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$G(X)$-QUASI INVO-CLEAN RINGS

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Abstract

Let $C(R)$ be the center of a ring R and $g(x) \in C(R)[x]$ be a fixed polynomial. In this paper, we introduce the notion of $g(x)$-quasi invo-clean rings where every element r can be written as $r = v + s$, where $v \in Qinv(R)$ and s is a root of $g(x)$. We study various properties of $g(x)$-quasi invo-clean rings. We prove that, for an even polynomial $g(x)$, the ring $R = \prod_{i \in I} R_i$ is $g(x)$-quasi invo-clean if and only if every R_i is $g(x)$-quasi invo-clean.

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Key Words and Phrases: Quasi involution elements, Quasi invo-clean rings, $g(x)$-quasi invo-clean rings.

1 Introduction

Let R be an associative ring with identity. An element $v \in R$ is called an involution when $v^2 = 1$. Moreover, v is said to be a

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quasi-involution if either v itself or its complement $1 - v$ is an involution [9]. We use the following notations throughout: $U(R)$ for the set of units of R, $Id(R)$ for the collection of idempotent elements, $Inv(R)$ for the set of involutions, and $Qinv(R)$ for the set of quasi-involutions. A ring R is termed clean if every $r \in R$ can be written as $r = u + e$, where $u \in U(R)$ and $e \in Id(R)$ [3, 20]. Various extensions of the concept of clean rings have been investigated in the literature [2, 5, 12, 13, 14, 15, 16, 17]. The ring R is called $g(x)$-clean whenever, for every $r \in R$ there exist a unit $u \in U(R)$ and a root s of the polynomial $g(x)$ such that $r = u + s$ [11]. Similarly, R is said to be invo-clean if for each $r \in R$ one can find $v \in Inv(R)$ and $e \in Id(R)$ satisfying $r = v + e$ [6, 7]. In the same manner, the ring R is defined as $g(x)$-invo-clean when for each $r \in R$ there exist $v \in Inv(R)$ and a root s of $g(x)$ with $r = v+s$ [10]. Finally, R is called quasi invo-clean if for every $r \in R$ there exist an element $v \in Qinv(R)$ together with an idempotent $e \in Id(R)$ such that $r = v + e$ [8]. In this paper, we introduce and investigate the concept of a $g(x)$-quasi invo-clean ring. Let R be a ring and $g(x) \in C(R)[x]$ be a fixed polynomial. An element $r \in R$ is said to be $g(x)$-quasi invo-clean if there exist some $v \in Qinv(R)$ and a root s of $g(x)$ such that $r = v + s$. The ring R itself is called $g(x)$-quasi invo-clean whenever every element of R admits such a decomposition. We study various properties of $g(x)$-quasi invo-clean rings. We establish a number of structural results concerning these rings. In particular, for a ring R, elements $a, b \in R$ and a natural number$n$, we prove that R is $(ax^{2n} - bx)$-quasi invo clean if and only if it is $(ax^{2n} + bx)$-quasi invo clean (Lemm 7). Moreover, it is shown that if $g(x)$ is an even polynomial, then the direct product ring $R = \prod_{i\in I} R_i$ is $g(x)$-quasi invo-clean precisely when each component ring R_i enjoys the same property (Lemma 13).

Finally, we demonstrate that for any commutative ring R, the polynomial ring $R[x]$ fails to be $(x^2 - x)$-quasi invo-clean (Theorem 17).

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2 Main Results

Definition 1. A ring R is said to be invo-clean if for each $r \in R$ one can find $v \in Inv(R)$ and $e \in Id(R)$ satisfying $r = v + e$ [6].

Definition 2. An element $v \in R$ is said to be a quasi-involution element if $v^2 = 1$ or $(1 - v)^2 = 1$. $Qinv(R)$ denotes the set of all quasi-involutions in R [8].

Definition 3. Let R be a ring and $g(x) \in C(R)[x]$ be a fixed polynomial. The ring R is called $g(x)$-invo-clean when for each $r \in R$ there exist $v \in Inv(R)$ and a root s of $g(x)$ with $r = v + s$ [10].

Definition 4. A ring R is called quasi invo-clean if for every $r \in R$ there exist an element $v \in Qinv(R)$ together with an idempotent $e \in Id(R)$ such that $r = v + e$ [8].

Definition 5. Let R be a ring and $g(x) \in C(R)[x]$ be a fixed polynomial. An element $r \in R$ is said to be $g(x)$-quasi invo-clean if there exist some $v \in Qinv(R)$ and a root s of $g(x)$ such that $r = v + s$. The ring R itself is called $g(x)$-quasi invo-clean whenever every element of R admits such a decomposition.

Every $g(x)$-invo-clean ring as well as every quasi-invo-clean ring automatically belongs to the class of $g(x)$-quasi invo-clean rings. However, the next example demonstrates that, in general, a $g(x)$-quasi invo-clean ring need not be $g(x)$-invo-clean nor quasi-invo-clean.

Example 6. *(i) Let Z denote the set of integers and $R = Z_5$. Then $Qinv(R) = \{0, 1, 2, 4\}$, $Inv(R) = \{1, 4\}$ and $Id(R) = \{0, 1\}$. Hence R is a quasi invo-clean ring which is not invo-clean. Then R is a $(x^2 - x)$-quasi invo-clean ring which is not $(x^2 - x)$-invo-clean.*

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(ii) *Let Z denote the set of integers and $R = Z_7$ and $g(x) = x^7+6x \in C(R)[x]$. Then $Qinv(R) = \{0, 1, 2, 6\}$, $Root(g(x)) = \{0, 2, 3, 5, 6\}$ and $Id(R) = \{0, 1\}$. Hence R is a $g(x)$-quasi invo-clean ring which is not quasi invo-clean.*

Lemma 7. *Suppose R is a ring and $a, b \in R$ with a natural number n. Then R is $(ax^{2n} - bx)$-quasi invo clean precisely when it is $(ax^{2n} + bx)$-quasi invo clean.*

Proof. Suppose that R is $(ax^{2n}-bx)$-quasi invo clean and $r \in R$. Hence $1 - r = v + s$ where $v \in Qinv(R)$ and $as^{2n} - bs = 0$. Then $r = (1-v)+(-s)$ such that $1-v \in Qinv(R)$ and $a(-s)^{2n}+b(-s) = 0$. Therefore R is $(ax^{2n} + bx)$-quasi invo clean.
Conversely, assume that R is $(ax^{2n}+bx)$-quasi invo clean and $r \in R$. Hence $1 - r = v + s$ where $v \in Qinv(R)$ and $as^{2n} + bs = 0$. Then $r = (1 - v) + (-s)$ such that $1 - v \in Qinv(R)$ and $as^{2n} - bs = 0$. Therefore R is $(ax^{2n} - bx)$-quasi invo clean. □

The next example illustrates that Lemma 7 fails to remain valid when odd powers are considered.

Example 8. *Let Z denote the set of integers. Then the ring Z_7 is a $(x^7 + 6x)$-quasi invo-clean ring which is not $(x^7 - 6x)$-quasi invo-clean.*

Corollary 9. *A ring R is quasi invo-clean precisely when it is $(x^2 + x)$-quasi invo clean.*

Proof. It follows from Lemma 7. □

Suppose R and S are two rings and $\phi : C(R) \longrightarrow C(S)$ is a ring homomorphism with $\phi(1_R) = 1_S$. If $g(x) = \sum_{i=0}^{n} r_i x^i \in C(R)[x]$, we let $g_\phi(x) := \sum_{i=0}^{n} \phi(r_i)x^i \in C(S)[x]$.

Lemma 10. *Let R and S be two rings, and let $\phi : R \longrightarrow S$ be a surjective ring homomorphism. Suppose $g(x) = \sum_{i=0}^{n} r_i x^i \in C(R)[x]$ is an even polynomial. If R is $g(x)$-quasi invo-clean, then the image ring S is $g_\phi(x)$-quasi invo-clean.*

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Proof. Let $g(x) = \sum_{i=0}^{n} r_i x^i \in C(R)[x]$ and define $g_\phi(x) := \sum_{i=0}^{n} \phi(r_i)x^i \in C(S)[x]$. Take any element $a \in S$. Then there exists $r \in R$ such that $1 - a = \phi(r)$. Since R is $g(x)$-quasi invo-clean, we can write $r = v+s$ with $v \in Qinv(R)$ and $s \in R$ satisfying $g(s) = 0$. Consequently, $1 - a = \phi(r) = \phi(v) + \phi(s)$, which implies $a = (-1 + \phi(v)) + \phi(-s) = \phi(-1 + v) + \phi(-s)$, where $\phi(-1 + v) \in Qinv(S)$ and

$$\begin{aligned} g_\phi(\phi(-s)) &= \sum_{i=0}^{n} \phi(r_i)(\phi(-s))^i \\ &= \sum_{i=0}^{n} \phi(r_i)\phi((-s)^i) = \sum_{i=0}^{n} \phi(r_i(-s)^i) \\ &= \phi(\sum_{i=0}^{n} r_i(-s)^i) = \phi(g(-s)) = \phi(0) = 0 \end{aligned}$$

Therefore S is $g_\phi(x)$-quasi invo-clean. □

Definition 11. Let R and S be two rings and $g(x) \in C(R)[x]$ be an even polynomial such that R is $g(x)$-quasi invo-clean. If there is an epimorphism $\phi : R \longrightarrow S$, then S is called a $\overline{g}(x)$-quasi invo-clean.

Corollary 12. *Let R and S be two rings and $g(x)$ be an even polynomial. Then the following statements hold.*

- *(i) Let I be an ideal of a $g(x)$-quasi invo-clean ring R. Then R/I is $\overline{g}(x)$-quasi invo-clean.*
- *(ii) Let the upper triangular matrix ring $T_n(R)$ is $g(x)$-quasi invo-clean. Then R is $\overline{g}(x)$-quasi invo-clean.*
- *(iii) Let the skew formal power series $R[[x, \alpha]]$ over R is $g(x)$-quasi invo-clean. Then R is $\overline{g}(x)$-quasi invo-clean.*
- *(iv) Let M be an (R, S)-bimodule and $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ be the*

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formal triangular matrix ring. If T is $g(x)$-quasi invo-clean. Then R and S are $\overline{g}(x)$-quasi invo-clean.

Proof. It follows from Lemma 10. □

Lemma 13. *Let $\{R_i\}_{i=1}^{n}$ be rings and $g(x)$ be an even polynomial. Then the direct product ring $R = \prod_{i\in I} R_i$ is $g(x)$-quasi invo-clean precisely when each component ring R_i enjoys the same property.*

Proof. Suppose that R is $g(x)$-quasi invo-clean. Since $\pi_j : \prod_{i=1}^{n} R_i \longrightarrow R_j$ by $\pi_j((r_i)) = r_j$ is a ring epimorphism, for every $1 \leq j \leq n$, R_j is $g(x)$-quasi invo-clean, by Corollary 12.
Conversely, Suppose that $r = (r_i) \in R$. For $1 \leq i \leq n$, write $r_i = v_i + s_i$ such that $v_i \in Qinv(R_i)$ and $g(s_i) = 0$. Then $r = (v_i) + (s_i)$ such that $(v_i) \in Qinv(R)$ and $g((s_i)) = 0$. Therefore R is $g(x)$-quasi invo-clean. □

Let R be a ring with an identity and S be a ring which is an R-R-bimodule such that $(s_1s_2)r = s_1(s_2r)$, $(s_1r)s_2 = s_1(rs_2)$ and $(rs_1)s_2 = r(s_1s_2)$ hold for all $s_1, s_2 \in S$ and $r \in R$. The ideal extension of R by S is defined to be the additive abelian group $I(R,S) = R \oplus S$ with multiplication $(r, s_1)(r', s_2) = (rr', rs_2 + s_1r' + s_1s_2)$. If $g(x) = (r_0, s_0) + (r_1, s_1)x + \cdots + (r_n, s_n)x^n \in C(I(R,S))[x]$, then $g_R(x) = r_0 + r_1x + \cdots + r_nx^n \in C(R)[x]$.

Lemma 14. *Let R be a ring with an identity, S be a ring which is an R-R-bimodule and $g(x) \in C(I(R,S))[x]$ be an even polynomial. If $I(R,S)$ is $g(x)$-quasi invo-clean, then R is $g_R(x)$-quasi invo-clean.*

Proof. Suppose that $\phi_R : I(R,S) \longrightarrow R$ by $\phi_R(r,s) = r$. Since ϕ_R is a ring epimorphism, R is $g_R(x)$-quasi invo-clean by Lemma 10. □

Let R be a ring and $\alpha : R \longrightarrow R$ be a ring endomorphism. The ring $R[[x,\alpha]]$ of skew formal power series over R; that is all formal power series in x with coefficients from R with multiplication

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defined by $xr = \alpha(r_x$ for all $r \in R$. It is clear that $R[[x]] = R[[x, 1_R]]$ and $R[[x, \alpha]] \cong I(R, \langle x \rangle)$ where $\langle x \rangle$ is the ideal generated by x.

Proposition 15. *Let R be a ring, $\alpha : R \longrightarrow R$ be a ring endomorphism and $g(x)$ be an even polynomial. If $R[[x, \alpha]]$ is $g(x)$-quasi invo-clean, then R is $g_\phi(x)$-quasi invo-clean such that $\phi : R[[x, \alpha]] \longrightarrow R$ is defined by $\phi(f) = f(0)$.*

Proof. It follows from Lemma 10. □

Lemma 16. *Let R be a commutative ring and $h = \sum_{i=0}^{n} r_i x^i \in Qinv(R[x])$. Then $r_0 \in Qinv(R)$ and $r_i \in Nil(R)$ for each $1 \leq i \leq n$.*

Proof. Since $h = \sum_{i=0}^{n} r_i x^i \in Qinv(R[x])$, $h^2 = 1$ or $(1-h)^2 = 1$. Hence $r_0^2 = 1$ or $(1-r_0)^2 = 1$, and so $r_0 \in Qinv(R)$. Suppose that P is a prime ideal of R. Hence $(R/P)[x]$ is an integral domain. Let $\psi : R[x] \longrightarrow (R/P)[x]$ by $\psi(\sum_{i=0}^{n} r_i x^i) = \sum_{i=0}^{n}(r_i + P)x^i$. Then ψ is a ring epimorphism. Since $\psi(h)\psi(h) = 1$ or $\psi(1-h)\psi(1-h) = 1$, $deg(\psi(h)\psi(h)) = deg(\psi(1))$ or $deg(\psi(1-h)\psi(1-h)) = deg(\psi(1))$. Then $r_1 + P = r_2 + P = \cdots = r_n + P = P$. Therefore $r_i \in Nil(R)$ for each $1 \leq i \leq n$. □

Theorem 17. *Let R be a commutative ring. Then the polynomial ring $R[x]$ fails to be $(x^2 - x)$-quasi invo-clean.*

Proof. Suppose that $R[x]$ is $(x^2 - x)$-quasi invo-clean. Hence $x = v + s$ where $v \in Qinv(R[x])$ and s is a root of $x^2 - x$. Then $x - s \in Qinv(R[x])$. So $1 \in Nil(R)$ and $-s \in Qinv(R)$ by Lemma 16, a contradiction. □

A Morita context is a 6-tuple $\mathcal{M}(R, M, K, S, \phi, \psi)$, where R and S are rings, M is an (R, S)-bimodule, K is a (S, R)-bimodule, and $\phi : M \otimes_S K \longrightarrow R$ and $\psi : K \otimes_R M \longrightarrow S$ are bimodule homomorphisms such that $T(\mathcal{M}) = \begin{pmatrix} R & M \\ K & S \end{pmatrix}$ is an associative ring with the obvious matrix operations. The ring $T(\mathcal{M})$ is the

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Morita context ring associated with $\mathcal{M}$. For more on Morita context rings see [1, 4, 18, 19]. If $g(x) = \begin{pmatrix} r_0 & m_0 \\ k_0 & s_0 \end{pmatrix} + \begin{pmatrix} r_1 & m_1 \\ k_1 & s_1 \end{pmatrix} x + \cdots + \begin{pmatrix} r_n & m_n \\ k_n & s_n \end{pmatrix} x^n \in C(T(\mathcal{M}))[x]$, then $g_R(x) = r_0 + r_1x + \cdots + r_nx^n \in C(R)[x]$ and $g_S(x) = rs_0 + s_1x + \cdots + s_nx^n \in C(S)[x]$.

Theorem 18. *Let $g(x)$ be an even polynomial and the Morita context ring $T(\mathcal{M}) = \begin{pmatrix} R & M \\ K & S \end{pmatrix}$ is $g(x)$-quasi invo-clean with $\phi, \psi = 0$. Then R is $g_R(x)$-quasi invo-clean and S is $g_S(x)$-quasi invo-clean.*

Proof. Suppose that $T(\mathcal{M})$ is $g(x)$-quasi invo-clean with $\phi, \psi = 0$. Hence $I = \begin{pmatrix} 0 & M \\ K & S \end{pmatrix}$ and $J = \begin{pmatrix} R & M \\ K & 0 \end{pmatrix}$ are two ideals of $T(\mathcal{M})$. Since $T(\mathcal{M})/I \cong R$ and $T(\mathcal{M})/J \cong S$, the assertion holds by Lemma 10. □

Corollary 19. *Let R and S be two rings, M be an (R, S)-bimodule and $g(x)$ be an even polynomial. Let $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ be the formal triangular matrix ring. If T is $g(x)$-quasi invo clean, then R is $g_R(x)$-quasi invo-clean and S is $g_S(x)$-quasi invo-clean.*

Proof. Follows from Theorem 18. □

Corollary 20. *Let R be a commutative ring, M be an (R, R)-bimodule such that $2M = 0$ and $g(x)$ be an even polynomial. Then $T = \begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$ is $g(x)$-quasi invo clean precisely when R is $g(x)$-quasi invo-clean.*

Proof. Follows from Lemma 10 and Corollary19. □

We close the article with the following two problems.

Problem 21. *What is the behaviour of the matrix rings over $g(x)$-quasi invo clean rings?*

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Problem 22. *Let R be a $g(x)$-quasi invo clean ring and $e \in Id(R)$. What is the behaviour of the corner ring eRe?*

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