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ALGORITHM OF THE REGULARIZATION METHOD FOR A SINGULARLY PERTURBED INTEGRO-DIFFERENTIAL SYSTEM WITH A RAPIDLY VARIABLE KERNEL AND WITH RAPIDLY OSCILLATING INHOMOGENEITIES

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Abstract

In this paper, we consider a singularly perturbed integro-differential system with a rapidly oscillating right-hand side,

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including an integral operator with a rapidly changing kernel. Earlier, a singularly perturbed integro-differential system with a rapidly oscillating inhomogeneity and with a slowly varying kernel was studied. The main goal of this work is to generalize the Lomov regularization method and to reveal the influence of a rapidly oscillating right-hand side and a rapidly changing kernel on the asymptotics of the solution to the original problem.

Math. Subject Classification: 34K26, 45J05.
Key Words and Phrases: singularly perturbation, rapidly oscillating inhomogeneity, rapidly changing kernel, regularization, iterative problem.

Introduction

In the description of mathematical models for preventing non-local epidemics, the spread of hepatitis, the mathematical theory of the boundary layer, quantum mechanics, the theory of phase transitions, natural and forced resonant oscillations of a medium with low viscosity, the calculation and design of transformers in which, when disconnected, a voltage exceeding the nominal occurs, etc., singularly perturbed equations with rapidly oscillating coefficients arise. A special place in this case is occupied by singularly perturbed differential equations with rapidly oscillating coefficients. The monographs [1, 2, 3, 4] are devoted to methods for solving problems with rapidly oscillating coefficients. The article develops and supplements the research of S.A. Lomov's school on the regularization method [3, 4], the scheme of which in general terms is as follows. By introducing regularizing functions that contain all the irregularities of the solution that arise due to the small parameter, the singular problem is reduced to a regular problem in a space of higher dimension. Using the scheme of classical perturbation theory, a theory of solvability of the corresponding iterative problems in a space of higher dimension is constructed, and then after constructing an asymptotic series in powers of the small parameter,

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a restriction to regularizing functions is made, which leads to the construction of an asymptotic solution to the original singularly perturbed problem.

It should be noted that singularly perturbed differential systems with rapidly oscillating coefficients were investigated in [1, 2, 3, 5, 6, 7]. The transition to integro-differential equations is associated with obvious difficulties, if only because in integro-differential equations rapidly oscillating components and rapidly changing factors at the integral operator generate new types of singularities that differ from the known ones. The splitting method [1, 2, 5], as far as we know, has not been applied to integro-differential equations. Singularly perturbed integro-differential equations with rapidly oscillating components (with rapidly oscillating coefficients and rapidly oscillating inhomogeneities) with a slowly changing kernel from the standpoint of the regularization method [3, 4] are studied in sufficient detail in [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. And problems with rapidly decreasing factors of the integral operator are considered in [20, 21, 22, 23, 24]. We also note that regularized asymptotic solutions for singularly perturbed integro-differential equations in partial derivatives are constructed in [25, 26, 27], and for singularly perturbed equations with fractional derivatives in [28, 29].

The paper considers the following problem

$$\begin{aligned} L_\varepsilon y(t,\varepsilon) &\equiv \varepsilon\tfrac{dy}{dt} - A(t)y - \int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t \mu(\theta)d\theta} K(t,s)y(s,\varepsilon)ds \\ &= h_1(t) + h_2(t)e^{\frac{i\beta(t)}{\varepsilon}}, \quad y(0,\varepsilon) = y^0, \;\; t\in[0,T] \end{aligned} \tag{1}$$

where $y = \{y_1(t,\varepsilon),\ldots,y_n(t,\varepsilon)\}$ is the unknown vector function, $A(t)$ is $(n\times n)-$ matrix, $\beta'(t) > 0$ is the frequency of a rapidly oscillating inhomogeneity, $h_j(t) = (h_{1j}(t),...,h_{nj}(t))$ is the known functions, $\mu(t) < 0$ is the given known function, y^0 is a constant number, $\varepsilon > 0$ is a small parameter, the number T is given. We will assume that the following conditions are met

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(i) $A(t) \in C^{\infty}\left([0,T], \mathbf{C}^{n\times n}\right)$, $\quad h_1(t), h_2(t) \in C^{\infty}\left([0,T], \mathbf{C}^{n}\right)$, $K(t,s) \in C^{\infty}\left(\{0 \le s \le t \le T\}, \mathbf{C}^{n\times n}\right)$;

(ii) the spectrum $\{\lambda_1(t), ..., \lambda_n(t)\}$ of the matrix $A(t)$, the frequency $\lambda_{n+1}(t) = i\beta'(t)$ of the rapidly oscillating inhomogeneity and the spectral value $\lambda_{n+2}(t) = \mu(t)$ of the rapidly varying kernel satisfy at each $t \in [0,T]$ the conditions:

$a)$ $\lambda_i(t) \ne \lambda_j(t), i \ne j, \lambda_j(t) \ne 0, i,j = \overline{1, n+2}$;

$b)$ $\mathrm{Re}\lambda_j(t) \le 0, j = \overline{1,n},$ $\mathrm{Re}\lambda_{n+2}(t) \le 0,$ $\beta'(t) > 0.$

Under the specified conditions, we will try to develop an algorithm for constructing a regularized asymptotic solution [1] of problem (1).

1 Space of solutions and regularization of the problem

We introduce regularizing functions of the form

$$\tau_j = \frac{1}{\varepsilon}\int_0^t \lambda_j(\theta)\, d\theta \equiv \frac{\psi_j(t)}{\varepsilon}, j = \overline{1, n+2}. \tag{2}$$

To shorten we introduce the notation $\tau = (\tau_1, \ldots, \tau_{n+2})$, $\psi_j(t,\varepsilon) = (\psi_1(t,\varepsilon), \ldots, \psi_{n+2}(t,\varepsilon))$ and instead of the desired solution $y(t,\varepsilon)$ to problem (1), we will study some extended function $u(t,\tau,\varepsilon)$ such that its restriction identically $u(t,\tau,\varepsilon)|_{\tau=\frac{\psi(t)}{\varepsilon}} \equiv y(t,\varepsilon)$ coincides with the desired solution to problem (1). We find the total derivative for the function $u\left(t, \frac{\psi(t)}{\varepsilon}, \varepsilon\right)$, and write it in the form

$$\dot{y}(t,\varepsilon) = \left(\dot{u} + \frac{1}{\varepsilon}\sum_{j=1}^{n+2} \lambda_j(t)\frac{\partial u}{\partial \tau_j}\right)_{\tau=\frac{\psi(t)}{\varepsilon}}, \quad \left(\dot{u} = \frac{\partial u}{\partial t}\right).$$

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Then the following problem is posed for the extended function

$$L_\varepsilon u(t,\tau,\sigma,\varepsilon) \equiv \varepsilon \frac{\partial u}{\partial t} + \sum_{j=1}^{n+2} \lambda_j(t) \frac{\partial u}{\partial \tau_j} - A(t)u$$
$$- \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_{n+2}(\theta)d\theta} K(t,s) u\left(s, \frac{\psi(s)}{\varepsilon}, \varepsilon\right) ds \tag{3}$$
$$= h_1(t) + h_2(t) e^{\tau_{n+1}} \sigma, \quad u(t,\tau,\sigma,\varepsilon)|_{t=0,\tau=0,\sigma=e^{i\beta(0)/\varepsilon}} = y^0.$$

However, problem (3) cannot be considered completely regularized, since the integral term

$$Ju \equiv J\left(u(t,\tau,\sigma,\varepsilon)|_{t=s,\tau=\psi(s)/\varepsilon}\right)$$
$$= \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_{n+2}(\theta)d\theta} K(t,s)\, u\left(s, \frac{\psi(s)}{\varepsilon}, \sigma, \varepsilon\right) ds$$

has not been regularized in it. To regularize it, we introduce a class M_ε that is asymptotically invariant with respect to the operator (see [1], p. 62).

Definition 1. We will say that a function $u(t,\tau,\sigma)$ belongs to the class U, if it can be represented as a sum

$$U = \left\{ u(t,\tau,\sigma): \quad u = u_0(t,\sigma) + \sum_{j=1}^{n+2} u_j(t,\sigma) e^{\tau_j}, \right.$$
$$\left. u_j(t,\sigma) \in C^\infty\left([0,T], \mathbf{C}^n\right), j = \overline{0, n+2} \right\}. \tag{4}$$

As a class M_ε we take restrictions of the class M_ε for $\tau = \frac{\psi(t)}{\varepsilon}$. We prove that $U|_{\tau=\frac{\psi(t)}{\varepsilon}}$ is invariant with respect to the integral operator J.

Theorem 2. *Let conditions (i) and (ii) be satisfied. Then the class $M_\varepsilon = U|_{\tau=\frac{\psi(t)}{\varepsilon}}$ is asymptotically invariant with respect to the integral operator J.*

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Proof. Substituting (4) into Ju, we have:

$$Ju(t,\tau) = \int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t \lambda_{n+2}(\theta)d\theta} K(t,s)u_0(s)ds$$

$$+\sum_{j=1}^{n+2}\int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t \lambda_{n+2}(\theta)d\theta} K(t,s)u_j(s)e^{\frac{1}{\varepsilon}\int\limits_0^s \lambda_j(\theta)d\theta} ds$$

$$= \int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t \lambda_{n+2}(\theta)d\theta} K(t,s)u_0(s)ds + e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_{n+2}(\theta)d\theta}\int\limits_0^t K(t,s)u_{n+2}(s)ds$$

$$+\sum_{j=1, j\neq n+2}^{n+2} e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_{n+2}(\theta)d\theta}\int\limits_0^t K(t,s)u_j(s)e^{\frac{1}{\varepsilon}\int\limits_0^s (\lambda_j(\theta)-\lambda_{n+2}(\theta))d\theta} ds.$$

It is necessary to show that integrals containing exponentials are expanded into an asymptotic series in powers ε (for $\varepsilon \to +0$). Applying the operation of integration by parts; we will have

$$J_0(t,\varepsilon) \equiv e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_{n+2}(\theta)d\theta}\int\limits_0^t K(t,s)u_0(s)e^{-\frac{1}{\varepsilon}\int\limits_0^s \lambda_{n+2}(\theta)d\theta} ds$$

$$= \varepsilon e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_{n+2}(\theta)d\theta}\int\limits_0^t \frac{K(t,s)u_0(s)}{-\lambda_{n+2}(s)} e^{-\frac{1}{\varepsilon}\int\limits_0^s \lambda_{n+2}(\theta)d\theta} ds$$

$$= \varepsilon e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_{n+2}(\theta)d\theta}\left[\left.\frac{K(t,s)u_0(s)}{-\lambda_{n+2}(s)} e^{-\frac{1}{\varepsilon}\int\limits_0^s \lambda_{n+2}(\theta)d\theta}\right|_{s=0}^{s=t}\right.$$

$$\left. -\int\limits_0^t \frac{\partial}{\partial s}\left(\frac{K(t,s)u_0(s)}{-\lambda_{n+2}(s)}\right) e^{-\frac{1}{\varepsilon}\int\limits_0^s \lambda_{n+2}(\theta)d\theta} ds\right]$$

$$= \varepsilon\left[\frac{K(t,t)u_0(t)}{-\lambda_{n+2}(t)} - \frac{K(t,0)u_0(0)}{-\lambda_{n+2}(0)} e^{-\frac{1}{\varepsilon}\int\limits_0^t \lambda_{n+2}(\theta)d\theta}\right]$$

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$$= \varepsilon e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_{n+2}(\theta)d\theta} \int\limits_0^t \frac{\partial}{\partial s}\left(\frac{K(t,s)u_0(s)}{-\lambda_{n+2}(s)}\right) e^{-\frac{1}{\varepsilon}\int\limits_0^s \lambda_{n+2}(\theta)d\theta} ds.$$

Thus, after a single integration by parts, the terms outside the integral are distinguished, which for $\tau = \psi$ have the form of summands (4), and the integral term is again an integral of the type $J_0(t,\varepsilon)$. Multiple integration by parts leads to a formal series

$$J_0(t,\tau) = \sum_{\nu=0}^{\infty} (-1)^{\nu} \varepsilon^{\nu+1} \left[\left(I_0^{\nu}(K(t,s)u_0(s))\right)_{s=t} - \left(I_0^{\nu}(K(t,s)u_0(s))\right)_{s=0} e^{\tau_{n+2}} \right], \tag{5_0}$$

where indicated: $\tau = \frac{\psi(t)}{\varepsilon}$, $I_0^0 = \frac{1}{-\lambda_{n+2}(s)}$, $I_0^{\nu} = \frac{1}{-\lambda_{n+2}(s)} \frac{\partial}{\partial s} I_0^{\nu-1}$, $\nu \geq 1$.

$$J_j(t,\varepsilon) \equiv e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_{n+2}(\theta)d\theta} \int\limits_0^t K(t,s)u_j(s) e^{\frac{1}{\varepsilon}\int\limits_0^s [\lambda_j(\theta)-\lambda_{n+2}(\theta)]d\theta} ds$$

$$= \varepsilon e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_{n+2}(\theta)d\theta} \int\limits_0^t \frac{K(t,s)u_j(s)}{\lambda_j(s)-\lambda_{n+2}(s)} de^{\frac{1}{\varepsilon}\int\limits_0^s [\lambda_j(\theta)-\lambda_{n+2}(\theta)]d\theta}$$

$$= \varepsilon e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_{n+2}(\theta)d\theta} \left[\left. \frac{K(t,s)u_j(s)}{\lambda_j(s)-\lambda_{n+2}(s)} e^{\frac{1}{\varepsilon}\int\limits_0^s [\lambda_j(\theta)-\lambda_{n+2}(\theta)]d\theta} \right|_{s=0}^{s=t} - \int\limits_0^t \frac{\partial}{\partial s}\left(\frac{K(t,s)u_j(s)}{\lambda_j(s)-\lambda_{n+2}(s)}\right) e^{\frac{1}{\varepsilon}\int\limits_0^s [\lambda_j(\theta)-\lambda_{n+2}(\theta)]d\theta} ds \right]$$

$$= \varepsilon \left[\frac{K(t,t)u_j(t)}{\lambda_j(t)-\lambda_{n+2}(t)} e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_j(\theta)d\theta} - \frac{K(t,0)u_j(0)}{\lambda_j(0)-\lambda_{n+2}(0)} e^{-\frac{1}{\varepsilon}\int\limits_0^s \lambda_{n+2}(\theta)d\theta} \right]$$

$$-\varepsilon e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_{n+2}(\theta)d\theta} \int\limits_0^t \frac{\partial}{\partial s}\left(\frac{K(t,s)u_j(s)}{\lambda_j(s)-\lambda_{n+2}(s)}\right) e^{\frac{1}{\varepsilon}\int\limits_0^s [\lambda_j(\theta)-\lambda_{n+2}(\theta)]d\theta} ds.$$

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Here, as in the integral $J_0(t,\varepsilon)$, the terms outside the integral are distinguished, which for $\tau=\psi$ have the form of summands (4), and the integral term is again an integral of the type $J_j(t,\varepsilon)$. Multiple integration by parts leads to a formal series

$$J_j(t,\tau)=\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\Big[\big(I_j^{\nu}(K(t,s)u_j(s))\big)_{s=t}e^{\tau_j} \\ -\big(I_j^{\nu}(K(t,s)u_j(s))\big)_{s=0}e^{\tau_{n+2}}\Big], \qquad j=\overline{1,n}\,,\ \ \nu\geq 1, \tag{5_j}$$

where indicated: $\tau=\frac{\psi(t)}{\varepsilon}$, $I_j^0=\frac{1}{\lambda_j(s)-\lambda_{n+2}(s)}$, $I_j^{\nu}=\frac{1}{\lambda_j(s)-\lambda_{n+2}(s)}\frac{\partial}{\partial s}I_j^{\nu-1}$, $\nu\geq 1$.

The series (5_0) and (5_j) converge asymptotically as $\varepsilon\to+0$ (uniformly in $t\in[0,T]$). We will show this, for example, for the series (5_0). The partial sum of this series

$$S_N(t,\varepsilon)=\sum_{\nu=0}^{N-1}(-1)^{\nu}\varepsilon^{\nu+1}\Big[\left(I_0^{\nu}(K(t,s)u_0(s))\right)_{s=t} \\ -\left(I_0^{\nu}(K(t,s)u_0(s))\right)_{s=0}e^{-\frac{1}{\varepsilon}\int\limits_0^t\lambda_{n+2}(\theta)d\theta}\Big]$$

satisfies the equality

$$J_0(t,\varepsilon)=S_N(t,\varepsilon) \\ +(-1)^N\varepsilon^N\int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t\lambda_{n+2}(\theta)d\varsigma}\frac{\partial}{\partial s}\left(I_0^{N-1}(K(t,s)u_0(s))\right)ds.$$

Let us integrate by parts the integral standing here, selecting one more power ε on the right-hand side:

$$J_0(t,\varepsilon)-S_N(t,\varepsilon) \\ +(-1)^N\varepsilon^{N+1}\int\limits_0^t\left(I_0^N(K(t,s)u_0(s))\right)d\left(e^{\frac{1}{\varepsilon}\int\limits_s^t\lambda_{n+2}(\theta)d\theta}\right)$$

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$$= (-1)^N \varepsilon^N \Big\{ \Big[\big(I_0^N (K(t,s)u_0(s))\big)_{s=t}$$

$$- \big(I_0^N (K(t,s)u_0(s))\big)_{s=0} e^{-\frac{1}{\varepsilon}\int\limits_0^t \lambda_{n+2}(\theta)d\theta} \Big]$$

$$- \int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t \lambda_{n+2}(z)dz} \frac{\partial}{\partial s} \big(I_0^N (K(t,s)u_0(s))\big)\, ds \Big\}.$$

Due to the infinite differentiability of the function $K(t,s)$ on $0 \leq \leq s \leq t \leq T$ and $\lambda_{n+2}(t)$ on $[0,T]$, and also due to the uniform boundedness of the function, the equality in the square brackets of the last one is uniformly bounded for $(t,\tau) \in [0,T] \times \{\tau : \mathrm{Re}\tau_j \leq 0, j = \overline{1, n+2}\}$. Consequently,

$$\|J_0(t,\varepsilon) - S_N(t,\varepsilon)\|_{C[0,T]} \leq C_N \varepsilon^{N+1}$$

where $C_N > 0$ is a constant independent of ε for $\varepsilon \in (0, \varepsilon_0]$ ($\varepsilon_0 > 0$ is sufficiently small). This means that the series (5_0) converges to the integral asymptotically for $\varepsilon \to +0$. The Theorem 1 is proved. □

Thus, the image $Ju(t,\varepsilon)$ is expanded into an asymptotic series

$$Ju(t,\tau) = e^{\tau_{n+2}} \int\limits_0^t K(t,s)u_{n+2}(s)ds$$

$$+ \sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} \left[\left(I_0^\nu \left(K(t,s)u_0(s)\right)\right)_{s=t} \right.$$

$$\left. - \left(I_0^\nu \left(K(t,s)u_0(s)\right)\right)_{s=0} e^{\tau_{n+2}} \right] \tag{6}$$

$$+ \sum_{j=1, j\neq n+2}^{n+2} \sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} \Big[\big(I_j^\nu \left(K(t,s)u_j(s)\right)\big)_{s=t} e^{\tau_j}$$

$$- \big(I_j^\nu \left(K(t,s)u_j(s)\right)\big)_{s=0} e^{\tau_{n+2}} \Big]$$

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where $\tau = \frac{\psi(t)}{\varepsilon}$. This proves that the class M_ε is asymptotically invariant with respect to the integral operator J.

Let us now construct an extension of the operator J.

If now the series

$$u(t,\tau,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k u_k(t,\tau), \quad u_k(t,\tau) \in U \tag{7}$$

on the narrowing $\tau = \frac{\psi(t)}{\varepsilon}$ is an asymptotic series for $\varepsilon \to +0$ (uniformly in $t \in [0,T]$), then the image of $Ju(t,\tau,\varepsilon)$, obviously, will be the same. To construct the series $Ju(t,\tau,\varepsilon)$, we introduce the so-called order operators . Let $u_k(t,\tau) \in U$ be an arbitrary element (4). Then we have the expansion (6), which can be written as

$$Ju(t,\tau) = R_0 u(t,\tau) + \sum_{\nu=0}^{\infty} \varepsilon^{k+1} R_{\nu+1} u_\nu(t,\tau) \tag{8}$$

where $\tau = \frac{\psi(t)}{\varepsilon}$, and the order operators R_ν have the form:

$$R_0 u(t,\tau) = e^{\tau_{n+2}} \int_0^t K(t,s) u_{n+2}(s)\, ds, \tag{9_0}$$

$$\begin{aligned} R_1 u(t,\tau) = {} & \Big[\left(I_0^0 \left(K(t,s) u_0(s) \right) \right)_{s=t} \\ & - \left(I_0^0 \left(K(t,s) u_0(s) \right) \right)_{s=0} e^{\tau_{n+2}} \Big] \\ & + \sum_{j=1}^{n+2} \Big[\left(I_j^0 \left(K(t,s) u_j(s) \right) \right)_{s=t} e^{\tau_j} \\ & - \left(I_j^0 \left(K(t,s) u_j(s) \right) \right)_{s=0} e^{\tau_{n+2}} \Big], \end{aligned} \tag{9_1}$$

$$\begin{aligned} R_{\nu+1} u(t,\tau) = {} & \left[\left(I_0^\nu \left(K(t,s) u_0(s) \right) \right)_{s=t} \right. \\ & \left. - \left(I_0^\nu \left(K(t,s) u_0(s) \right) \right)_{s=0} e^{\tau_{n+2}} \right] \\ & + \sum_{j=1}^{n+2} \Big[\left(I_j^\nu \left(K(t,s) u_j(s) \right) \right)_{s=t} e^{\tau_j} \end{aligned}$$

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$$-\left(I_j^{\nu}\left(K(t,s)u_j(s)\right)\right)_{s=0} e^{\tau_{n+2}}\Big], \nu \geq 1. \qquad (9_{\nu+1})$$

The $R_\nu : U \to U$ operators are called order operators because they extract the sum of the terms of the order ν with respect to the parameter ε in the expression $Ju(t,\tau)$. Applying the operator J to the series (7), and then using formulas (8),$(9_0), \ldots, (9_{\nu+1})$, and collecting the coefficients at the same powers of ε, we arrive at the following expression for $Ju(t,\tau,\varepsilon)$:

$$Ju(t,\tau,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k Ju_k(t,\tau) = \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^{r} R_{r-s} u_s(t,\tau)|_{\tau=\psi(t)/\varepsilon}.$$

This equality is the basis for defining the extension of the operator J.

Definition 3. The formal extension of the integral operator J is the operator $\tilde{J}$, which acts on each function of the form (6) according to the law

$$\tilde{J}u(t,\tau,\varepsilon) \equiv \tilde{J}\left(\sum_{k=0}^{\infty} \varepsilon^k u_k(t,\tau)\right) \stackrel{def}{=} \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^{r} R_{r-s} u_s(t,\tau). \qquad (10)$$

The operator $\tilde{J}$ is defined at least in the class of series (7) (with coefficients $u_k(t,\tau) \in U$) converging asymptotically at $\varepsilon \to +0$ (uniformly in $(t,\tau) \in [0,T] \times \left\{\tau : \mathrm{Re}\tau_j \leq 0, j = \overline{1, n+2}\right\}$.

Now we can write the system, completely regularized with respect to the original problem (2):

$$\begin{aligned} L_\varepsilon u(t,\tau,\varepsilon) &\equiv \varepsilon \tfrac{\partial u}{\partial t} + \sum_{j=1}^{n+2} \lambda_j(t) \tfrac{\partial u}{\partial \tau_j} - A(t)u - \tilde{J}u \\ &= h_1(t) + h_2(t)e^{\tau_{n+2}}\sigma, \quad u(t,\tau,\varepsilon)|_{t=0,\tau=0} = y^0 \end{aligned} \qquad (11)$$

where the operator $\tilde{J}$ has the form (10).

The original problem (1) was singularly perturbed: at $\varepsilon = 0$ we could not in general subordinate the solutions to the initial

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condition in (1). Problem (11), in which we consider the variables τ as equal to the t independent variables, is regularly perturbed. In it we can set $\varepsilon = 0$ and subordinate the solution of the resulting equation to the point initial condition.

2 Iterative problems and their solvability in the space U

We will define the solution of system (11) in the form of series (7). Substituting (7) into (11) and equating $(9_0), \ldots, (9_{\nu+1})$ the coefficients ε at the same powers (taking into account the formulas $(9_0), \ldots, (9_{\nu+1})$), we obtain the following systems of equations:

$$\begin{aligned} Lu_0(t,\tau) \equiv \sum_{j=1}^{n+2} \lambda_j(t) \frac{\partial u_0}{\partial \tau_j} - A(t)u_0 - R_0 u_0 \\ = h_1(t) + h_2(t) e^{\tau_{n+2}} \sigma, \quad u_0(0,0) = y^0; \end{aligned} \tag{12_0}$$

$$Lu_1(t,\tau) = -\frac{\partial u_0}{\partial t} + R_1 u_0, \quad u_1(0,0) = 0; \tag{12_1}$$

$$Lu_2(t,\tau) = -\frac{\partial u_1}{\partial t} + R_1 u_1 + R_2 u_0, \quad u_2(0,0) = 0; \tag{12_2}$$

$$\cdots$$

$$\begin{aligned} Lu_k(t,\tau) = -\tfrac{\partial u_{k-1}}{\partial t} + R_k u_0 + ... + R_1 u_{k-1}, \\ u_k(0,0) = 0, \quad k \geq 1. \end{aligned} \tag{12_k}$$

The original problem (1) is a system of ordinary integro-differential equations, if the space $U = \mathrm{C}^n$, and the equations (12_0),$\ldots$, (12_k), are systems of partial integro-differential equations for vector functions. But this should not confuse us, since these are systems with constant coefficients, in which is included t as a parameter. And we will solve them, roughly speaking, according to the same rules by which systems of ordinary integro-differential equations with constant coefficients are solved.

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To calculate solutions to iterative problems of the $(12_0),\ldots,(12_k)$ we need to study the solvability of the general iterative system

$$Lu(t,\tau) \equiv \sum_{j=1}^{n+2} \lambda_j(t)\frac{\partial u}{\partial \tau_j} - A(t)u - R_0 u = H(t,\tau), u(0,0) = y_* \quad (13)$$

where right side $H(t,\tau) = H_0(t) + \sum\limits_{j=1}^{n+2} H_j(t)e^{\tau_j}$ is a known function of the space U, $y_* \in \mathbf{C}$ is constant, the operator R_0 has the form (see (9_0)):

$$R_0 u \equiv R_0\left(u_0(t) + \sum_{j=1}^{n+2} u_j(t)e^{\tau_j}\right) = e^{\tau_{n+2}}\int_0^t K(t,s)u_{n+2}(s)ds$$

in the U space.

Let us identify the conjugate space to the space with the space of functions

$$U^* = \left\{ z(t,\tau.\sigma) : \quad z = z_0(t,\sigma) + \sum_{j=1}^{n+2} z_j(t,\sigma)e^{\tau_j}, \right.$$

$$\left. z_j(t,\sigma) \in C^\infty\left([0,T], \mathbf{C}^n\right), j = \overline{0, n+2} \right\}.$$

Let us define for each fixed $t \in [0,T]$ the scalar product of the elements of space U by the elements of space U^* as follows:

$$\langle z(t,\tau), w(t,\tau)\rangle \equiv \left\langle z_0(t) + \sum_{j=1}^{n+2} z_j(t)e^{\tau_j}, w_0(t) + \sum_{j=1}^{n+2} w_j(t)e^{\tau_j} \right\rangle$$

$$\stackrel{def}{=} (z_0(t), w_0(t)) + \sum_{j=1}^{n+2} (z_j(t), w_j(t))$$

where $(*,*)$ denotes the usual scalar product in the complex space $\mathbf{C}^n$.

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In the future, we will need the eigenvectors $\varphi_k(t)$ of the matrix $A(t)$, corresponding to the eigenvalues $\lambda_k(t)$ respectively, $k=\overline{1,n}$, and the eigenvectors $\chi_j(t)$ of the conjugate matrix $A^*(t)$, corresponding to the eigenvalues $\bar{\lambda}_j(t)$ respectively, $j=\overline{1,n}$. Moreover, the system $\{\varphi_k(t)\}$ is biortonormal with respect to the system $\{\chi_j(t)\}$, i.e.

$$(\varphi_k(t),\chi_j(t))=\begin{cases}1, & k=j,\\ 0, & k\neq j.\end{cases}$$

The normal solvability theorem is valid for the system (13).

Theorem 4. *Let conditions (i) and (iia) be satisfied and the right-hand side* $H(t,\tau)=H_0(t)+\sum\limits_{j=1}^{n+2}H_j(t)e^{\tau_j}$ *of system (13) belongs to the space* U. *Then, for the solvability of system (13) in the space* U, *it is necessary and sufficient that the identities*

$$\langle H(t,\tau),\chi_k(t)e^{\tau_k}\rangle\equiv 0\quad \forall t\in[0,T],\quad k=\overline{1,n} \tag{14}$$

hold.

Proof. We will seek the solution to system (13) in the space U of non-resonant solutions. The elements of U have the form:

$$u(t,\tau)=u_0(t)+\sum_{j=1}^{n+2}u_j(t)e^{\tau_j}. \tag{15}$$

Substituting (15) into system (13), we will have

$$\sum_{j=1}^{n+2}\left[\lambda_j(t)I-A(t)\right]u_j(t)e^{\tau_j}-A(t)u_0(t)-e^{\tau_{n+2}}\int\limits_0^t K(t,s)u_{n+2}(s)ds$$

$$=H_0(t)+\sum_{j=1}^{n+2}H_j(t)e^{\tau_j}.$$

Equating here separately the free terms and coefficients at the same exponents, we obtain the following systems:

$$-A(t)u_0(t)=H_0(t), \tag{16_0}$$

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$$[\lambda_j(t)I - A(t)]\,u_j(t) = H_j(t), j = \overline{1, n+1}, \tag{16_j}$$

$$[\lambda_{n+2}(t)I - A(t)]\,u_{n+2}(t) - \int\limits_0^t K(t,s)u_{n+2}(s)ds = H_{n+2}(t). \tag{16_{n+2}}$$

Due to the fact that $\det A(t) \neq 0 \quad \forall t \in [0,T]$, the system (16_0) has a unique solution $u_0(t) = -A^{-1}(t)H_0(t)$. Since $\det[\lambda_{n+2}(t)I - -A(t)] \neq 0 \quad \forall t \in [0,T]$, the system (16_{n+2}) can be written in the form

$$\begin{aligned} u_{n+2}\,(t) = \int\limits_0^t \left([\lambda_{n+2}(t)I - A(t)]^{-1} K(t,s)\right) u_{n+2}(s)ds \\ - [\lambda_{n+2}(t)I - A(t)]^{-1} H_{n+2}(t). \end{aligned} \tag{17}$$

Due to the smoothness of the kernel $[\lambda_{n+2}(t)I - A(t)]^{-1} K(t,s)$ and inhomogeneity $[\lambda_{n+2}(t)I - A(t)]^{-1} H_{n+2}(t)$, this Volterra integral system has unique solution $u_{n+2}(t) \in C^{\infty}([0,T], \mathbf{C}^n)$. Since $\lambda_{n+1}(t) = i\beta'(t)$ it is not an eigenvalue of the matrix $A(t)$, the system (16_j) at $j = n+1$ is solvable in the space $C^{\infty}([0,T], \mathbf{C}^n)$:

$$u_{n+1}(t) = [\lambda_{n+1}(t)I - A(t)]^{-1} H_{n+1}(t).$$

The system (16_j) at $j = 1, 2, ..., n$ is solvable in the space $C^{\infty}([0,T], \mathbf{C}^n)$ if and only if the identities

$$\langle H(t), \chi_k(t) \equiv 0 \rangle \quad \forall t \in [0,T], \;\; k = \overline{1,n}$$

hold. It is easy to see that these identities coincide with the identities (14). Thus, condition (14) is necessary and sufficient for the solvability of system (13) in the space U. Theorem 4 is proved. $\square$

Remark 5. If identity (14) holds, then under conditions (i) and (ia) system (13) has the following solution in space:

$$u(t,\tau) = u_0(t) + \sum_{k=1}^{n} \left[\alpha_k(t)\varphi_k(t) + \sum_{s=1, s\neq k}^{n} \frac{(H_k(t), \chi_s(t))}{\lambda_k(t) - \lambda_s(t)} \varphi_s(t) \right] e^{\tau_k}$$

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$$+\left[\lambda_{n+1}(t)I - A(t)\right]^{-1} H_{n+1}(t)e^{\tau_{n+1}} + u_{n+2}(t)e^{\tau_{n+2}} \tag{18}$$

where $\alpha_k(t) \in C^\infty\left([0,T], \mathbf{C}^n\right)$ are arbitrary functions, $k = \overline{1,n}$, $u_0(t) = -A^{-1}(t)H_0(t)$, $u_{n+2}(t)$ is the solution to the integral system (17).

3 Unique solvability of the general iterative problem in space U. Remaining term theorem

As can be seen from (18), the solution to system (13) is determined ambiguously. However, if it is subject to additional conditions:

$$\begin{aligned} &u(0,0) = y_*, \\ &\left\langle -\tfrac{\partial u}{\partial t} + R_1 u + Q(t,\tau), \chi_k(t)e^{\tau_k} \right\rangle \equiv 0 \ \forall t \in [0,T], \quad k = \overline{1,n} \end{aligned} \tag{19}$$

where $Q(t,\tau) = Q_0(t) + \sum\limits_{j=1}^{n+2} Q_j(t)e^{\tau_j}$ is the known function of the space U, y_* is a constant vector of the complex space $\mathbf{C}^n$, then problem (13) will be uniquely solvable in the space U. Note that conditions (19) are natural for the entire series of iterative problems (12_k) and arise as solvability conditions (14) when moving from problem (12_k) to problem (12_{k+1}). The following theorem on the unique solvability of the system (13) holds:

Theorem 6. *Let conditions (i) and (ii) be satisfied, the right-hand side $H(t,\tau)$ of the system (13) belongs to the space U and satisfies the orthogonality condition (14). Then system (13) under additional conditions (19) is uniquely solvable in U.*

Proof. Under condition (14), system (13) has a solution (18) in the space U, where the functions $\alpha_k(t) \in C^\infty\left([0,T], \mathbf{C}^1\right), k = \overline{1,n}$, are still arbitrary. Subordinate (18) to the first condition (19), i.e.

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$u(0,0)=y_*$. We obtain $\sum\limits_{k=1}^{n}\alpha_k(0)\varphi_k(0)=y^*$, where is denoted:

$$y^* = y_* + A^{-1}(0)H_0(0) - \sum_{k=1}^{n}\sum_{s=1,s\neq k}^{n}\frac{(H_k(0),\chi_s(0))}{\lambda_k(0)-\lambda_s(0)}\varphi_s(0)$$

$$-\left[\lambda_{n+1}(0)I - A(0)\right]^{-1}H_{n+1}(0) + \left[\lambda_{n+2}(0)I - A(0)\right]^{-1}H_{n+2}(0).$$

Multiplying the scalar equality $\sum\limits_{k=1}^{n}\alpha_k(0)\varphi_k(0)=y^*$ by $\chi_s(0)$ and taking into account the biorthogonality of the systems $\{\varphi_k(t)\}$ and $\{\chi_j(t)\}$, we find the values

$$\alpha_k(0) = \left(y^*,\chi_s(0)\right), k=\overline{1,n}. \tag{20}$$

Let us now subject solution (18) to the second condition (19). The right side of this equation:

$$\begin{aligned}
-\frac{\partial u_0}{\partial t} + R_1u_0 + Q\left(t,\tau\right) &= -\dot{u}_0(t) - \sum_{k=1}^{n}\left(\alpha_k(t)\varphi_k(t)\right)^{\bullet}e^{\tau_k}\\
&- \sum_{s=1,s\neq k}^{n}\left(\frac{(H_k(t),\chi_s(t))}{\lambda_k(t)-\lambda_s(t)}\varphi_s(t)\right)^{\bullet}e^{\tau_k}\\
&+\left([\lambda_{n+2}(t)I - A(t)]^{-1}H_{n+2}(t)\right)^{\bullet}e^{\tau_{n+2}}\\
&+\sum_{k=1}^{n}\left[\frac{K(t,t)\alpha_k(t)}{\lambda_k(t)-\lambda_{n+2}(t)}e^{\tau_k} - \frac{K(t,0)\alpha_k(0)}{\lambda_k(0)-\lambda_{n+2}(0)}\right]\\
&+\frac{K(t,t)u_{n+1}(t)}{\lambda_{n+1}(t)-\lambda_{n+2}(t)}e^{\tau_{n+1}} - \frac{K(t,0)u_{n+1}(0)}{\lambda_{n+1}(0)-\lambda_{n+2}(0)}\\
&+\frac{K(t,0)u_0(0)}{\lambda_{n+2}(0)}e^{\tau_{n+2}} - \frac{K(t,t)u_0(t)}{\lambda_{n+2}(t)} + Q(t,\tau).
\end{aligned} \tag{21}$$

Now multiplying (21) scalarly by $\chi_k(t)e^{\tau_k}, k=\overline{1,n}$, we obtain the system of ordinary differential equations

$$\dot{\alpha}_k(t) + \left[(\dot{\varphi}_k(t),\chi_k(t)) + \sum_{s=1,s\neq k}^{n}\left(\frac{(H_k(t),\chi_s(t))}{\lambda_k(t)-\lambda_s(t)}\right)(\dot{\varphi}_s(t),\chi_k(t))\right.$$

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$$-\frac{K(t,t)}{\lambda_k(t)-\lambda_{n+2}(t)}\Bigg]\,\alpha_k(t)+Q_k(t)=0,\quad k=\overline{1,n}.$$

Adding the initial condition (20) to it, we can uniquely find the functions $\alpha_k(t), k=\overline{1,n}$, and, hence, construct a solution (18) of the problem (13) in the space U in a unique way. Theorem 6 is proved. □

Applying Theorems 4 and 6 to iterative problems (12_k), we find uniquely their solutions in the space U and construct series (7). Just as in [3, 30] we prove the following statement.

Theorem 7. *Let conditions (i) and (ii) be satisfied for system (1). Then, for* $\varepsilon\in(0,\varepsilon_0]$ *(*$\varepsilon_0>0$ *is sufficiently small), system (1) has a unique solution* $y(t,\varepsilon)\in \mathrm{C}^1([0,T],\mathbf{C}^n)$; *in this case, the estimate*

$$||y(t,\varepsilon)-u_{\varepsilon N}(t)||_{C[0,T]}\le c_N\varepsilon^{N+1}, N=0,1,2,\ldots$$

holds. Here $u_{\varepsilon N}(t)$ *is the restriction (at* $\tau=\frac{\psi(t)}{\varepsilon}$*) of the* $N-th$ *partial sum of the series (7) (with coefficients* $u_k(t,\tau)\in U$*, satisfying the iterative problems (*12_k*)), and the constant* $c_N>0$ *does not depend on* ε *at* $\varepsilon\in(0,\varepsilon_0]$.

4 Constructing a solution to the first iterative problem

Using Theorem 4, we will try to find a solution to the first iteration problem (12_k). Since the right-hand side $h_1(t)+h_2(t)e^{\tau_{n+2}}\sigma$ of the system (12_0), satisfy condition (14), this system has (according to (18)) a solution in the space U in the form

$$u_0(t,\tau)=u_0^{(0)}(t)+\sum_{k=1}^{n}\alpha_k^{(0)}(t)\varphi_k(t)e^{\tau_k}+u_{n+2}(t)\sigma e^{\tau_{n+2}}\qquad(22)$$

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where $\alpha_k^{(0)}(t) \in C^\infty([0,T], \mathbf{C}^n)$ are arbitrary functions, $k = \overline{1,n}$, $u_0^{(0)}(t) = -A^{-1}(t)h_1(t)$, $u_{n+2}^{(0)}(t)$ is the solution of the integral system

$$u_{n+2}^{(0)}(t) = \int_0^t \left([\lambda_{n+2}(t)I - A(t)]^{-1} K(t,s) \right) u_{n+2}(s)ds$$

$$- [\lambda_{n+2}(t)I - A(t)]^{-1} h_2(t)\sigma.$$

Submitting (22) to the initial condition $u_0(0,0) = y^0$, we will have

$$u_0^{(0)}(0) + \sum_{k=1}^n \alpha_k^{(0)}(0)\varphi_k(0) + u_{n+2}^{(0)}(0)\sigma = y^0 \quad \Leftrightarrow$$

$$\Leftrightarrow \sum_{k=1}^n \alpha_k^{(0)}(0)\varphi_k(0) = y^0 + A^{-1}(0)h_1(0) + [\lambda_{n+2}(0)I - A(0)]^{-1} h_2(0)\sigma.$$

Multiplying this equality scalarly by $\chi_j(0), j = \overline{1,n}$, uniquely we find $\alpha_k^{(0)}(0)$:

$$\begin{aligned} \alpha_k^{(0)}(0) = ([y^0 + A^{-1}(0)h_1(0) \\ +[\lambda_{n+2}(0)I - A(0)]^{-1} h_2(0)\sigma], \chi_j(0)), k = \overline{1,n}. \end{aligned} \tag{23}$$

For a complete calculation of the functions $\alpha_k^{(0)}(t)$, we pass to the next iterative problem (12_1). Substituting the solution (22) of the system (12_0) into it, we obtain the following system of equations:

$$Lu_1(t,\tau) = -\frac{d}{dt}u_0^{(0)}(t) - \sum_{k=1}^n \frac{d}{dt}\left(\alpha_k^{(0)}(t)\varphi_k(t)\right) e^{\tau_k}$$

$$-\frac{d}{dt}u_{n+2}^{(0)}(t)\sigma e^{\tau_{n+2}} = -\dot{u}_0^{(0)}(t)$$

$$-\sum_{k=1}^n \left(\dot{\alpha}_k^{(0)}(t)\varphi_k(t) + \alpha_k^{(0)}(t)\dot{\varphi}_k(t)\right) e^{\tau_k} - \dot{u}_{n+2}^{(0)}(t)\sigma e^{\tau_{n+2}}$$

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$$+\sum_{k=1}^{n}\left[\frac{K(t,t)\alpha_k^{(0)}(t)}{\lambda_k(t)-\lambda_{n+2}(t)}\varphi_k(t)e^{\tau_k}-\frac{K(t,0)\alpha_k^{(0)}(0)}{\lambda_k(0)-\lambda_{n+2}(0)}\varphi_k(0)\right]$$

$$+\frac{K(t,0)u_{n+1}^{(0)}(0)}{\lambda_{n+1}(0)-\lambda_{n+2}(0)}e^{\tau_{n+1}}-\frac{K(t,t)u_{n+1}^{(0)}(t)}{\lambda_{n+1}(t)-\lambda_{n+2}(t)}.$$

Performing here scalar multiplication by $\chi_j(t), j=\overline{1,n}$, we obtain the following system of ordinary differential equations

$$-\frac{d\alpha_k^{(0)}(t)}{dt}-\left[(\dot{\varphi}_k(t),\chi_j(t))-\frac{K(t,t)}{\lambda_k(t)-\lambda_{n+2}(t)}\right]\alpha_k^{(0)}(t)=0, k=\overline{1,n}.$$

Adding the initial condition (23) to this equation, we find $\alpha_k^{(0)}(t)$:

$$\alpha_k^{(0)}(t)=\alpha_k^{(0)}(0)e^{-\int\limits_0^t\left[\left(\varphi_k'(x),\chi_j(x)\right)-\frac{K(x,x)}{\lambda_k(x)-\lambda_{n+2}(x)}\right]dx}, k=\overline{1,n}$$

and hence the solution (22) of the problem (12_0) will be found uniquely in the space U.

In this case, the leading term of the asymptotics has the following form:

$$y_{\varepsilon 0}(t)=u_0^{(0)}(t)+u_{n+2}^{(0)}(t)e^{+\frac{i\beta(t)}{\varepsilon}}+$$

$$+\sum_{k=1}^{n}\left[\left(y^0+A^{-1}(0)h_1(0)+\left[\lambda_{n+2}(0)I-A(0)\right]^{-1}h_2(0)\sigma\right),\chi_j(0)\right]\times$$

$$\times\varphi_k(t)e^{-\int\limits_0^t\left[(\varphi'_k(x),\chi_j(x))-\frac{K(x,x)}{\lambda_k(x)-\lambda_{n+2}(x)}\right]dx+\frac{1}{\varepsilon}\int\limits_0^t\lambda_k(x)dx}. \qquad (24)$$

Conclussion

It can be seen from expression (24) for $y_{\varepsilon 0}(t)$ that the construction of the leading term of the asymptotics of the solution to problem (1) is significantly influenced by both the rapidly oscillating inhomogeneity and the kernel of the integral operator.

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