

GRAPHS WITH METRIC-LOCATION -DOMINATION NUMBER ATTAIN UPPER BOUNDS AND LOWER BOUNDS

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Abstract

For a connected graph G of order n , the relationship among the metric-location-domination number $\gamma_M(G)$, the domination number $\gamma(G)$ and the metric dimension $\beta(G)$ is known to satisfy the inequality: $\max\{\gamma(G), \beta(G)\} \leq \gamma_M(G) \leq \min\{n - 1, \gamma(G) + \beta(G)\}$. In this paper, we investigate connected graphs whose metric-location-domination number attains the lower bound of the known inequality by

exploring its relationship with earrings and full-degree vertices. Furthermore, we characterize graphs of full-degree vertices with metric-location-domination number 2.

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1 Introduction

Let $G = (V, E)$ be a simple connected graph. A set $D \subseteq V$ is called a *dominating set* of G if every vertex not in D is adjacent to at least one vertex in D . The *dominating number* $\gamma(G)$ of G is defined as the minimum size of such a dominating set of G , see [8, 9]. For an ordered subset $R = \{r_1, \dots, r_p\}$ of V and $v \in V$, the *representation* of a vertex v with respect to R is the p -vector $(d(v, r_1), d(v, r_2), \dots, d(v, r_p))$, where $d(x, y)$ denote the shortest path distance between vertices x and y in G . A set R is called a *resolving set* of G if every vertex of G has unique representation relative to R . The *metric dimension* $\beta(G)$ is the size of the smallest resolving set of G , see [3, 2, 10].

Research on determining the metric dimension and domination numbers in graphs has seen significant developments in recent decades. The metric dimension, which measures the ability of a set of vertices to distinguish all other vertices based on distance, has become a central focus in graph theory due to its wide range of applications. At the same time, the domination number, which represents the size of the smallest set of vertices that dominates the entire graph, has also been extensively studied as an important parameter in graph structure.

The combination of the concepts of metric dimension and domination number was first introduced by Brigham and colleagues in 2003 [1] under the term dominating resolving set. This concept integrates both notions to find a set that simultaneously functions as a dominating set and a resolving set.

In the following year, Henning and Oellermann [4] conducted a study on a similar concept, referring to it as the metric-location-domination number. In their research, Brigham et al. investigated the relationship between the domination number $\gamma(G)$, the metric dimension $\beta(G)$, and the metric-location-domination number $\gamma_M(G)$, which is expressed by the following relation:

$$\max\{\gamma(G), \beta(G)\} \leq \gamma_M(G) \leq \min\{n - 1, \gamma(G) + \beta(G)\}$$

This result simultaneously establishes lower and upper bounds for the metric-location-domination number of a graph. Meanwhile, Henning and Oellermann focused their attention on characterizing graphs of order n that have a metric-location-domination number equal to $n - 2$ and $n - 1$.

Bounds that connect domination numbers and metric dimension in graphs fundamentally support the characterization of graphs whose metric-location-domination number attains either the upper or lower bound. The metric domination number and metric dimension can be related in such a way to the extremal properties of these graphs [5]. Other studies examining these results have been presented in [6, 7, 13, 11, 12]

In the effort to characterize graphs that satisfy these extremal conditions, the role of graph structures such as ears and full-degree vertices becomes very important. An *earring* a vertex adjacent to at least two vertices of degree one can determine how a graph attains the lower bound of the metric-location-domination number. Meanwhile, the presence of a *full-degree vertex*, that is, a vertex adjacent to all other vertices in G strongly influences the attainment of the upper bound. Analyzing the existence and properties of these two elements allows for the identification of sufficient or necessary conditions for a graph to have its metric-location-domination number reach these bounds.

The resolving set R and the dominating set D of any graph G satisfy one of the following conditions:

1. $R \subseteq D$, for some R that is a subset of a minimum dominating set D .

2. $D \subset R$, for some D that is a proper subset of a minimum resolving set R .
3. $D \cap R = \emptyset$, for every minimum dominating set D and every minimum resolving set R .
4. $D \cap R \neq \emptyset$, for some minimum dominating set D and some minimum resolving set R , where neither is a subset of the other

If condition (1) or (2) holds, the graph G has a metric-location-domination number equal to its lower bound, namely $\gamma_M(G) = \max\{\gamma(G), \beta(G)\}$. This is because its resolving set can also act as its dominating set, or conversely, its dominating set can also serve as its resolving set. In other words, one of these parameters "dominates" the other in determining the domination-location-metric set of graph. Examples of graphs that satisfy this are the path graph P_n ($n \geq 4$) for condition (1) and the complete bipartite graph $K_{m,n}$ ($m, n \geq 2$) for condition (2). If condition (3) holds, the metric-location-domination number of G attains its upper bound, namely, $\gamma_M(G) = \min\{\gamma(G) + \beta(G), n - 1\}$. This happens because in determining the minimum domination-location-metric set of G , one must consider both the resolving set and the dominating set. In other words, these two parameters resolving set and dominating set are disjoint. An example of a graph in this category is the star graph $K_{1,n}$ ($n \geq 2$). Furthermore, if condition (4) is satisfied, the graph G has a metric-location-domination number $\gamma_M(G)$ that lies strictly between the above bounds: $\max\{\gamma(G), \beta(G)\} < \gamma_M(G) < \min\{\gamma(G) + \beta(G), n - 1\}$.

Note that two twin vertices in a graph G have the same neighbors in G ; therefore, two twin vertices have the same distance to every other vertex in G . To have different representations, at least one of the two twin vertices must be included in a resolving set of G . More generally, if a graph has p twin vertices, then every resolving set must include at least $p - 1$ of those twin vertices

Theorem 1. *If S is a set containing $p \geq 2$ twin vertices in a connected graph G then every resolving set must include at least $p - 1$ vertices of S .*

2 Upper and Lower Bounds of Metric-Location Domination Number with earrings and Full degree Vertex

An earring plays a special role in the relationship between dominating sets and resolving sets. Its presence can influence how we select vertices to dominate or distinguish other vertices. In many cases, vertices like earrings have unique characteristics that make them relevant in determining minimum dominating sets and minimum resolving sets. Therefore, it is interesting to understand how earrings relate to these two important concepts: the minimum dominating set and the minimum resolving set.

Lemma 1. *If G has a earring then:*

- (a) *Every dominating set of G contains all earring of G .*
- (b) *Every resolving set of G does not contain any earring of G .*

Proof. Suppose graph G has earrings $a_1, a_2, \dots, a_r$, $r \geq 1$. Let D and R be a minimum dominating set and a minimum resolving set of G , respectively.

- (a) Let $|D| = |A| + |B|$, where A is the set of earrings of G , and B is the minimum dominating set of $G - (A \cup A')$, with A' is the adjacent set of earring a_i , $i = 1, 2, \dots, r$.
 Suppose D does not contain a_i for some $i = 1, 2, \dots, r$. Consider the set of degree one vertices $V = \{v_{i1}, v_{i2}, \dots, v_{ik}\}$, $k \geq 2$ with $v_{ij} \sim a_i$ for every $j = 1, 2, \dots, k$. Note that V is the set are only adjacent to a_i . Since $a_i \notin D$, therefore $V \subseteq D$. Hence $|V| \geq |A|$, which implies $|D| = |V| + |B| > |A| + |B|$, a contradiction.

- (b) Suppose R contains a earring a of G . Consider the set of degree-one vertices $V = \{v_1, v_2, \dots, v_k\}$, $k \geq 2$ with $v_i \sim a$ for every $i = 1, 2, \dots, k$. Note that $v_1, v_2, \dots, v_k$ form the set twin vertices, based on 1, $k-1$ the vertices must be contained in R . Furthermore, note that vertices in V are only adjacent to a . This means a cannot resolve vertices in V , a contradiction.

□

Lemma 2. *In a graph G order n , $\gamma(G) = 1$ if and only if there exists a number $k < n$ of vertices in G of full degree.*

Proof. Let G be a graph order n . Let $\gamma(G) = 1$. Since $\gamma(G) = 1$ this means there is exists a vertex v that is adjacent to every other vertex in G . Therefore G has a full degree vertex. Conversely, let G has $k < n$ vertices of full degree. This means there exist a vertex v adjacent to other $n - 1$ vertices. Consequently $\gamma(G) = 1$. □

The parameter called the metric-location-domination number is important in analyzing the structure of a graph by combining the concepts of domination and the ability to distinguish vertices based on their positions. Achieving the lower bound value of this parameter indicates the presence of certain structural characteristics in the graph under consideration. The following theorem states that if the metric-location-domination number of a graph attains this lower bound, then the graph contains no earring

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Theorem 2. *Let G be a connected graph other than P_3 . If*

$\gamma_M(G) = \max \{\gamma(G), \beta(G)\}$ then G contains no earring.

Proof. Let G has a earring a with adjacent to vertices of degree one $v_1, v_2, \dots, v_k$, $k \geq 2$. Suppose $\gamma_M(G) = \max \{\gamma(G), \beta(G)\}$, consider two cases:

Case 1: $\gamma_M(G) = \gamma(G)$ then $a \in D$ and $v_i \notin D$, this means $D \subset M$. Hence $|D| < |M|$, a contradiction.

Case 2: $\gamma_M(G) = \beta(G)$ then $a \notin R$ and there is exist i such that $v_i \notin D$, therefore $R \subset M$, hence $|R| < |M|$, a contradiction $\square$

Theorem 3. If graph G has $k \geq 2$ full degree vertices then $\gamma_M(G) = \beta(G)$.

Proof. Let graph G has full degree vertices $a_1, a_2, \dots, a_k$, $k \geq 2$. $a_1, a_2, \dots, a_k$ form a clique in G , therefore $k - 1$ these vertices are contained in a minimum resolving set R of G and R can be dominate all vertices in G . Hence, $D \subset R$, therefore $\gamma_M(G) = \beta(G)$. $\square$

Theorem 4. If graph G order $n \geq 3$ has a earring and full degree vertices then $\gamma_M(G) = \gamma(G) + \beta(G)$.

Proof. Let a be a earring and a full degree vertex of G , with adjacent of degree one $v_1, v_2, \dots, v_k$, $k \geq 2$. Then $a \in D$ but $a \notin R$. Since G has a full degree vertex then $\gamma(G) = 1$. Therefore $D \cap R = \emptyset$. Hence $\gamma_M(G) = \gamma(G) + \beta(G)$. $\square$

Theorem 5. Let G be a graph of full degree vertex. If $\gamma_M(G) = 2$ then $n \leq 5$.

Proof. Let G be a graph of order n Let $\gamma_M(G) = 2$. Let vertex v is a full degree vertex and $M = \{v_1, v_2\}$ is a minimum metric-location-domination set of G . Since G has a full degree vertex then diameter of G is at most 2.

Case 1. $v \in M$. Since length $n - 2$ vertices to vertex v is 1 then for $n - 2$ others vertex of at most unique distance, $n - 2 \leq 2$. So that $n \leq 4$.

Case 2. $v \notin M$. Since distance $n - 2$ vertices to vertex v is 1

then for representation unique $n - 2$ other vertices then maximal 2 distance $n - 2 \leq 2$, since vertex v is vertex of G then $n \leq 5$. $\square$

Theorem 6. *Let G be a graph of full degree vertex. Then $\gamma_M(G) = 2$ if and only if $G = K_1 + H$, where $H \in \{P_2, 2K_1, P_2 \cup K_1, P_3, P_4, 2P_2, C_5\}$.*

Proof. Let G is a graph that has a full degree vertex. Since G a full degree vertex, we have $\gamma(G) = 1$, Let the minimum dominating set of G be $D = \{x\}$. Since $\gamma_M(G) = 2$ then there is exist a set $M = \{u, v\} \subset V(G)$ which is a minimum metric-locating dominating set of G .

If $x \in M$, write $M = \{x, u\}$. Based on Theorem ?? $d_x \leq 3$, hence $|V(H)| \leq 3$. Furthermore, since vertices in H must distinguish other vertices in H then H is at most one twin vertex. Therefore $H \cong 2K_1, H \cong K_2, H \cong K_1 \cup P_2, H \cong P_3$.

If $x \notin M$, write $M = \{u, v\}$. Since x is a full degree vertex then $R(x|M) = (1, 1)$, therefore $R(y|M) = (1, 2)$ or $(2, 1)$ for every vertex $y \in V(G) - M - \{x\}$. So That $2 \leq V(H) \leq 4$. Furthermore, since two vertices in H must distinguish other vertices in H then is at most two twin vertices, therefore $H \cong K_2, H \cong 2K_1, H \cong K_1 \cup P_2, H \cong P_3, H \cong 2P_2, H \cong P_4, H \cong C_4$. $\square$

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