equations (ODEs) with significant applications in control theory, physics, and finance, often solvable through transformation to linear second-order ODEs or by finding particular solutions.

Fractional Riccati differential equations extend this concept using fractional derivatives to model systems with memory effects, finding applications in diverse areas and requiring specialized analytical and numerical techniques for their solution. Recent research focuses on developing more efficient and accurate methods to solve these nonlinear fractional-order equations due to their increasing importance in modeling complex real-world phenomena

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Fuzzy Fractional Riccati differential equations (FFRDEs) form a precise non-linear model for defining a certain type of physical scheme in the diffusion process, optimal control, optical networks and artificial intelligence. Ercan Çelik, Mesut Karabacak, and Ekhtiar Khodadadi [11] used the variation iteration method (VIM) under Caputo H-differentiability to solve fuzzy fractional Riccati differential equations.
The main objective of the research described in this study is always to extend the application of the RPS method and provide a power series solution to the FFRDEs under strongly generalized differentiation. Consider the following nonlinear fuzzy fractional riccati differential equation

$$D_t^{\alpha}\tilde{y}(t) = U\tilde{y}^2(t) + V\tilde{y}(t) + W, t > 0, 0 < \alpha < 1. \quad (1.1)$$

with the fuzzy initial condition

$$\tilde{y}(0) = \tilde{y}_0 \quad (1.2)$$

where $\tilde{y}^2(t) \neq 0$, U, V and W∈R, $\tilde{y}_0$ is arbitrary fuzzy number, D^{α} is fractional derivative for order α and $\tilde{y}(t)$ is an unknown fuzzy function of the crisp variable.
The section 2 provides preliminary definitions and theorems for Caputo's fractional derivative operator and residual power series. Section 3 introduce the concept, which is a formulation of the fuzzy fractional Riccati differential equation. The basic methodology discussed in Section 4, where the residual power series approach for the fuzzy fractional Riccati differential equations is used to demonstrate the proposed procedure's high performance and dependability. The paper concludes with some numerical examples.

2. Preliminaries

In this section, briefly elaborate some definitions and important concepts of fuzzy sets and fractional derivatives.

Definition 2.1 [12]: A mapping z: R → [0, 1] is called a fuzzy number if the following properties are satisfied

1. $z(\lambda s+(1-\lambda) t) \geq \min\{z(s), z(t)\}$ for each $s, t \in R$ and $\lambda \in [0, 1]$, which is called a convex property.
2. $\exists s \in$ R such that $z(s) = 1$, which is called a normal property.
3. The set $\{s \in \mathrm{R}|\mathrm{z(s)} > \alpha\}$ is closed for each $\alpha \in [0, 1]$, which is called an upper semi continuous property.
4. The set $\{s \in \mathrm{R}|\mathrm{z(s)} > 0\}$ is compact, where $\overline{\{\cdot\}}$ is the closure of $\{\cdot\}$.

For $0 < \alpha \leq 1$,
$[z]_{\alpha} = \{s \in \mathrm{R} \mid z(s) \geq \alpha\}$, $[z]_0 = \{s \in \mathrm{R} \mid \quad z(s) > 0\}$, $[z]_1 \neq \phi$.

Thus, if z is a fuzzy number, then $[z]_{\alpha} = [\underline{z}(\alpha), \overline{z}(\alpha)]$, where $\underline{z}(\alpha) = \min\{s| \ s \in [z]_{\alpha}\}$ and $\overline{z}(\alpha) = \max\{s| \ s \in [z]_{\alpha}\}$ for each $\alpha \in [0,1]$. Hence, the notation $[z]_{\alpha}$ is called the α-cut representation or parametric form of a fuzzy number z.

Definition 2.2 [12]: A mapping z: R → [0, 1] is a fuzzy number with α-cut representation $[\underline{z}(\alpha), \overline{z}(\alpha)]$ if and only if the following conditions are satisfied

1. The function $\underline{z}$: [0, 1] → R is bounded and increasing.
2. The function $\overline{z}$: [0, 1] → R is bounded and decreasing.
3. $\underline{z}(\alpha) \leq \overline{z}(\alpha)$ for all $\alpha \in [0, 1]$.
4. For each $r \in (0, 1]$, $lim_{\alpha\to r^-}\ \underline{z}(\alpha) = \underline{z}(r)$ and $lim_{\alpha\to r^-}\ \overline{z}(\alpha) = \overline{z}(r)$.
5. For each $r \in [0, 1)$, $lim_{\alpha\to 0^+}\ \underline{z}(\alpha) = \underline{z}(0)$ and $lim_{\alpha\to 0^+}\ \overline{z}(\alpha) = \overline{z}(0)$.

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Definition 2.3 [12]: Let $H_d: \mathcal{R}_F \times \mathcal{R}_F \to \mathcal{R}_+ \cup \{0\}$. The Hausdorff metric H_d is a function defined by

$$H_d(w,z) = \sup_{\alpha\in[0,1]} \max\{|w(\alpha) - z(\alpha)|, |w(\alpha) - z(\alpha)|\}$$

for each $w, z \in \mathcal{R}_F \backslash$

Definition 2.4 [13]: A fractional power series (FPS), where $0 < n \leq 1, x \in I$ and $t \geq t_0$,

$$\sum_{i=0}^{\infty} z_i(x)(t-t_0)^i = z_0(x) + z_1(x)(t-t_0) + z_2(x)(t-t_0)^2 + \cdots$$

is called a multiple fractional power series (MFPS) about t_0.

Definition 2.5[14]: Let $z: [a,b] \to \mathcal{R}_F$ be a fuzzy-valued function and $t_0 \in [a,b]$. Where z is strongly generalized differentiable at t_0, if there exists an element $z'(t_0) \in \mathcal{R}_F$ such that

1. For all $h > 0$, the H-differences $z(t_0+h) \ominus z(t_0), z(t_0) \ominus z(t_0-h)$ exist, and

$$z'(t_0) = \lim_{h\to 0^+} \frac{z(t_0+h) \ominus z(t_0)}{h} = \lim_{h\to 0^+} \frac{z(t_0) \ominus z(t_0-h)}{h}$$

where $z'(t_0)$ is denoted by $D_1^1 z(t_0)$.

2. For all $h < 0$, the H-differences $z(t_0+h) \ominus z(t_0), z(t_0) \ominus z(t_0-h)$ exist, and

$$z'(t_0) = \lim_{h\to 0^-} \frac{z(t_0) \ominus z(t_0+h)}{-h} = \lim_{h\to 0^-} \frac{z(t_0-h) \ominus z(t_0)}{-h}.$$

where $z'(t_0)$ is denoted by $D_1^2 z(t_0)$.

Theorem 2.6[14]: For all real values of a and b and each $r \in [0,1]$, $z: [a,b] \to \mathbb{R}^F$ and $[z(t)]_r = [z_{1r}(t), z_{2r}(t)]$, such that $z_{1r}(t)$ and z_{2r}(t) are differentiable functions on $[a,b]$, the following holds

1. If z is (1)-differentiable on $[a,b]$, then $[z'(t)]_r = [z'_{1\mathrm{r}}(t), z'_{2\mathrm{r}}(t)]$
2. If z is (2)-differentiable on $[a,b]$, then $[z'(t)]_r = [z'_{2\mathrm{r}}(t), z'_{1\mathrm{r}}(t)]$

Definition 2.7[15]: The Riemann-Liouville fractional integral of x of order $\alpha > 0$ with $a \geq 0$ is defined as

$$I_a^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s) ds, t > a$$

where $\Gamma(\alpha)$ is the well-known gamma function defined as

$$\Gamma(\alpha) = \int_0^\infty t^{\beta-1} e^{-t} dt$$

For $\alpha = 0$, the Riemann-Liouville fractional integral of x is

$$I_a^0 x(t) = x(t)$$

Definition 2.8 [15]: The Caputo fractional derivative of x of order $\alpha > 0$ with $a \geq 0$ is defined as

$${}^C D_a^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds$$

for $n-1 < \alpha \leq n, n \in \mathbb{N}$ and $t \geq a$.

3.Mathematical Formulation of Fuzzy Fractional Riccati Differential Equation

Consider the following nonlinear Fuzzy Fractional Riccati Differential Equation

$$D_t^\alpha \tilde{y}(t) = U\tilde{y}^2(t) + V\tilde{y}(t) + W, t > 0, 0 < \alpha < 1. \tag{3.1}$$

with the fuzzy initial condition

$$\tilde{y}(0) = \tilde{y}_0 \tag{3.2}$$

where $\tilde{y}^2(t) \neq 0$, $U, V, W \in \mathcal{R}$, $\tilde{y}(t): [0,T] \to \mathcal{R}_F$ and $\tilde{y}(0) \in \mathcal{R}_F$ are constant coefficients.

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The following are the conditions and assumptions. The FFRDEs are expressed in Eq. (3.2) has a unique solution along with the initial conditions. Second, Assume the r-level $r \in [0,1]$representation of $D_t^\alpha \tilde{y}(t), \tilde{y}^2(t), \tilde{y}(t)$ and $\tilde{y}(0)$ as $[D_t^\alpha \tilde{y}_{1r}(t),\ D_t^\alpha \tilde{y}_{2r}(t)]$, $[\tilde{y}_{1r}^2(t),\ \tilde{y}_{1r}^2(t)]$, $[\tilde{y}_{1r}(t),\ \tilde{y}_{2r}(t)]$, $\left[\tilde{y}_{0,1r}(t), \tilde{y}_{0,2r}(t)\right]$ respectively. Thus, the Fuzzy Fractional Riccati Differential Equations (FFRDEs)can be in the parametric form as follows:

$$[D_t^\alpha \tilde{y}(t)]_r = U[\tilde{y}^2(t)]_r + V[\tilde{y}(t)]_r + W, t > 0, 0 < \alpha < 1. \tag{3.3}$$

with the fuzzy initial condition

$$[\tilde{y}(0)]_r = [\tilde{y}_0]_r \tag{3.4}$$

Convert the FFRDEs, as in Eqs. (3.3) and (3.4), into crisp ordinary differential equations (ODEs) systems. The fuzzy solution $\tilde{y}(t)$ is obtained by using fuzzy initial condition with respect to the type of differentiability, where $\tilde{y}(t)$ is either (1)-differentiable or (2)-differentiable. Case 1: If $\tilde{y}(t)$ is (1)-differentiable, then FFRDEs (3.3) and (3.4) can be converted into the following crisp system.

$$\begin{aligned} D_t^\alpha \tilde{y}_{1r}(t) &= U\tilde{y}_{1r}^2(t) + V\tilde{y}_{1r}(t) + W \\ D_t^\alpha \tilde{y}_{2r}(t) &= U\tilde{y}_{2r}^2(t) + V\tilde{y}_{2r}(t) + W \end{aligned} \tag{3.5}$$

with the fuzzy initial condition

$$\begin{aligned} \tilde{y}_{1r}(0) &= \tilde{y}_{0,1r} \\ \tilde{y}_{2r}(0) &= \tilde{y}_{0,2r} \end{aligned} \tag{3.6}$$

The process to get the equations are as follows:
(i) Generate the Residue Power Series method for the system in Equations (3.5) and (3.6).
(ii) Examine the accuracy of the solutions $[\tilde{y}_{1r}(t), \tilde{y}_{2r}(t)]$ and $[D_t^\alpha \tilde{y}_{1r}(t), D_t^\alpha \tilde{y}_{2r}(t)]$ as r-level sets, where $r \in [0,1]$
(iii) Then $[\ \tilde{y}_{1r}(t), \tilde{y}_{2r}(t)]$ are the (1)-solutions $\tilde{y}(t)$ in a r-level representation.

Case 2: If $\tilde{y}(t)$ is (2)-differentiable, then FFRDEs (3.3) and (3.4) can be converted into the following crisp system.

$$\begin{aligned} D_t^\alpha \tilde{y}_{1r}(t) &= U\tilde{y}_{2r}^2(t) + V\tilde{y}_{2r}(t) + W \\ D_t^\alpha \tilde{y}_{2r}(t) &= U\tilde{y}_{1r}^2(t) + V\tilde{y}_{1r}(t) + W \end{aligned} \tag{3.7}$$

with the fuzzy initial condition

$$\begin{aligned} \tilde{y}_{1r}(0) &= \tilde{y}_{0,1r} \\ \tilde{y}_{2r}(0) &= \tilde{y}_{0,2r} \end{aligned} \tag{3.8}$$

The process to get the equations are as follows:
(i) Generate the Residue Power Series method for the system in Equations (3.7) and (3.8).
(ii) Examine the accuracy of the solutions $[\tilde{y}_{1r}(t), \tilde{y}_{2r}(t)]$ and $[D_t^\alpha \tilde{y}_{1r}(t), D_t^\alpha \tilde{y}_{2r}(t)]$ as r-level sets, where $r \in [0,1]$.
(iii) Then $[\tilde{y}_{2r}(t), \tilde{y}_{1r}(t)]$ are the (2)-solutions $\tilde{y}(t)$ in a r-level representation.

4.The Residual Power Series Method for the Fuzzy Fractional Riccati Differential Equation

In this section, the (1)-solution for the fuzzy fractional Riccati differential equation (3.3) and (3.4) is obtained by implementing the procedures of the residual power series method. Further, the same procedure can be followed for attaining the (2)-differentiable solution. Assume that $\tilde{y}(t)$ is (1)-differentiable, therefore the solutions of equations (3.5) and (3.6) at $t = 0$ have the following forms,

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$$y_{1r}(t)=\sum_{k=0}^{\infty} a_k \frac{t^{k\alpha}}{\Gamma(1+k\alpha)}$$
$$y_{2r}(t)=\sum_{k=0}^{\infty} b_k \frac{t^{k\alpha}}{\Gamma(1+k\alpha)} \tag{4.1}$$

By using the initial conditions $y_{1r}(0)=y_{0,1r}=a_0$ and $y_{2r}(0)=y_{0,2r}=b$ as initial approximations, the expression of (4.1) can be written as

$$y_{1r}(t)=a_0+\sum_{k=1}^{\infty} a_k \frac{t^{k\alpha}}{\Gamma(1+k\alpha)}$$
$$y_{2r}(t)=b_0+\sum_{k=1}^{\infty} b_k \frac{t^{k\alpha}}{\Gamma(1+k\alpha)} \tag{4.2}$$

Consequently, the i th-truncated series solutions of $y_{1r}(t)$ and $y_{2r}(t)$ can be written as

$$y_{i,1r}(t)=a_0+\sum_{k=1}^{i} \frac{a_k t^{k\alpha}}{\Gamma(1+k\alpha)}$$
$$y_{i,2r}(t)=b_0+\sum_{k=1}^{i} \frac{b_k t^{k\alpha}}{\Gamma(1+k\alpha)} \tag{4.3}$$

According to the residual power series approach, the i th-residual functions of the system (3.5) and (3.6) are defined by

$$\mathrm{Res}_{i,1r}(t)=D^{\alpha}y_{1r}(t)-Uy_{1r}^2(t)-Vy_{1r}(t)-W$$
$$\mathrm{Res}_{i,2r}(t)=D^{\alpha}y_{2r}(t)-Uy_{2r}^2(t)-Vy_{2r}(t)-W \tag{4.4}$$

where the ∞ th-residual functions are given by

$$\mathrm{Res}_{\infty,1r}(t)=\lim_{i\to\infty}\mathrm{Res}_{i,1r}(t)=D^{\alpha}y_{1r}(t)-Uy_{1r}^2(t)-Vy_{1r}(t)-W$$
$$\mathrm{Res}_{\infty,2r}(t)=\lim_{i\to\infty}\mathrm{Res}_{i,2r}(t)=D^{\alpha}y_{2r}(t)-Uy_{2r}^2(t)-Vy_{2r}(t)-W \tag{4.5}$$

As in the residual power series method, let $\mathrm{Res}_{\infty,ir}(t)=0, \forall t\in[0,R]$, where R is the radius of convergence and $i=\{1,2\}$, which are infinitely differentiable functions at $t=0$. Then $\frac{d^{k-1}}{dt^{k-1}}\mathrm{Res}_{k,ir}(0)=0, k=1,2,3,\dots,j$. The residual power series algorithm provides the parameters a_k and b_k, where $k\geq 1$.

To find the coefficients a_1 and b_1, substitute $y_{1,1r}(t)=a_0+a_1\frac{t^{\alpha}}{\Gamma(1+\alpha)}, y_{1,2r}(t)=b_0+b_1\frac{t^{\alpha}}{\Gamma(1+\alpha)}$, into the residual functions $\mathrm{Res}_{1,1r}(t)$ and $\mathrm{Res}_{1,2r}(t)$ for $i=1$ in (4.4), then

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$$\mathrm{Res}_{1,1r}(t) = D^{\alpha} y_{1,1r}(t) - U y_{1,1r}^{2}(t) - V y_{1,1r}(t) - W$$
$$= \mathrm{D}^{\alpha}\left(a_0 + a_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) - U\left(a_0 + a_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^2 - V\left(a_0 + a_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) - W$$
$$\mathrm{Res}_{1,2r}(t) = D^{\alpha} y_{1,2r}(t) - U y_{1,2r}^{2}(t) - V y_{1,2r}(t) - W$$
$$= \mathrm{D}^{\alpha}\left(b_0 + b_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) - U\left(b_0 + b_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^2 - V\left(b_0 + b_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) - W$$

Using the residual power series fact that $\alpha = 1$, and setting $\mathrm{Res}_{1,1r}(0) = 0$ and $\mathrm{Res}_{1,2r}(0) = 0$, in(4.6), it gives that $a_1 = U\,{a_0}^2 + Va_0 + W, b_1 = U{b_0}^2 + Vb_0 + W$. Then, the first approximations are

$$y_{1,1r}(t) = a_0 + (Ua_0^2 + Va_0 + W)\frac{t^{\alpha}}{\Gamma(1+\alpha)}$$
$$y_{1,2r}(t) = b_0 + (Ub_0^2 + Vb_0 + W)\frac{t^{\alpha}}{\Gamma(1+\alpha)} \tag{4.6}$$

For $i = 2$, the second approximations are

$$y_{2,1r}(t) = a_0 + (Ua_0^2 + Va_0 + W)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + a_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$
$$y_{2,2r}(t) = b_0 + (Ub_0^2 + Vb_0 + W)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + b_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \tag{4.7}$$

The residual functions $\mathrm{Res}_{2,1r}(t)$ and $\mathrm{Res}_{2,2r}(t)$ from equation (4.4) are as follows

$$\mathrm{Res}_{2,1r}(t) = D^{\alpha} y_{2,1r}(t) - U y_{2,1r}^{2}(t) - V y_{2,1r}(t) - W,$$
$$= \mathrm{D}^{\alpha}\left(a_0 + (Ua_0^2 + Va_0 + W)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + a_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)$$
$$-\mathrm{U}\left(a_0 + (Ua_0^2 + Va_0 + W)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + a_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2$$
$$-\mathrm{V}\left(a_0 + (Ua_0^2 + Va_0 + W)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + a_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) - W$$
$$\mathrm{Res}_{2,2r}(t) = D^{\alpha} y_{2,2r}(t) - U y_{2,2r}^{2}(t) - V y_{2,2r}(t) - W,$$
$$= \mathrm{D}^{\alpha}\left(b_0 + (Ub_0^2 + Vb_0 + W)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + b_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)$$
$$-\mathrm{U}\left(b_0 + (Ub_0^2 + Vb_0 + W)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + b_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2$$
$$-\mathrm{V}\left(b_0 + (Ub_0^2 + Vb_0 + W)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + b_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) - W \tag{4.8}$$

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Differentiating both sides of the residual functions yields

$$\frac{d}{dt}\mathrm{Res}_{2,1r}(t) = \frac{d}{dt}\left[D^\alpha y_{2,1r}(t) - Uy_{2,1r}^2(t) - Vy_{2,1r}(t) - W\right]$$

and

$$\frac{d}{dt}\mathrm{Res}_{2,2r}(t) = \frac{d}{dt}\left[D^\alpha y_{2,2r}(t) - Uy_{2,2r}^2(t) - Vy_{2,2r}(t) - W\right]$$

Using the residual power series facts ($\alpha = 1$),

$$\frac{d}{dt}\mathrm{Res}_{2,1r}(0) = 0, \frac{d}{dt}\mathrm{Res}_{2,2r}(0) = 0$$

Therefore, from equation (4.8),

$$\begin{aligned} a_2 &= Ua_0a_1 + \frac{1}{2}Va_1 \\ b_2 &= Ub_0b_1 + \frac{1}{2}Vb_1 \end{aligned} \tag{4.9}$$

The second approximations are then

$$\begin{aligned} y_{2,1r}(t) &= a_0 + (Ua_0^2 + Va_0 + W)\frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{\left(Ua_0a_1 + \frac{1}{2}Va_1\right)t^{2\alpha}}{\Gamma(1+2\alpha)} \\ y_{2,2r}(t) &= b_0 + (Ub_0^2 + Vb_0 + W)\frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{\left(Ub_0b_1 + \frac{1}{2}Vb_1\right)t^{2\alpha}}{\Gamma(1+2\alpha)} \end{aligned} \tag{4.10}$$

For $i = 3$, the third approximations $y_{3,1r}(t)$ and $y_{3,2r}(t)$ are substituted into the residual functions $\mathrm{Res}_{3,1,r}(t)$ and $\mathrm{Res}_{3,2r}(t)$ of equation (4.4). Utilizing the residual power series facts ($\alpha = 1$), and the conditions,

$$\frac{d^2}{dt^2}\mathrm{Res}_{3,1r}(0) = 0, \frac{d^2}{dt^2}\mathrm{Res}_{3,2r}(0) = 0.$$

Then, the third coefficients are given by

$$\begin{aligned} a_3 &= \frac{1}{3}U(2a_0a_2 + a_1^2) + \frac{1}{3}Va_2 \\ b_3 &= \frac{1}{3}U(2b_0b_2 + b_1^2) + \frac{1}{3}Vb_2 \end{aligned} \tag{4.11}$$

For $i = 4$, the fourth approximations $y_{4,1r}(t)$ and $y_{4,2r}(t)$ are substituted into the residual functions $\mathrm{Res}_{4,1r}(t)$ and $\mathrm{Res}_{4,2r}(t)$ of equation (4.4). Using the residual power series facts ($\alpha = 1$), and the conditions,

$$\frac{d^3}{dt^3}\mathrm{Res}_{4,1r}(0) = 0, \frac{d^3}{dt^3}\mathrm{Res}_{4,2r}(0) = 0.$$

The fourth coefficients are given by

$$\begin{aligned} a_4 &= \frac{1}{2}U(a_1a_2 + a_0a_3) + \frac{1}{4}Va_3 \\ b_4 &= \frac{1}{2}U(b_1b_2 + b_0b_3) + \frac{1}{4}Vb_3 \end{aligned} \tag{4.12}$$

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By continuing the same procedure up to arbitrary coefficients of order $i = n$, and using the residual power series facts,

$$\frac{d^{n-1}}{dt^{n-1}}\mathrm{Res}_{n,1r}(0) = 0, \frac{d^{n-1}}{dt^{n-1}}\mathrm{Res}_{n,2r}(0) = 0.$$

Then, the coefficients a_n and b_n can be obtained.
Similarly, $\tilde{y}(t)$ is the (2)-solution for the (2)-differentiable fuzzy fractional Riccati differential equation (3.3) and (3.4).

5.Numerical Example

Validate the accuracy of our proposed algorithm, examples are numerically examined in this section.

Example 5.1

Consider the fuzzy fractional Riccati differential equation

$$D_{0^+}^{\alpha}\tilde{y}(t) = 2\tilde{y}(t) - \tilde{y}^2(t) + 1, 0 \le t \le R, 0 < \alpha < 1. \qquad (5..1.1)$$

with the fuzzy initial condition

$$[\tilde{y}(0)]_\gamma = \tilde{0} = [\gamma - 1, 1 - \gamma], \gamma \in [0,1] \qquad (5.1.2)$$

The exact solution of the FFRDEs in Eqs. (5.1.1) and (5.1.2) is

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right), \text{when } \gamma = 1$$

The Taylor expansion of $\tilde{y}(t)$ around $t = 0$ is

$$y(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{2}t^4 - \frac{7}{15}t^5 - \frac{7}{45}t^6 + \frac{53}{315}t^7 + \frac{71}{315}t^8 + \cdots. \qquad (5.1.3)$$

The FFRDEs, as in Eqs. (5.1.1) and (5.1.2), can be represented by the set of ODEs corresponding to their parametric forms as

$$\begin{aligned} D^{\alpha}y_{1r}(t) &= 2\tilde{y}_{1r}(t) - \tilde{y}_{1r}^2(t) + 1 \\ D^{\alpha}y_{2r}(t) &= 2\tilde{y}_{2r}(t) - \tilde{y}_{2r}^2(t) + 1 \end{aligned} \qquad (5.1.4)$$

with fuzzy initial conditions

$$\begin{aligned} y_{1r}(0) &= \gamma - 1 \\ y_{2r}(0) &= 1 - \gamma \end{aligned}$$

By using the initial conditions $y_{1r}(0) = y_{1,r}(0) = a_0$ and $y_{2r}(0) = y_{2,r}(0) = b_0$ as initial approximations, the expression of (5.1.5) can be written as $y_{1r}(0) = \gamma - 1$ and $y_{2r}(0) = 1 - \gamma$. The residual power series solutions $D^{\alpha}y_{1r}(t)$ and $D^{\alpha}y_{2r}(t)$ of system (5.1.4) can be written as

$$\begin{aligned} D^{\alpha}y_{1r}(t) &= \gamma - 1 + \frac{a_1 t^{\alpha}}{\Gamma(1+\alpha)} + \frac{a_2 t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots + \frac{a_n t^{n\alpha}}{\Gamma(1+n\alpha)} + \cdots \\ D^{\alpha}y_{2r}(t) &= 1 - \gamma + \frac{b_1 t^{\alpha}}{\Gamma(1+\alpha)} + \frac{b_2 t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots + \frac{b_n t^{n\alpha}}{\Gamma(1+n\alpha)} + \cdots \end{aligned} \qquad (5.1.5)$$

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The coefficients a_n and b_n for the residual power series solutions are calculated iteratively by using the residual power series method $\frac{d^{n-1}}{dt^{n-1}}\mathrm{Res}_{n,1r}(0) = 0$ and $\frac{d^{n-1}}{dt^{n-1}}\mathrm{Res}_{n,2r}(0) = 0$, for $\mathrm{n} = 1,2$ ...,

$a_0 = \gamma - 1$

$a_1 = -2 + 4\gamma - \gamma^2$

$a_2 = \gamma^3 - 6\gamma^2 + 10\gamma - 4$

$a_3 = \frac{1}{3}(-20 + 64\gamma - 64\gamma^2 + 24\gamma^3 - 3\gamma^4)$

$a_4 = \frac{1}{3}(-32 + 128\gamma - 180\gamma^2 + 110\gamma^3 - 30\gamma^4 + 3\gamma^5 - 15\gamma^6)$

$a_5 = \frac{1}{15}(-256 + 1232\gamma - 2228\gamma^2 + 1920\gamma^3 - 840\gamma^4 + 180\gamma^5 - 15\gamma^6)$

and

$b_0 = 1 - \gamma$

$b_1 = 2 - \gamma^2$

$b_2 = -\gamma^3 + 2\gamma$

$b_3 = \frac{1}{3}(-4 + 8\gamma^2 - 3\gamma^4)$

$b_4 = \frac{1}{3}(-8 + 10\gamma^3 - 3\gamma^5)$

$b_5 = \frac{1}{15}(16 - 68\gamma^2 + 60\gamma^4 - 15\gamma^6)$

The Residue Power Series solution to Eq. (5.1.4) in an infinite series is as follows

$$\tilde{y}_{1r}(t) = (\gamma - 1) + (-2 + 4\gamma - \gamma^2)t + (\gamma^3 - 6\gamma^2 + 10\gamma - 4)t^2 + \cdots$$
$$\tilde{y}_{2r}(t) = (1 - \gamma) + (2 - \gamma^2)t + (-\gamma^3 + 2\gamma)t^2 + \cdots$$

As a special case, when $\gamma = 1$, the residual power series solution becomes coincides with the exact solution in terms of the infinite series

$$y(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{2}t^4 - \frac{7}{15}t^5 - \frac{7}{45}t^6 + \frac{53}{315}t^7 + \frac{71}{315}t^8 + \cdots$$

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Table 1 shows the numerical results of the exact and approximate solutions using the RPS method for Example 5.1 at $\gamma = 1$. Moreover, the relative and absolute errors at some chosen grid points t_i in $[0,1]$ with a step-size of 0.1 are also given in Table 1 for $\mathrm{n} = 51$ and $\gamma = 1$. The results indicate that the RPS approximate solutions are almost as close as to the exact solutions comparative VIM. However, a more accurate solution can be obtained using more iterations

Table 1: Comparison of Exact, VIM, and RPS Solutions for Example 5.1 at $\gamma = 1$

t_i	Exact Solution	VIM Solution	RPS Solution	Absolute error	Relative error
0.1	0.110295196916	0.110278530190	0.110295196917	1.52656×10^{-16}	1.38406×10^{-15}
0.2	0.241976799621	0.241710108444	0.241976799621	5.55112×10^{-17}	2.29407×10^{-16}
0.3	0.395104848660	0.393754185429	0.395104848660	2.77556×10^{-16}	7.02486×10^{-16}
0.4	0.567812166293	0.563540894476	0.567812166293	3.33067×10^{-16}	5.86579×10^{-16}
0.5	0.756014393431	0.745597718254	0.756014393431	3.33067×10^{-16}	4.40556×10^{-11}
0.6	0.953566216472	0.932150235429	0.953566216472	1.11022×10^{-16}	1.16429×10^{-16}
0.7	1.152948966979	1.114399827333	1.152948966980	5.32907×10^{-14}	4.62212×10^{-14}
0.8	1.346363655368	1.285272104635	1.346363655422	5.39830×10^{-11}	4.00954×10^{-11}
0.9	1.526911313280	1.444220694000	1.526911336846	2.35658×10^{-8}	1.54336×10^{-8}

Example 5.2

Consider the fuzzy fractional Riccati differential equation

$$^{C}D_{0^+}^{\alpha}\tilde{y}(t) = 2\tilde{y}(t) - \tilde{y}^2(t) + 1, 0 \leq t \leq R, 0 < \alpha < 1. \quad (5.2.1)$$

with the fuzzy initial condition

$$[\tilde{y}(0)]_\gamma = \tilde{0} = [0.1 + 0.1\gamma, 0.3 - 0.1\gamma], \gamma \in [0,1]. \quad (5.2.2)$$

For $\gamma = 1$, the initial condition in Eq. (5.2.2) becomes $y(0) = 0.2$. Therefore, the exact solution to Eq. (5.1.1) can be expressed as

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{5\sqrt{2} - 4}{5\sqrt{2} + 4}\right)\right) \quad (5.2.3)$$

The FIVP, as in Eqs. (5.2.1) and (5.2.2), can be represented by the set of ODEs corresponding to their parametric forms as

$$\begin{aligned} D^\alpha y_{1r}(t) &= 2\tilde{y}_{1r}(t) - \tilde{y}_{1r}^2(t) + 1 \\ D^\alpha y_{2r}(t) &= 2\tilde{y}_{2r}(t) - \tilde{y}_{2r}^2(t) + 1 \end{aligned} \quad 5.2.4)$$

with fuzzy initial conditions

$$\begin{aligned} y_{1r}(0) &= 0.1 + 0.1\gamma \\ y_{2r}(0) &= 0.3 - 0.1\gamma \end{aligned} \quad (5.2.5)$$

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By using the initial conditions $y_{1r}(0) = y_{1,r}(0) = a_0$ and $y_{2r}(0) = y_{2,r}(0) = b_0$ as initial approximations, the expression of (5.2.5) can be written as $y_{1r}(0) = 0.1 + 0.1\gamma$ and $y_{2r}(0) = 0.3 - 0.1\gamma$. The residual power series solutions $D^\alpha y_{1r}(t)$ and $D^\alpha y_{2r}(t)$ of system (5.2.4) can be written as

$$\begin{aligned} D^\alpha y_{1r}(t) &= 0.1 + 0.1\gamma + \frac{a_1 t^\alpha}{\Gamma(1+\alpha)} + \frac{a_2 t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots + \frac{a_n t^{n\alpha}}{\Gamma(1+n\alpha)} + \cdots \\ D^\alpha y_{2r}(t) &= 0.3 - 0.1\gamma + \frac{b_1 t^\alpha}{\Gamma(1+\alpha)} + \frac{b_2 t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots + \frac{b_n t^{n\alpha}}{\Gamma(1+n\alpha)} + \cdots \end{aligned} \tag{5.2.6}$$

The first few terms a_k are given as

$$\begin{aligned} a_0 &= 0.1 + 0.1\gamma \\ a_1 &= \frac{1}{102}(119 + 18\gamma - \gamma^2) \\ a_2 &= \frac{1}{103}(1071 + 43\gamma - 27\gamma^2 + \gamma^3) \\ a_3 &= \frac{1}{3\times 10^4}(5117 - 5652\gamma - 658\gamma^2 + 108\gamma^3 - 3\gamma^4) \\ a_4 &= \frac{1}{3\times 10^5}(-168147 - 64585\gamma + 5130\gamma^2 + 1430\gamma^3 - 135\gamma^4 + 3\gamma^5) \\ a_5 &= \frac{1}{3\times 10^6}(-1537123 + 11682\gamma + 151955\gamma^2 + 540\gamma^3 - 2445\gamma^4 + 162\gamma^5 - 3\gamma^6) \\ a_6 &= \frac{1}{9\times 10^7}(695079 + 18187783\gamma + 2825739\gamma^2 - 719285\gamma^3 - 40635\gamma^4 + 11109\gamma^5 - 567\gamma^6 + 9\gamma^7) \end{aligned}$$

The first few terms b_k are given as

$$\begin{aligned} b_0 &= 0.3 - 0.1\gamma \\ b_1 &= \frac{1}{102}(151 - 14\gamma - \gamma^2) \\ b_2 &= \frac{1}{103}(1057 + 53\gamma - 21\gamma^2 - \gamma^3) \\ b_3 &= \frac{1}{3\times 10^4}(-8003 + 7084\gamma - 82\gamma^2 - 84\gamma^3 - 3\gamma^4) \\ b_4 &= \frac{1}{3\times 10^5}(-267421 + 30985\gamma + 10710\gamma^2 - 470\gamma^3 - 105\gamma^4 - 3\gamma^5) \\ b_5 &= \frac{1}{3\times 10^6}(-935747 - 560126\gamma + 108755\gamma^2 + 13020\gamma^3 - 1005\gamma^4 - 126\gamma^5 - 3\gamma^6) \\ b_6 &= \frac{1}{9\times 10^7}(42289513 - 20342887\gamma - 1706523\gamma^2 + 685685\gamma^3 + 38955\gamma^4 - 5061\gamma^5 - 441\gamma^6 - 9\gamma^7) \end{aligned}$$

As a special case, when $\gamma = 1$, the residual power series solution becomes coincides with the exact solution in terms of the infinite series

$$y(t) = \frac{1}{5} + \frac{34}{25}t + \frac{136}{125}t^2 - \frac{68}{1875}t^3 - \frac{7072}{9365}t^4 - \frac{21488}{46875}t^5 + \frac{163744}{703125}t^6 + \frac{11459632}{24609375}t^7 + \frac{16217728}{123046875}t^8 + \cdots$$

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Table 2: Comparison of Exact, VIM, and RPS Solutions for Example 5.2 at $\gamma = 1$

t_i	Exact Solution	VIM Solution	RPS Solution	Absolute error	Relative error
0.1	0.346763995050	0.213819197444	0.346763995059	1.11022×10^{-16}	3.20167×10^{-16}
0.2	0.513897275840	0.354019060301	0.513897275841	1.11022×10^{-16}	2.16040×10^{-16}
0.3	0.697990872510	0.514101858879	0.697990872514	2.22045×10^{-16}	3.18120×10^{-16}
0.4	0.893472160240	0.691903840329	0.893472160244	3.33067×10^{-16}	3.72778×10^{-16}
0.5	1.093134993060	0.883224192910	1.093134993060	2.22045×10^{-16}	2.03126×10^{-16}
0.6	1.289137171940	1.081929582737	1.289137171947	2.22045×10^{-16}	1.72243×10^{-16}
0.7	1.474194376960	1.280538738237	1.474194376963	1.05871×10^{-12}	7.18161×10^{-13}
0.8	1.642601362550	1.471235301080	1.642601633577	1.02182×10^{-9}	6.22073×10^{-10}
0.9	1.790797918330	1.647062909193	1.790798347506	4.29173×10^{-7}	2.39655×10^{-7}

The fuzzy fractional Riccati differential equation has been solved using two different methods, the Residual Power Series (RPS) method and the Variational Iteration Method (VIM) [11]. The RPS method constructs an analytical power series solution with accuracy, closely matching the exact solution and showing low errors comparatively VIM method. It ensures systematic convergence by minimizing residuals at each step.

6.Conclusion

This paper focused on the application of the Residual Power Series (RPS) method under the framework of strongly generalized differentiability to derive exact solutions for fuzzy fractional Riccati differential equations (FFRDEs). A suitable fuzzy rule base was carefully selected, extracted, and applied within defined constraints. The findings demonstrated that the RPS method significantly reduces simulation time for nonlinear fuzzy fractional differential equations while preserving accuracy. Moreover, the RPS results were found to be both reliable and comparable with those obtained through alternative methods.

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