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ESTIMATIONS OF CLASSICAL SOLUTION OF QUASI-INVERSE PROBLEM

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Abstract

In [1], the authors established a prior estimation for the solution of quasi-inverse problem. In this paper, we get some estimations of the classical solution of the same problem from series representation of solution [10]. Also we establish a prior estamation of solution of higher order than that one in [1]. Our results play an important role in optimal control theory. Many works were devoted to this problem and the control problem of heat conduction with the inverse direction of time and integral boundary conditions. [2, 3, 4, 5, 6, 7, 8, 9, 11].

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1 Introduction

Consider the following quasi-inverse problem:

$$\frac{\partial u_\epsilon}{\partial t} - \frac{\partial^2 u_\epsilon}{\partial x^2} - \epsilon \frac{\partial^3 u_\epsilon}{\partial x^2 \partial t} = 0, \quad (x,t) \in Q, t < T$$
$$u_\epsilon(x,T) = X(x), \quad 0 \le x \le 1$$
$$u_\epsilon(0,t) = 0$$

$$\int_0^1 u_c(x,t)dx = 0, \quad 0 \le t \le T \tag{1}$$

The solution of Eq. 1 can be written in a bi-orthogonal series as follows:

$$u_\epsilon(x,t) = \sum_{k=1}^{\infty} \left[\left(\frac{2\sqrt{\lambda_k}}{(1+\lambda_k\epsilon)^2} \rho_{2k-1} t + \rho_{2k} \right) X_{2k}(x) + \rho_{2k-1} X_{2k-1}(x) \right] e^{\frac{\lambda_t}{1+\lambda_k \epsilon} t} \tag{2}$$

where $X_{2k}(x) = \sin(2\pi kx), X_{2k-1}(x) = x\cos(2\pi kx), \lambda_k = (2\pi k)^2$ (formula (14), [10])

Proposition 1. For the solution of (1), the following estimation holds

$$\int_0^1 (u_\epsilon(x,t))^2 \, dx \le a \|X\|^2_{L_2(0,1)} \tag{3}$$

where

$$a = \frac{2R}{r} \left(\frac{t^2}{16\pi^2 \epsilon^4} + 1 \right) e^{\frac{2t}{\epsilon}} \tag{4}$$

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Proof. For any function $\rho(x) \in L_2(0,1)$, it is proved that the following estimation holds:

$$r\|\rho\|_{L_2(0,1)} \leq \sum_{k=1}^{\infty} \rho_k^2 \leq R\|\rho\|^2_{L_2(0,1)} \tag{5}$$

where ρ_k are the expansion coefficients of this function in a biorthogonal series by basis $X_k(x), r = \frac{3}{4}, R = 16$. It is obvious that now the series (2) is biorthogonal for $u_\epsilon(x,t)$, where the variable t is considered as a parameter. Using both sides of (5), and elementary transformations, we have:

$$\begin{aligned}
\int_0^1 (u_\epsilon(x,t))^2 &\stackrel{\text{def}}{=} \|u_\epsilon(x,t)\|^2_{L_2(0,1)} \\
&\leq \frac{1}{r}\sum_{k=1}^{\infty}\left[\left(\frac{2\sqrt{\lambda_k}}{(1+\lambda_k\epsilon)^2}\rho_{2k-1}t + \rho_{2k}\right)^2 e^{\frac{2\lambda_k}{1+\lambda_k\epsilon}t} + \rho_{2k-1}^2 e^{\frac{2\lambda_k}{1+\lambda_k\epsilon}t}\right] \\
&\leq \frac{2}{r}\sum_{k=1}^{\infty}\left[\rho_{2k}^2 + \rho_{2k-1}^2\left(\frac{4t^2}{\lambda_k^3\epsilon^4}+1\right)\right] e^{\frac{2t}{\epsilon}} \\
&\leq \frac{2R}{r}\left(\frac{t^2}{16\pi^6\epsilon^4}+1\right) e^{\frac{2t}{\epsilon}} \|x\|^2_{L_2(0,1)}
\end{aligned} \tag{6}$$

□

Proposition 2. For the solution of (1), the following estimation holds:

$$\int_0^1 \left(\frac{\partial^2 u_\epsilon(x,t)}{\partial x^2}\right)^2 dx \leq b\|x\|^2_{W_2^2(0,1)} \tag{7}$$

where

$$b = \frac{3.2^7}{r}\left(\frac{t^2}{16\pi^2\epsilon^4} + \frac{1}{\pi^2} + 1\right) e^{\frac{2t}{\epsilon}} \tag{8}$$

Proof. We set $\frac{\partial^2 u_\epsilon(x,t)}{\partial x^2}$ in the following form:

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$$\frac{\partial^2 u_\epsilon(x,t)}{\partial x^2} = -\sum_{k=1}^{\infty}\left[\left(\frac{2\sqrt{\lambda_k}}{(1+\lambda_k\epsilon)^2}\rho_{2k-1}t+\rho_{2k}+\frac{2}{\sqrt{\lambda_k}\rho_{2k-1}}\right)\lambda_k X_{2k}(x)+\rho_{2k-1}\lambda_k X_{2k-1}(x)\right]e^{\frac{\lambda_k}{1+\lambda_k\epsilon}t} \tag{9}$$

where t is a parameter. Applying elementary transformations from the left-hand side of (5), we get:

$$\begin{aligned}
\int_0^1\left(\frac{\partial^2 u_\epsilon}{\partial x^2}(x,t)\right)^2 dx &\overset{\text{def}}{=} \left\|\frac{\partial^2 u_\epsilon}{\partial x^2}(x,t)\right\|^2_{L_2(0,1)} \\
&\leq \frac{1}{r}\sum_{k=1}^{\infty}\lambda_k^2\left[\left(\frac{2\sqrt{\lambda_k}}{(1+\lambda_k\epsilon)^2}\rho_{2k-1}t+\rho_{2k}+\frac{2}{\sqrt{\lambda_k}}\rho_{2k-1}\right)^2+\rho_{2k-1}^2\right]e^{\frac{2\lambda_k}{1+\lambda_k\epsilon}t} \\
&\leq \frac{1}{r}\sum_{k=1}^{\infty}\lambda_k^2\left[\left(3\frac{4\lambda_k}{(1+\lambda_k\epsilon)^4}\rho_{2k-1}^2t^2+3\rho_{2k}^2+3\frac{4}{\lambda_k}\rho_{2k-1}^2\ +3\rho_{2k-1}^2\right)\right]e^{\frac{2\lambda_k}{1+\lambda_k\epsilon}t} \\
&\leq \frac{3}{r}\sum_{k=1}^{\infty}\lambda_k^2\left[\rho_{2k}^2+\rho_{2k-1}^2\left(\frac{4t^2}{\lambda_k^3\epsilon^4}+\frac{4}{\lambda_k}+1\right)\right]e^{\frac{2t}{\epsilon}} \\
&\leq \frac{3}{r}\left(\frac{t^2}{16\pi^6\epsilon^4}+\frac{1}{\pi^2}+1\right)e^{\frac{2t}{\epsilon}}\sum_{k=1}^{\infty}\lambda_k^2\left(\rho_{2k}^2+\rho_{2k-1}^2\right)
\end{aligned} \tag{10}$$

If we integrate ρ_{2k} and ρ_{2k-1} from (formula (9), [10]) twice by parts, and taking into account that $X(0)=0$, and $X'(0)=X'(1)$, we get

$$\begin{aligned}
\rho_{2k-1} &= \frac{-1}{\pi^2k^2}\int_0^1 X''(x)\cos(2\pi kx)dx \\
\rho_{2k} &= \frac{-1}{\pi^2k^2}\int_0^1(1-x)X''(x)\sin(2\pi kx)dx+\frac{2}{\pi^2k^2}\int_0^1 X'(x)\sin(2\pi kx)dx
\end{aligned} \tag{11}$$

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or, we set ρ_{2k}, ρ_{2k-1} in the form

$$\begin{aligned} \rho_{2k-1} &= \frac{-1}{\sqrt{2}\pi^2 k^2}\alpha_k \\ \rho_{2k} &= \frac{-1}{\sqrt{2}\pi^2 k^2}\left(\beta_k - 2\gamma_k\right) \end{aligned} \tag{12}$$

where

$$\begin{aligned} \alpha_k &= \int_0^1 X''(x)\sqrt{2}\cos(2\pi kx)dx \\ \beta_k &= \int_0^1 (1-x)X''(x)\sqrt{2}\cos(2\pi kx)dx \\ \gamma_k &= \int_0^1 X'(x)\sqrt{2}\sin(2\pi kx)dx \end{aligned} \tag{13}$$

Fourier coefficients of the functions $X''(x), (1-x)X''(x)$ and $X'(x)$ are as in (formula (25),[10]). We estimate the series (10):

$$\sum_{k=1}^{\infty} \lambda_k^2\left(\rho_{2k}^2 + \rho_{2k-1}^2\right) \le 8\sum_{k=1}^{\infty}\left(\alpha_k^2 + 2\beta_k^2 + 8\gamma_k^2\right) = 4^3\sum_{k=1}^{\infty}\left(\alpha_k^2 + \beta_k^2 + \gamma_k^2\right) \tag{14}$$

and using Bessel's inequality from (10) we get:

$$\begin{aligned} \int_0^1 \left(\frac{\partial^2 u_\epsilon}{\partial x^2}(x,t)\right)^2 dx &\le \frac{3}{r}4^3\left(\frac{t^2}{16\pi^2\epsilon^4} + \frac{1}{\pi^2} + 1\right)e^{\frac{2t}{\epsilon}}\sum_{k=1}^{\infty}\left(\alpha_k^2 + \beta_k^2 + \gamma_k^2\right) \\ &\le \frac{3}{r}2^7\left(\frac{t^2}{16\pi^2\epsilon^4} + \frac{1}{\pi^2} + 1\right)e^{\frac{2t}{\epsilon}}\left[\|X''\|_{L_2}^2 + \|X'\|_{L_2}^2\right] \end{aligned} \tag{15}$$

from where implies estimation (7). $\square$

It is obvious that the following estimation is correct:

$$\int_0^1 \left(\frac{\partial^2 u_\epsilon}{\partial x^2}\right)^2 dx \le b_1\|X\|_{L_2(0,1)}^2, \tag{16}$$

where

$$b_1 = \frac{3}{r}2^7\int_0^T\left(\frac{t^2}{16\pi^2\epsilon^4} + \frac{1}{\pi^2} + 1\right)e^{\frac{2t}{\epsilon}}dt. \tag{17}$$

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Proposition 3. For the solution of (1), the following estimation holds:

$$\int_0^1 \left(\frac{\partial^2 u_\epsilon}{\partial x^2}(x,t)\right)^2 dx \le c\|X\|^2_{W_2^3(0,1)}, \tag{18}$$

where

$$c = 2^3 3^2 R\left(\frac{t^2}{16\pi^6\epsilon^4} + \frac{9}{4\pi^2} + 1\right) e^{\frac{2t}{\epsilon}}. \tag{19}$$

Proof. Formally, we set $\frac{\partial^3 u^3}{\partial x^3}(x,t)$ in the following form:

$$\begin{aligned}\frac{\partial^3 u_\epsilon}{\partial x^3}(x,t) = &-\frac{1}{4}\sum_{k=1}^{\infty}\Bigg[\left(\frac{2\sqrt{\lambda_k}}{(1+\lambda_k\epsilon)^2}\rho_{2k-1}t + \rho_{2k} + \frac{3}{\sqrt{\lambda_k}}\rho_{2k-1}\right)\\ &\lambda_k\sqrt{\lambda_k}Y_{2k-1}(x) + \rho_{2k-1}\lambda_k\sqrt{\lambda_k}Y_{2k}(x)\Bigg]\cdot e^{\frac{\lambda_k}{1+\lambda_k\epsilon}t}\\ &+\sum_{k=1}^{\infty}\rho_{2k-1}\lambda_k\sqrt{\lambda_k}X_{2k}(x)e^{\frac{\lambda_k}{1+\lambda_k\epsilon}t}\end{aligned} \tag{20}$$

where the first series is a bi-orthogronal expansion of the function $F_1(x,t)$ defined by this series with basis:

$$\begin{aligned}Y_{2k-1}(x) &= 4\cos(2\pi kx),\\ Y_{2k}(x) &= 4(1-x)\sin(2\pi kx),\end{aligned}$$

$$\begin{aligned}F_1(x,t) = -\frac{1}{4}\sum_{k=1}^{\infty}\Bigg[&\left(\frac{2\sqrt{\lambda_k}}{(1+\lambda_k\epsilon)^2}\rho_{2k-1}t + \rho_{2k} + \frac{3}{\sqrt{\lambda_k}}\rho_{2k-1}\right)\cdot\\ &\lambda_k\sqrt{\lambda_k}Y_{2k-1}(x) + \rho_{2k-1}\lambda_k\sqrt{\lambda_k}Y_{2k}(x)\Bigg]\cdot e^{\frac{\lambda_k}{1+\lambda_k\epsilon}t},\end{aligned} \tag{21}$$

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and the second series is an expansion defined by this series by the functions $F_2(x,t)$ with the functions $X_{2k}(x) = \sin 2\pi kx$:

$$F_2(x,t) = \sum_{k=1}^{\infty} \rho_{2k-1}\lambda_k\sqrt{\lambda_k}X_{2k}(x)e^{\frac{\lambda_k}{1+\lambda_k\epsilon}}t \tag{22}$$

Next, for the functions $\rho(x) \in L_2(0,1)$, with the coefficients $\bar{\rho}_k$ an expansion in bi-orthogonal series with conjugate basis $Y_k(x)$, the following eatimation holds:

$$\frac{1}{R}\|\rho\|^2_{L_2(0,1)} \leq \sum_{k=1}^{\infty} \bar{\rho}_k^2 \leq \frac{1}{r}\|\rho\|^2_{L_2(0,1)}. \tag{23}$$

We estimate the norms of the functions $F_1(x,t)$ and $F_2(x,t)$ from the left-hand-side of (23) and froms Parseval equalities, considering t as a parameter:

$$\begin{aligned}
\|F_1(\cdot,t)\|^2_{L_2(0,1)} &\leq \frac{R}{16}\sum_{k=1}^{\infty}\left[\left(\frac{2\sqrt{\lambda_k}}{(1+\lambda_k\epsilon)^2}\rho_{2k-1}t + \rho_{2k} + \frac{3}{\sqrt{\lambda_k}}\rho_{2k-1}\right)^2\lambda_k^3 + \rho_{2k-1}^2\lambda_k^3\right]e^{\frac{2\lambda_k}{1+\lambda_k\epsilon}t} \\
&\leq \frac{3R}{16}\sum_{k=1}^{\infty}\lambda_k^3\left[\rho_{2k}^2 + \left(\frac{4t^2}{\lambda_k^3\epsilon^4} + \frac{9}{\lambda_k} + 1\right)\rho_{2k-1}^2\right]e^{\frac{2t}{\epsilon}} \\
&\leq \frac{3R}{16}\left(\frac{t^2}{16\pi^6\epsilon^4} + \frac{9}{4\pi^2} + 1\right)e^{\frac{2t}{\epsilon}}\sum_{k=1}^{\infty}\lambda_k^3\left(\rho_{2k}^2 + \rho_{2k-1}^2\right).
\end{aligned} \tag{24}$$

Then, using the results of [10], it is not hard to get the following estimations:

$$\|F_1(\cdot,t)\|^2_{L_2(0,1)} \leq 2\cdot 3^2 R\left(\frac{t^2}{16\pi^6\epsilon^4} + \frac{9}{4\pi^2} + 1\right)e^{\frac{2t}{\epsilon}}\|X\|^2_{W_2^3(0,1)}, \tag{25}$$

and for the function $F_2(x,t)$, we get that:

$$\|F_2(\cdot,t)\|^2_{L_2(0,1)} \leq \sum_{k=1}^{\infty}\lambda_k^3\rho_{2k-1}^2 e^{\frac{2t}{\epsilon}} \leq 8e^{\frac{2t}{\epsilon}}\|X'''\|^2_{L_2(0,1)} \tag{26}$$

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Next, using an obvious inequality:

$$\left\| \frac{\partial^3 u_\epsilon}{\partial x^3}(x,t) \right\|^2_{L_2(0,1)} \leq 2 \left(\|F_1(x,t)\|^2_{L_2(0,1)} + \|F_2(x,t)\|^2_{L_2(0,1)} \right), \quad (27)$$

We get the required estimation (18). □

It is obvious that

$$\int_Q \left(\frac{\partial^3 u_\epsilon}{\partial x^3}(x,t) \right)^2 dxdt \leq c_1 \|X\|^2_{W_2^3(0,1)}, \quad (28)$$

where

$$c_1 = 2^3 \cdot 3^2 R \int_0^T \left(\frac{t^2}{16\pi^6 \epsilon^4} + \frac{9}{4\pi^2} + 1 \right) e^{\frac{2t}{\epsilon}} dt. \quad (29)$$

From propositions 2 and 3, implies the following theorem:

Theorem 1. For the classical solution of problem (1), the following estimation holds:

$$\max_{0 \leq x \leq 1} \left| \frac{\partial^2 u_\epsilon}{\partial x^2}(x,t) \right| \leq \alpha \|X\|_{W_2^3(0,1)} \quad (30)$$

where $\alpha = 4c$.

Proof. We set $x_0 \in (0,1)$ and $\forall x, 0 < x_0 < x < 1$. We write the following equity,

$$\frac{\partial^2 u_\epsilon}{\partial x^2}(x,t) = \int_{x_0}^x \frac{\partial^3 u_\epsilon}{\partial x^3}(\xi,t) d\xi + \frac{\partial^2 u_\epsilon}{\partial x^2}(x_0,t). \quad (31)$$

Using Cauchy inequality, we have:

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$$\begin{aligned}\left|\frac{\partial^2 u_\epsilon}{\partial x^2}(x,t)\right|^2 &\leq 2\left(\left(\int_{x_0}^{x}\frac{\partial^3 u_\epsilon}{\partial x^3}(\xi,t)d\xi\right)^2+\left(\frac{\partial^2 u_\epsilon}{\partial x^2}(x_0,t)\right)^2\right)\\ &\leq 2\left(\int_{x_0}^{x}d\xi\int_{x_0}^{x}\left(\frac{\partial^3 u_\epsilon}{\partial x^3}(\xi,t)\right)^2d\xi+\left(\frac{\partial^2 u_\epsilon}{\partial x^2}(x_0,t)\right)^2\right)\\ &\leq 2\left(\int_0^1\left(\frac{\partial^3 u_\epsilon}{\partial x^3}(\xi,t)\right)^2d\xi+\left(\frac{\partial^2 u_\epsilon}{\partial x^2}(x_0,t)\right)^2\right).\end{aligned} \tag{32}$$

Integrating both sides with respect to x_0, we get

$$\left|\frac{\partial^2 u_\epsilon}{\partial x^2}(x,t)\right|^2 \leq 2\left(\int_0^1\left(\frac{\partial^3 u_\epsilon}{\partial x^3}(\xi,t)\right)^2d\xi+\int_0^1\left(\frac{\partial^2 u_\epsilon}{\partial x^2}(\xi,t)\right)^2d\xi\right). \tag{33}$$

From propositions 2 and 3 , implies that

$$\left|\frac{\partial^2 u_\epsilon}{\partial x^2}(x,t)\right|^2 \leq 4c\left\|X'''\right\|^2_{W_2^3(0,1)}, \tag{34}$$

from where implies the theorem's statement.

Corollary 1.

$$\max_{(x,t)\in\bar{Q}}\left|\frac{\partial^2 u_\epsilon}{\partial x^2}(x,t)\right| \leq \alpha_1\|X\|_{W_2^3(0,0)}, \tag{35}$$

where

$$\alpha_1 = 2^3\cdot 3^2 4R\left(\frac{T^2}{16\pi^6\epsilon^4}+\frac{9}{4\pi^2}+1\right)e^{\frac{2T}{\epsilon}}. \tag{36}$$

Theorem 2. For the classical solution of (1), the following estimation holds:

$$\int_0^1\left(\frac{\partial^4 u_\epsilon}{\partial x^3\partial t}\right)^2dx \leq c\|X\|^2_{W_2^3(0,1)} \tag{37}$$

where

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$$C = \left(\frac{c}{\epsilon^2} + \frac{4}{\epsilon^4} e^{\frac{2t}{\epsilon}} \right). \tag{38}$$

Proof. We set $\frac{\partial^4 u_\epsilon(x,t)}{\partial x^3 \partial t}$ in the following form:

$$\frac{\partial^4 u_\epsilon(x,t)}{\partial x^3 \partial t} = V_\epsilon(x,t) + W_\epsilon(x,t), \tag{39}$$

where

$$V_\epsilon(x,t) = \sum_{k=1}^{\infty} \frac{\lambda_k}{1+\lambda_k \epsilon} \frac{\partial^3 u_{\epsilon,k}}{\partial x^3}(x,t), \tag{40}$$

$\Big(\frac{\partial^3 u,^2}{\partial x^3}(x,t)$ is the k^{th} term of the series (20) ,

$$W_c(x,t) = -\sum_{k=1}^{\infty} \frac{2\lambda_k^2}{(1+\lambda_k \epsilon)^2} e^{\frac{\lambda_k}{1+\lambda_k \epsilon} t} \rho_{2k-1} Y_{2k-1}(x). \tag{41}$$

From the estimations (5), (18), (23), and Parseval's equality, we estimate the norms

$$\|V_\epsilon(x,t)\|^2_{L_2(0,1)} \le \frac{C}{\epsilon} \|X\|^2_{L_2(0,1)},$$

$$\|W_\epsilon(x,t)\|^2_{L_2(0,1)} \le \frac{2}{\epsilon^4} e^{\frac{2t}{\epsilon}} \|X\|^2_{L_2(0,1)}. \tag{42}$$

Using triangle inequality for the norm, we get the required estimation (37).

Corollary 2. For the classical solution of problem (1), the following estimation holds:

$$\int_Q \left(\frac{\partial^4 u_\epsilon}{\partial x^3 \partial t} \right)^2 dx \le c_1 \|X\|^2_{w_2^3(a,1)} \tag{43}$$

where

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$$c_1 = \frac{2}{\epsilon^2}\left(2^3 \cdot 3^2 R \int_0^T \left(\frac{t^2}{16\pi^6\epsilon^4} + \frac{9}{4\pi^2} + 1\right) e^{\frac{2t}{\epsilon}} dt + \frac{1}{\epsilon}\left(e^{\frac{2t}{\epsilon}} - 1\right)\right). \tag{44}$$

Note that, the estimation (44) of the classical solution of problem (1) is higher than the order of the a priori estimation in [1].

References

[1] A. Almomani, R. Almomani, Higher order estimation of the solution of quasi-inverse problem, International Journal of Applied Mathematics 32, No4 (2019), 623-626; DOI: 10.12732/ ijam. v $32i4 \cdot 6$.

[2] A. Bouziani, Mixed problem with boundary integral conditions for a certain parabolic equation, J. Appl. Math. Stochastic Anal., 9 (1990), 323-330.

[3] A. Bowziani, Strong nolution for a mixad problem with nonlocal condition for a certain pluriparabolic equations, Hiroshima Math, J., 27 (1997). 373 390.

[4] H. Almefleh, R. Almomani, New a priori estimations of the solution of quasi-inverse problem, International Journal of Applied Mathematics, 27, No. $3(2014), 205 - 210$; DOI: 10.12732/ijam. V27i3.1.

[5] H. Almefleh, R. Almomani, A priori estimate of the solution of quasi-inverse problem, European Intern. J. of Sci and Technology (EIJST), 2, No. 2 (2013), $78 - 80$.

[6] M. Feng, B. Du, W. Ge, Impulsive boundary value problems with integral boundary conditions and one-dimensional p-Laplacian, Nonlinear Anal, $70(2009), 3119 - 3126$.

Received: July 16, 2025

International Journal of Applied Mathematics
Volume 38 No. 2s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

[7] N. E. Benouar and N. I. Yurchuk. Mixed problem with an integral condition for parabolic equations with the Bessel operator, Dierentsial'nye Uravneniya, 27 (1991), 2094-2098.

[8] N.I. Ionkin, Solution of a boundary-value problem in heat conduction with a nonclassical boundary condition, Diff. Eqs., 13, No 2 (1977), 294-304.

[9] N.I. Yurchuk, Mixed problem with an integral condition for certain parabolic equations, Diff. Eqs., 22. No 12 (1986), 2117-2126.

[10] Raid Almomani, Ahmad Almomani; Sufficient conditions for the Existence of Classical Solution of Quasi- Inverse Problem, International Journal of Applied Mathematics, 33 No. 3 (2020), 393-401; 1) DOI: 10.12732 / ijam. V $33i3.3$.

[11] R. Almomani, H. Almefleh, On heat conduction problem with integral boundary condition. J. of Emerging Trends in Eng and Appl. Sci. (JETEAS), 3, No. 6 (2012) 977-979.

Received: July 16, 2025