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Comparative study of two identical synchronization between two chaotic models

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Abstract

In this paper we have done a comparative study between two identical synchronizations; identical synchrinization using Carroll and Pecora method and identical synchronization using Continuous Control method, we have used for this comparative study two chaotic models wich have almost the same behavior.

Key Words: *Identical Synchronization, Carroll and Pecora method, Continuous Control method.*

1 Introduction

The synchronization problem is a very important topic in the non-linear science. Carroll and Pecora [8] has introduced the idea of chaotic synchronization in 1990. Two or more coupled chaotic systems are called synchronized if their behaviors are closely related. Chaos synchronization has received much attention due to its applications in many area such secure communication, information processing, biological systems and chemical reactions. Usually two dynamical systems are called synchronized if the distance between their corresponding states converges to zero as time goes to infinity. This type of synchronization is called identical synchronization.

Since the result of Pecora and Carroll [8], a multitude of articles have been devoted to the synchronization of chaotic systems, in the vast majority of cases, these

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articles consider Master/Slave type configurations, synonymous with unidirectional couplings (only the Master system transmitting information or commands to the Slave charged to synchronize). With regard to this form of coupling and the pairing, identical or heterogeneous, of various chaotic systems (such as those of Lorenz, Chua, Rossler, Chen...), many synchronisation criteria were then defined, whether in the context of free or forced synchronization.

However, to our knowledge, a part from the investigation of certain approaches such as the theory of graphs, very few studies relating to the case of a bidirectional coupling (where each system is both Master and Slave) with control of each system, have been carried out in considering control Theory for support. However, in such configuration, each system influences the dynamics of the other part through the bias of the couplings. Therefore, synchronization is part of the evolution of each system towards a consensual behavior; thus increasing the complexity of its analysis compared to the case of a simple follow-up of a reference trajectory.

In this paper we present a comparative study between two identical synchronisation of two coupled models which are the Chen's model and four-scroll's model using two identical synchronisations methods, Carroll and Pecora method and Active Control method

2 Identical Synchronization

2.1 Identical Synchronization by Carroll and Pecora method :

As stated previously, chaotic systems are dynamical systems which define synchronization because of their sensitivity to initial conditions. The evolution of two identical chaotic systems, which begin almost in the same initial points, becomes uncorrelative over time. Hence the practical impossibility of building identical chaotic systems, synchronized in the laboratory. In 1990 Pecora and Carroll reported [8], in an article, their discovery that two identical chaotic systems synchronize under certain conditions. So they considered an autonomous chaotic system of dimension n:

$$\dot{x} = f(x) \tag{1}$$

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In 1990 Carroll and Pecora [8] showed that: if we substitute a variable of a chaotic system by the decomposition of this system, then we can synchronize it with another identical system. Suppose that we have two identical chaotic dynamical systems of dimension n presented by:

$$\dot{X} = f(X) \tag{2}$$

$$\dot{Y} = f(Y) \tag{3}$$

One of the two systems is the master and the other is the slave, the idea of this method is to decompose the master system into two subsystems as follows:

$$\dot{X}_1 = F_1(X_1, X_2) \tag{4}$$

$$\dot{X}_2 = F_2(X_1, X_2) \tag{5}$$

and we will consider one of the two systems as a transmitting signal and it will be injected into the slave system, then we obtain the following system:

$$\dot{Y} = F_2(X_1, Y) \tag{6}$$

As slave or receiver system, and X_1 is transmitting signal. The following diagram represents the synchronization of Caroll and Pecora. In this method, complete synchronization is defined as the identity between the trajectories of the systems slave and master system for the same chaotic transmitter signal. The existence of competent synchronization implies that the slave system is asymptotically stable: $e(t) \to 0$ when $t \to \infty$ with $e(t)$ is represents the error system of the synchronization defined by: $e(t) = Y - X_2$

Théorème 1. *The master system and the slave system are synchronized if and only if all the Lyapunov exponents of the slave system, called the conditional Lyapunov exponents, are negative*

2.2 Identical Synchronization using continuous control method :

The principle of this method is : Suppose we have two identical chaotic systems defined by [9]:

$$\dot{X} = AX + \phi(X) \tag{7}$$

$$\dot{Y} = AY + \phi(Y) + M_{n\times n}(Y - X) \tag{8}$$

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the two systems are linked by a unidirectional coupling, with X and Y being variables of the master and slave systems respectively, and $M_{n\ timesn}$ a diagonal square matrix of order n defined by:

$$M_{n\times n} = \begin{pmatrix} P_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P_n \end{pmatrix}$$

A is a square constant matrix of order n, and $\ phi$ a continuous function which represents the nonlinear part of each system satisfying the following Lipschitz condition:

$$\|\phi(X) - \phi(Y)\| \leq \rho\|X - Y\| \tag{9}$$

ρ is called the Lipschitz constant, and $\ |.\ |$ the Euclidean norm, then we have the following theorem:

Théorème 2. *: Let E be the unit matrix of order n, and $\ lambda_i$ are the eigenvalues of the symmetric matrix*

$$\frac{(A+M_{n\times n})+(A+M_{n\times n})^T}{2} + \rho \times E$$

If $max(\ lambda_i) < 0$, then the master system and the slave system are synchronized in the sense that the error system tends towards zero exponentially. [2]

3 Identical synchronization of two identical chaotic systems by Carroll and Pecora method

3.1 Chen's attractor

The Chen system was discovered thanks to the chaotification (anti-control) of the controlled Lorenz system, that means : from a non-chaotic system we will try to create a chaotic system by using the feedback method. state (state feedback). The controlled Lorenz system is given by:

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = (c - a)x - xz + cy \\ \dot{z} = xy - bz \end{cases} \tag{10}$$

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With $a; b; c$ are constants, and u the linear feedback controller of the form:

$$u = k_1 x + k_2 y + k_3 z$$

with k_1, k_2, k_3 constants to be determined. We want the eqref eq10 system to become chaotic; so it suffices that $k_1; k_2; k_3$ take values so that this system has a chaotic behavior. The equilibrium points of the system (3.1) are given by: $P_1(0;0;0)$;

$$P_{2;3}(\pm\sqrt{b(2c-a)}; \pm\sqrt{b(2c-a)}; 2c-a)$$

The Jacobian matrix of the system (3.1) at the point $(x_0; y_0; z_0)$, is given by:

$$J_{(x_0;y_0;z_0)} = \begin{pmatrix} -a & a & 0 \\ c + k_1 - z_0 & k_2 - 1 & k_3 - x_0 \\ y_0 & x_0 & -b \end{pmatrix}$$

We notice that at the equilibrium point $P_1(0;0;0)$ the associated Jacobian matrix is:

$$J_{(0;0;0)} = \begin{pmatrix} -a & a & 0 \\ c + k_1 & k_2 - 1 & k_3 \\ 0 & 0 & -b \end{pmatrix}$$

k_3 does not contribute to its eigenvalues so it does not contribute to Lyapunov exponents, so for simplicity, Chen chose $k_3 = 0$. To have the chaotic behavior, it is wrong at least to have an unstable eigenvalue for each of the other points of equilibrium. By applying the Routh-Herwitz criteria, we can take $k_1 = -a$; $k_2 = 1 + c$ as a simple choice among several others. Then the feedback controller is given by:

$$u = -ax + (1 + c)y$$

so the system (10) becomes:

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = (c - a)x - xz + cy \\ \dot{z} = xy - bz \end{cases}$$

called the Chen system.

We will synchronize two identical systems of Chen by the method of Carroll and Pecora method, and we will study all the possible cases.

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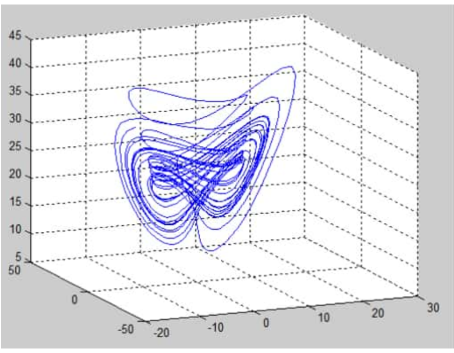

Figure 1: Chen's attractor $a = 35$, $b = 3$, $c = 28$

3.2 The case where the transmitting signal is x

We will assume in this part that the sender system is given by [5]:

$$\begin{cases} \dot{x}_1 = a(y_1 - x_1) \\ \dot{y}_1 = (c-a)x_1 - x_1 z_1 + c y_1 \\ \dot{z}_1 = x_1 y_1 - b z_1 \end{cases}$$

that we can decompose into two-systems by the following substitution:

$$X_1 = (x_1)$$

$$X_2 = (y_1; z_1)$$

In this case the sending system is

$$\begin{cases} \dot{y}_1 = (c-a)x_1 - x_1 z_1 + c y_1 \\ \dot{z}_1 = x_1 y_1 - b z_1 \end{cases} \tag{11}$$

while the receiving system will be present by:

$$\begin{cases} \dot{y}_2 = (c-a)x_1 - x_1 z_2 + c y_2 \\ \dot{z}_2 = x_1 y_2 - b z_2 \end{cases} \tag{12}$$

Let's introduce the following values:

$$e_y = y_2 - y_1$$

$$e_z = z_2 - z_1$$

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in systems (11); (12); we get the following system error:

$$\begin{cases} \dot{e}_y = ce_y - x_1 e_z \\ \dot{e}_z = x_1 e_y - be_z \end{cases} \tag{13}$$

We will choose the function:

$$V_1 = \frac{1}{2}(e_y^2 + e_z^2)$$

as a Lyapunov function, its derivative is given by:

$$\dot{V}_1 = ce_y^2 - be_z^2$$

for the system (13) to be exponentially stable in the neighborhood of the origin, it suffices that $dotV < 0$; for that we get :

$$e_y < \sqrt{\frac{b}{c} e_z}$$

Numerical simulation

For parameter values $(a; b; c) = (35; 3; 28)$; and for the initial values $(x_1; y_1; z_1) = (50; 10; 10)$; et $(y_2; z_2) = (-10; -10)$ we get :

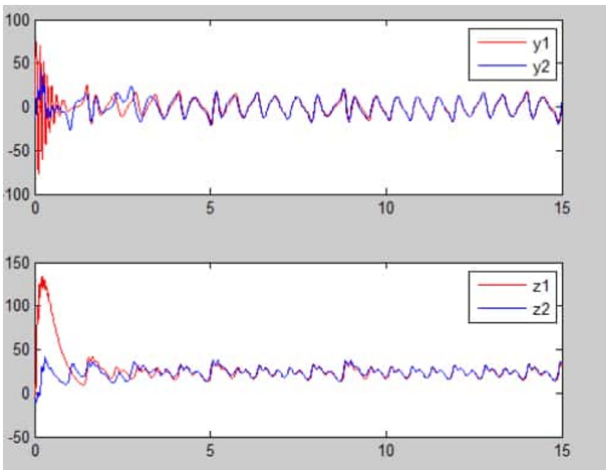

Figure 2: Identical synchronization of Chen's model using Carroll and Pecora method where the signal trasmetter is x

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3.3 The case where the transmitting signal is z

In this case the sending system is given by:

$$\begin{cases} \dot{x}_1 = a(y_1 - x_1) \\ \dot{y}_1 = (c-a)x_1 - x_1 z + c y_1 \end{cases} \tag{14}$$

Then the receiving system is represented by:

$$\begin{cases} \dot{x}_2 = a(y_2 - x_2) \\ \dot{y}_2 = (c-a)x_2 - x_2 z + c y_2 \end{cases} \tag{15}$$

From the equations (14); (15) we get the following error system :

$$\begin{cases} \dot{e}_x = a(e_y - e_x) \\ \dot{e}_y = (c-a)e_x - e_x z + c e_y \end{cases}$$

with : $e_x = x_2 - x_1$ et $e_y = y_2 - y_1$

The Lyapunov function, in this part, will be chosen by :

$$V_3 = \frac{1}{2}(e_y^2 + e_z^2)$$

Its derivative is given by:

$$\dot{V}_3 = -ae_2 x + e_y^2(c - z) + c e_x e_y$$

For the two systems (14) and (15) to be synchronized, it suffices that :

$$\dot{V}_3 < 0$$

For parameter values $(a; b; c) = (35; 3; 28)$; and for the initial values $(x_1; y_1; z_1) = (50; 35; 50)$; et $(x_2; y_2) = (50; 29)$. We get:

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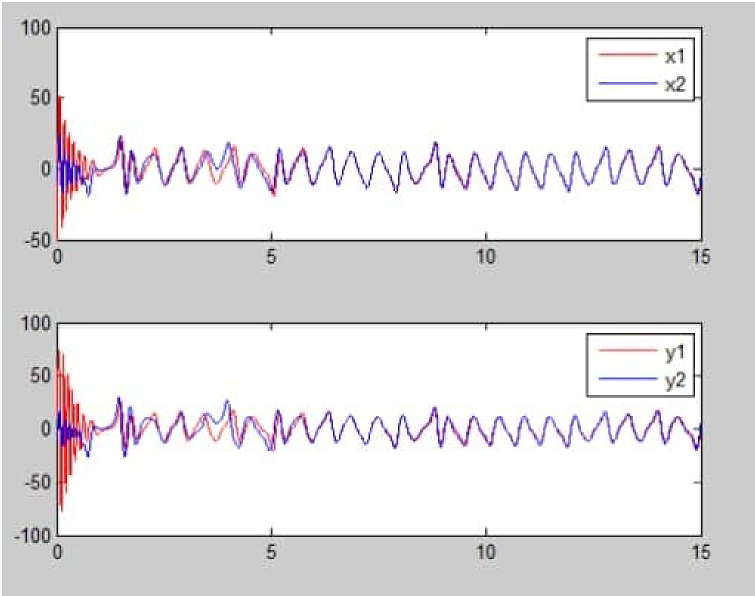

Figure 3: Identical synchronization of chen's model by Carroll and Pecora method where the signal trasmetter is z

4 Identical Synchronization of two identical chaotic systems by active control

4.1 Four-scroll's attractor

Consider the following Four-scroll attractor [4] :

$$\begin{cases} \dot{x}_1 = ax_1 + y_1 z_1 \\ \dot{y}_1 = -by_1 + x_1 z_1 \\ \dot{z}_1 = -cz_1 + x_1 y_1 \end{cases} \tag{16}$$

x, y and z are state variables, and a, b and c real parameters, consistent and positives. In the following numerical data : $a = 0,4,\ \ b = 12\ \ et\ \ c = 5$, the system four scroll (16) have the chaotic attractor of Figure 4.

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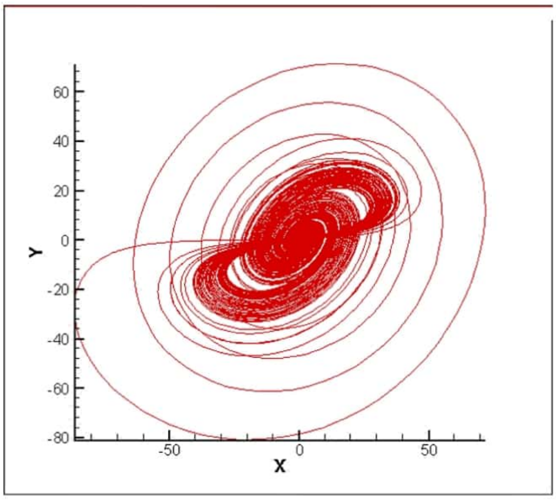

Figure 4: Four-scroll attractor where $a = 0.4$, $b = 12$ et $c = 5$

To observe the synchronization behavior for Four-scroll attractor, we have two identical Four-scroll attractor systems, master system with three state variables denoted by the subscript 1 and slave system having identical equations denoted by the subscript 2. The master and slave systems are defined as follows [9] :
The master system :

$$\begin{cases} \dot{x}_2 = ax_2 + y_2 z_2 + u_1 \\ \dot{y}_2 = -by_2 + x_2 z_2 + u_2 \\ \dot{z}_2 = -cz_2 + x_2 y_2 + u_3 \end{cases} \tag{17}$$

By substracting (16) from (17) we get:

$$\begin{cases} \dot{x}_3 = ax_3 - y_2 z_2 + y_1 z_1 + u_1 \\ \dot{y}_3 = -by_3 + x_2 z_2 - x_1 z_1 + u_2 \\ \dot{z}_3 = -cz_3 + x_2 y_2 - x_1 y_1 + u_3 \end{cases} \tag{18}$$

By suitable choice of u_1, u_2 and u_3 , the system (18) becomes :

$$\begin{cases} \dot{x}_3 = ax_2 + V_1 \\ \dot{y}_3 = -by_3 + V_2 \\ \dot{z}_3 = -cz_3 + V_3 \end{cases} \tag{19}$$

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Where :

$$\begin{cases} \dot{u}_1 = y_2 z_2 - y_1 z_1 + V_1 \\ \dot{u}_2 = -x_2 z_2 + x_1 z_1 + V_2 \\ \dot{u}_3 = -x_2 y_2 + x_1 y_1 + V_3 \end{cases} \tag{20}$$

There are many possible choices for controller V_1, V_2 and V_3. We choose :

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = A \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$$

Where A is a constant matrix. Let us choose a particular form of the matrix A that is given by :

$$\begin{pmatrix} -(a+1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore difference signal system becomes :

$$\begin{cases} \dot{x}_3 = a x_2 \\ \dot{y}_3 = -b y_3 \\ \dot{z}_3 = -c z_3 \end{cases} \tag{21}$$

4.2 Numerical simulation

We select the parameters for Four-scroll attractor systems: $a = 0.4, b = 12, c = 5$. To ensure the chaotic behavior shown in Fig.4. The initial values of coupled Four-scroll attractor system are taken as:

$$x_1(0) = 0.4, y_1(0) = 12, z_1(0) = 5$$

$$x_2(0) = -5.6, y_2(0) = 13.6, z_2(0) = -12.5$$

The results of the simulation of coupled Four-scroll attractor system with the active controller desactivated are shown in Fig.5.

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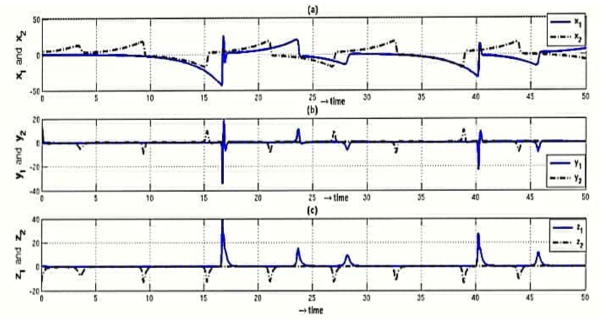

Figure 5: presentation of tow four scroll coupled attractors where the active control is desactivated.

Fig.6 presents the sequence of signals when active control activated and difference signals are shown in Fig.7. With the active controller connected into the circuit.

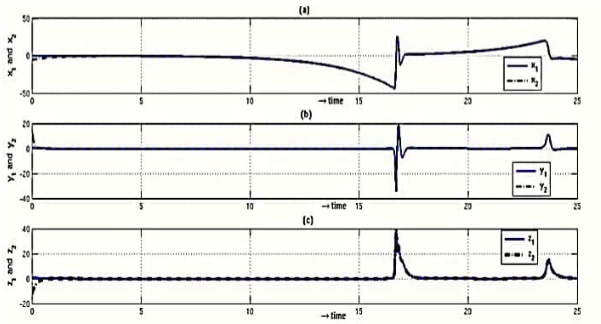

Figure 6: presentation of two four scroll coupled attractor where the active control is activated.

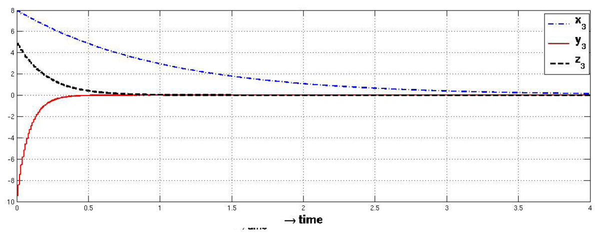

Figure 7: presentation of different signals x_3, y_3 et z_3 where the active control is activated.

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4.3 Results and Discussion

In this paper, we reviewed a comparative study between two types of unilateral synchronization, and we devoted this study to each of the two models, Four-scroll's model and chen's model as they are have the same behavior.

We noticed at the beginning that the first model, which was a synchronization study using a carroll and Pecora method, we have obtained a very good results, and we got synchronization after about 5 minutes.

As for the study of synchronization in the second method, which is the continuuous control method, it relied a lot on the choosing of the coefficient of synchronization , but despite that, we got a very good synchronization for this model as well, and that was after just 3 minutes had passed. In summary, the seconde method relies a lot on the synchronization coefficient, but in the end it gives very good results and faster than the first method, which was based mainly on the passage of time as well as the initial values, but it was easier as there is no synchronization coefficient only the initial values.

5 Conclusion

Through this study, we formed the idea that both methods of identical synchronization gives good results, but the second method was faster and more reliable in terms of the accuracy of the results.

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