International Journal of Applied Mathematics
Volume 38 No. 2s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

VERTEX COLOURING OF FINITE NETWORKS WITH RESPECT TO AVERAGE DISTANCE

Priyanka Pandey [1] , Mayamma Joseph [2]

[1] CHRIST (Deemed to be University), Bengaluru,
Karnataka - 560029, INDIA
e-mail: priyanka.pandey@christuniversity.in

[2] CHRIST (Deemed to be University), Bengaluru,
Karnataka - 560029, INDIA
e-mail: mayamma.joseph@christuniversity.in

Abstract

For a finite network, represented as a graph $G = (V, E)$ with average distance $\mu(G)$, the *average distance colouring* of G is a function c from V to the set of non-negative integers, such that for any $v \in V$, $|c(v) - c(u)| \geq 1$ for all $u \in V$ such that $d(u, v) \leq \lceil \mu \rceil$. In this paper, we find the average distance colouring number of some special types of networks and present a greedy algorithm to colour any graph with average distance colouring constraint.

Math. Subject Classification 2020: 05C12, 05C15

Key Words and Phrases: graph colouring, average distance, average distance colouring

Received: July 26, 2025

International Journal of Applied Mathematics
Volume 38 No. 2s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

1 Introduction

A network is often modelled as a graph, wherein the vertices represent entities of the network and the edges represent connections or interactions between them. Several problems related to networks can be solved using the idea of graph colouring, a core concept in graph theory. The colouring could be that of vertices, edges or both. Among several variants of graph coluring, a proper vertex colouring (proper colouring) of a graph G is an assignment of colours to the vertices of G such that no two adjacent vertices receive the same colour.

Graph colouring has widespread applications in problems such as scheduling, frequency assignment, and even solving puzzles like Sudoku. Extensive work has been carried out in graph colouring by employing constraints based on various factors and properties. One such factor is the distance[2] between vertices, where vertex colours are dependent on the distance. Kramer and Kramer, have provided an extensive survey on distance-colouring in graphs [5], which was complemented by the study of *k-distance* chromatic numbers in trees and cycles by Niranjan and Kola [6]. We are interested in examining a vertex colouring problem with respect to the concept of average distance in graphs. The average distance $\mu(G)$ of a graph measures how spread out the vertices are, which is crucial in applications like sensor networks, distributed systems, and social network analysis. This concept of average distance colouring was introduced in [8] and the current study is an extension of the same by obtaining an algorithm to colour the vertices of any network using this new colouring protocol.

Before proceeding with the definition of average distance colouring, we first define the average distance in a graph. For a graph G with n vertices, the average distance

$$\mu(G) = \frac{1}{\binom{n}{2}} \sum_{u,v \in V} d(u,v). \tag{1}$$

Received: July 26, 2025

International Journal of Applied Mathematics
Volume 38 No. 2s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

The average distance colouring of $G = (V, E)$ is a function c from V to the set of non-negative integers, such that for any $v \in V$, $|c(v) - c(u)| \geq 1$ for all u such that $d(u, v) \leq \lceil \mu \rceil$ [8]. The minimum number of distinct colours required to colour any graph G such that G admits an average distance colouring is called average distance colouring number, χ_μ, of G.

Bounds for the colouring number and colouring for few of the graph classes is obtained in [8]. In this article we investigate the colouring problem for few other relavant networks and present a Python program to calculate average distance of any graph G. Further, we present a greedy algorithm to colour any graph G with average distance colouring constraint.
Throughout the paper, we consider connected graphs denoted by $G = (V, E)$ with vertex set V and edge set E. For the definitions not defined in this paper we refer West[4].

2 Exact value of $\chi_\mu(G)$ for some special networks

We obtain value of $\chi_\mu(G)$ of networks for applications related to communication networks. These networks have applications in various field importantly in area of hub-and-spoke network model, which is considered for our first result where each group of nodes (complete graphs) are independent but each of this group communicate via one common vertex/node. This could also be used in resource allocation where the overall work in managed by central authority.

Theorem 1. *For a Windmill Graph W_s^k,*

$$\chi_\mu(W_s^k) = ks - k + 1$$

Proof. Consider the graph G obtained from k copies of a complete graph K_s by adjoining the complete graphs to one vertex. This is called windmill graph W_s^k which finds its application in

Received: July 26, 2025

International Journal of Applied Mathematics
Volume 38 No. 2s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

communcation network. Generalized windmill graphs are a generalization of the classic windmill graphs with applications in network communication, optimization, wireless sensor networks, ad-hoc networks, and distributed computing systems[10].

In this graph, if two vertices belongs to different complete graphs they will be distance 2 apart and if they are from same complete graph they are at distance one. Using this fact, For complete graph K_s,

$$\sum_{u,v \in V(K_s)} d(u,v) = \frac{s(s-1)}{2}.$$

Since we have k copies of K_s,

$$\sum_{u,v \in V(W_s^k)} d(u,v) = \frac{ks(s-1)}{2} + k(k-1)(s-1)^2.$$

Further, the total number of vertices in (W_s^k) is $sk - k + 1$ and thus

$$\mu(W_s^k) = \frac{\sum_{u,v \in V(W_s^k)} d(u,v)}{\binom{ks-k+1}{2}}$$

which can be simplified to the following expression

$$\mu(W_s^k) = \frac{2(k-1)(s-1) + s}{(ks-k+1)}.$$

We will show that for any value of $k \geq 2$ and $s \geq 3$ $\mu(W_s^k) \leq 2$. On the contrary, assume that $\mu(W_s^k) > 2$.

$$\implies s < 0$$

which is a contradiction to the fact that $s \geq 3$. Therefore, $\mu(W_s^k) \leq 2$. Further, we know that for any graph which is not a complete graph $\mu(G) > 1$. Thus definition f or a verage d istance colouring reduces to a colouring c such that $|c(u) - c(v)| \geq 1$ for all u such that $d(u,v) \leq 2$ for every $v \in V$.
Since the diameter of $W_s^k = 2$, each vertex should be assigned

Received: July 26, 2025

International Journal of Applied Mathematics
Volume 38 No. 2s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

a distinct colour giving $\chi_\mu(W_s^k) \geq ks - k - 1$ which can be attained by assigning the central vertex 0 and the remaining vertices $1, 2, ...k(s-1)$ giving $\chi_\mu(W_s^k) \leq ks - k + 1$. Hence the result.

□

For the next result, we consider a graph $G = K_s \circ \overline{K_r}$. This structure plays an important role in network security analysis. We will use the following theorem to prove our next result.

Theorem 2. *[8] For a graph G on n vertices with average distance $\mu(G)$, $\chi_\mu = n$ if and only if $d = \lceil \mu(G) \rceil$ where d denotes the diameter of G.*

Theorem 3. *For a graph $G = K_s \circ \overline{K_r}$ obtained by considering complete graph K_s and adding r pendant edges to each of the vertices of K_s,*

$$\chi_\mu(G) = \begin{cases} s + r & for\ s \geq 3\ and\ r = 1 \\ sr + s & otherwise. \end{cases}$$

Proof. For the graoh $G = K_s \circ \overline{K_r}$, on calculating the sum of the distances between any two pair of vertices in G, we get the following expression.

$$\sum_{u,v \in V(G)} d(u,v) = sr^2 + s(s-1)\frac{(3r^2 + 4r + 1)}{2}$$

Further, we obtain the expression for average distance of graph which is given below.

$$\mu(G) = \frac{2sr^2 + s(s-1)(3r^2 + 4r + 1)}{(rs + s - 1)(rs + s)}$$

We know that the average distance of G must be greater than 1. Next, we determine the cases for which the average distance is less than or equal to 2. If $\mu(G)$ is less than or equal to 2, then

$$\frac{2sr^2 + s(s-1)(3r^2 + 4r + 1)}{(rs + s - 1)(rs + s)} \leq 2$$

Received: July 26, 2025

International Journal of Applied Mathematics
Volume 38 No. 2s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

On simplification, we get

$$r(sr - r - 2) \leq (s - 1)$$

Given that s considered for this theorem is always greater than or equal to three, it is easy to verify that, this is true if $r = 1$ and $s \geq 3$. Otherwise, $\mu(G) > 2$.

When $r = 1$ and $s \geq 3$, we colour graph G such that $|c(u) - c(v)| \geq 1$, if $d(u, v) \leq 2$. This could be achieved by colouring r vertices adjacent to any vertex of K_s with distinct colours $1, 2, ..., r$. Since each set of pendant vertices is at distance 3, the same set of colours can be used for all s sets of r vertices. Further, we can assign colours $r + 1, r + 2, ..., r + s$ to the vertices of K_s. Thus, we get $\chi_\mu \leq r + s$. Since it is impossible to repeat the colours in the set of pendant vertices and the colours of pendant vertices can never be assigned to the vertices of K_s, the above colouring is optimum, giving $\chi_\mu \geq s + r$. Hence, the result.

For the remaining values of s and r, $\lceil \mu(G) \rceil = 3$. Thus, the number of colours required to colour such a graph is equal to the number of vertices of the graph (using Theorem 2). Hence the result. □

Obtaining exact value of χ_μ becomes easier for the graphs which are of smaller diameter and also where the sum of the distances between any two pairs of vertices depends only on one variable n. On considering other graphs where there value of average distance keeps changing, it becomes difficult to find the bound for $\mu(G)$ of such graphs for example, Dutch windmill graph D_r^k, which is obtained by taking k copies of C_r with a common vertex. This leads us to approach this problem using programming and calculate the value of $\mu(G)$ for any arbitrary graph G which helps in finding $\chi_\mu(G)$.

For a graph G, with adjacency list, following program gives the value of $\mu(G)$.

```
from queue import Queue
myGraph = {0:[1,2,4],1:[0,2],2:[0,1,5],3:[4],4:[0,3,5,6],\
```

Received: July 26, 2025

International Journal of Applied Mathematics
Volume 38 No. 2s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

```
5:[4,2],6:[4]}
nver=7
def leastDistance(graph, source):
    Q = Queue()
    distance = {k: 9999999 for k in myGraph.keys()}
    visited_vertices = set()
    Q.put(source)
    visited_vertices.update({source})
    while not Q.empty():
        vertex = Q.get()
        if vertex == source:
            distance[vertex] = 0
        for u in graph[vertex]:
            if u not in visited_vertices:
                # update the distance
                if distance[u] > distance[vertex] + 1:
                    distance[u] = distance[vertex] + 1
                Q.put(u)
                visited_vertices.update({u})
    lis= list(distance.values())
    return distance, lis
sum=0
for i in range(nver):
    print("Least distance of vertices from vertex", i," is:")
    g2= leastDistance(myGraph, i)
    h2=g2[1]
    print(h2)
    for j in range(i,nver):
        sum=sum+h2[j]
print()

print("sum of all distance",sum)

print("mu of graph", sum/(nver*(nver-1)/2))
```

Received: July 26, 2025

International Journal of Applied Mathematics
Volume 38 No. 2s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

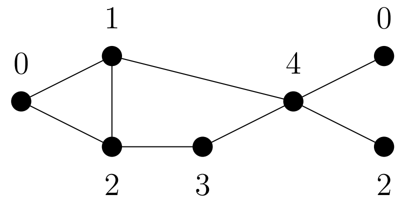

Figure 1: Average distance colouring of graph G with $\mu(G) = 1.809$

In the above Python program, we have obtained the average distance of the graph (Figure 1) with an adjacency list given in the form of a Python dictionary.

Adjacency list = $\{0:[1,2,4], 1:[0,2], 2:[0,1,5], 3:[4], 4:[0,3,5,6], 5:[4,2], 6:[4]\}$

Output:

Least distance of vertices from vertex 0 is:
[0, 1, 1, 2, 1, 2, 2]
Least distance of vertices from vertex 1 is:
[1, 0, 1, 3, 2, 2, 3]
Least distance of vertices from vertex 2 is:
[1, 1, 0, 3, 2, 1, 3]
Least distance of vertices from vertex 3 is:
[2, 3, 3, 0, 1, 2, 2]
Least distance of vertices from vertex 4 is:
[1, 2, 2, 1, 0, 1, 1]
Least distance of vertices from vertex 5 is:
[2, 2, 1, 2, 1, 0, 2]
Least distance of vertices from vertex 6 is:
[2, 3, 3, 2, 1, 2, 0]
sum of all distance 38
mu of graph 1.8095238095238095

Received: July 26, 2025

International Journal of Applied Mathematics
Volume 38 No. 2s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

3 Greedy algorithm for average distance colouring of graph G

From the program above, we can obtain the list of distances from each vertex to every other vertex of the graph G. Using this information, we present a greedy algorithm to colour any graph G which gives an approximate solution for the average distance colouring problem. We require the following notations for further study.

The diameter of graph G is the greatest distance between any two vertices of G [9]. The end vertices of the path whose length is equal to the diameter of G are considered as *peripheral vertices*. For any vertex v of G, $N_k[v]$ denotes the set of vertices which are at a distance less than or equal to k in G. We use a list denoted by $D_i = [d_1, d_2, ..., d_n]$ which represents the distance of vertex i from vertices $v_1, v_2, .., v_n$ respectively.

Algorithm 1: GADC Algorithm

Consider a graph G on n vertices with vertices ordered $v_1, v_2, ..., v_n$.

1. Find the average distance of graph G.
2. Select any peripheral vertex *say* v for one of the diametral paths of graph G.
3. Colour vertex v with colour c_1. Colour the vertices in μ neighbourhood of G with colours $c_2, c_3, \ldots c_l$ where $l = |N_\mu[v]|$.
4. Select an uncoloured vertex u adjacent to one of the coloured vertices. Consider the list of distances $[d_1, d_2, ..., d_n]$ corresponding to u. Examine the value of d_i for coloured vertices and inspect if there exist $d_i > \lceil \mu \rceil$. If it exists, select the colour *say* c_k of such vertex if c_k is not assigned to any vertex in $N_\mu(u)$ and assign it to vertex u. Else, assign the vertex u with a new colour.
5. Repeat Step 4 till there exists no uncoloured vertex in G.

We illustrate the above algorithm for the graph G shown in Figure 2.

Received: July 26, 2025

International Journal of Applied Mathematics
Volume 38 No. 2s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

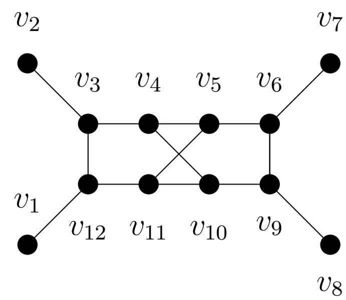

Figure 2: Illustration of average distance colouring greedy algorithm

From the program given above, we obtain the value of $\mu(G) = 2.484$. Thus, $\lceil \mu(G) \rceil = 3$. (Step 1)
From the definition of average distance colouring $|c(u) - c(v)| \geq 1$ if $d(u, v) \leq 3$ for the given graph G. Next, we choose the peripheral vertex say v_2 (Step 2) for the diametral path $v_2, v_3, v_4, v_5, v_6, v_7$. Step 3 gives the colouring of G as shown in Figure 3.

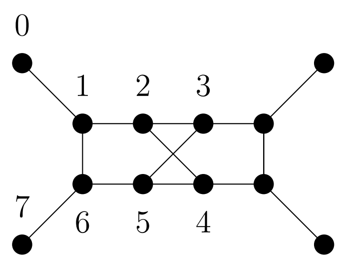

Figure 3: Illustration of Step 3

Further, Figures 4, 5, 5, 6, 7 show the implementation of Step 4 to achieve the average distance colouring of G.

Received: July 26, 2025

International Journal of Applied Mathematics
Volume 38 No. 2s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

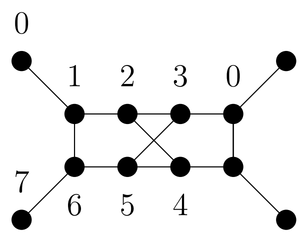

Figure 4: Illustration of Step 4 - Part 1

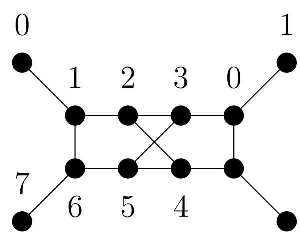

Figure 5: Illustration of Step 4 - Part 2

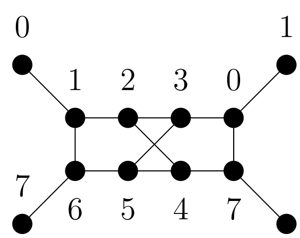

Figure 6: Illustration of Step 4 - Part 3

Received: July 26, 2025

International Journal of Applied Mathematics
Volume 38 No. 2s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

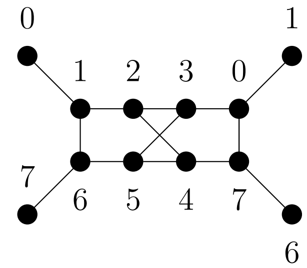

Figure 7: Illustration of Step 4 - Part 4

The concept of the average distance colouring of a graph restricts the nodes from receiving the same colour if they are within the range of average distance of the graph undertaken for the study. Using the program given in this article, one can also look into the problem of examining the values or bounds for $\chi_\mu(G)$ in terms of the order and size for different classes of graphs G.

References

[1] F. Harary, Addison Wesley, Reading, Massachusetts, 1969.

[2] W. Goddard and O. R. Oellermann, Structural Analysis of Complex Networks: 49-72, 2011.

[3] H. Wiener, Structural determination of paraffin boiling points, *J. Am. Chem. Soc.* (1947), 69, 1, 17–20.

[4] D. B. West, *Introduction to Graph Theory*, 2nd ed., Prentice Hall, Upper Saddle River, NJ, 2001.

[5] F. Kramer and H. Kramer, A survey on the distance-colouring of graphs, *Discrete Mathematics*, vol. 308, no. 2, pp. 422–426, 2008.

Received: July 26, 2025

International Journal of Applied Mathematics
Volume 38 No. 2s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

[6] P. K. Niranjan and S. R. Kola, The k-distance chromatic number of trees and cycles, *AKCE International Journal of Graphs and Combinatorics*, vol. 16(2), 2019

[7] L. March and P. Steadman, *The Geometry of Environment: An introduction to spatial organization in design*, M.I.T. Press, Cambridge, Mass., 1974.

[8] P. Pandey and M. Joseph, *Average distance colouring of graphs*, Acta Univ. Sapientiae Informatica, vol. 15, no. 2, pp. 205–220, 2023, doi: 10.2478/ausi-2023-0014.

[9] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & Digraphs*, 6th ed., Chapman and Hall/CRC, London, 2015.

[10] W. Wang, M. Vladimir, M. Abisado and N. Mujahid, "Characterizing Generalized Windmill Graphs in terms of Complexity Measures via Eccentricity Dependent Topological Indices," 2023 *3rd International Symposium on Computer Technology and Information Science (ISCTIS)*, Chengdu, China, 2023, pp. 539-546

Received: July 26, 2025