International Journal of Applied Mathematics

Volume 38 No. 10s, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

On 3-variable (p, q)-Hermite polynomials

Mohammed Fadel [1] , Clemente Cesarano [2],
William Ramírez [*2,3], Pablo Buitrón [4]

[1] Department of Mathematics, University of Lahej,
Lahej - 73560, Yemen
e-mail: mohdfadel80@gmail.com

[2] Section of Mathematics, International Telematic University Uninettuno,
Corso Vittorio Emanuele II,39,
Roma - 00186Roma, Italy
e-mail:c.cesarano@uninettuno.it

[3] Departamento de Ciencias Naturales y Exactas,
Universidad de la Costa,
Barranquilla, - 00186, Colombia.
e-mail:wramirez4@cuc.edu.co

[4] Section of Mathematics,
International Telematic University Uninettuno,
Corso Vittorio Emanuele II,39,
Roma - 00186, Italy
e-mail: p.buitronespinoza@students.uninettunouniversity.net

Received: November 06, 2025

International Journal of Applied Mathematics

Volume 38 No. 10s, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Abstract

Throughout this work, we explore 3-variable (p, q)-Hermite polynomials and then gain their properties, including generating function, series definition, (p, q)-partial derivatives, (p, q)-partial differential equations and (p, q)-integral representations. Moreover, we investigate their pure recurrence relation, operational identity and addition formulas.

Math. Subject Classification: 34C05; 33C45; 34C07; 11B83.

Key Words and Phrases: (p, q)-Hermite polynomials; (p, q)-operational identity; (p, q)-recurrence relation.

1 Introduction

The mathematical framework of post-quantum calculus, often known as (p, q)-calculus, was created in the late 20^{th} century. The (p, q)-calculus is complemented for the quantum calculus, sometimes known as the q-calculus. Whenever $p = 1$, the (p, q)-calculus diminishes to the q-calculus. Recently, (p, q)-calculus is attracting attention among researchers in several domains of mathematics and physics. The (p, q)-analogues for multiple ordinary special functions, as well as polynomials like Beta, Gamma, Euler, Bernoulli, and other polynomials, have been explored and investigated (see [1, 2, 3, 6, 9, 10, 11, 15, 16, 17, 18]).

We explore various terms and symbols in (p, q)-calculus (see[3, 2, 18]).

Let $0 < q < p \leq 1$. The (p, q)-number $[\theta]_{p,q}$ corresponds to outlined below [3]:

$$[\theta]_{p,q} = \frac{p^\theta - q^\theta}{p - q}, \tag{1}$$

assuming every integer that is positive θ.

International Journal of Applied Mathematics
Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

The (p,q)-factorial appears as stated [3]:

$$[\theta]_{p,q}! = \prod_{\upsilon=1}^{\theta} [\upsilon]_{p,q}, \qquad \theta \geq 1, \qquad [0]_{p,q}! = 1.$$

The expression that follows represents the (p,q)-binomial coefficient [3, 18]:

$$\begin{bmatrix} \upsilon \\ \theta \end{bmatrix}_{p,q} = \frac{[\upsilon]_{p,q}!}{[\upsilon-\theta]_{p,q}![\theta]_{p,q}!}. \tag{2}$$

According to [13], each of the (p,q)-exponential expressions $e_{p,q}(x)$ and $E_{p,q}(x)$ are outlined below:

$$e_{p,q}(x) = \sum_{\theta=0}^{\infty} p^{\binom{\theta}{2}} \frac{x^{\theta}}{[\theta]_{p,q}!} \tag{3}$$

and

$$E_{p,q}(x) = \sum_{\theta=0}^{\infty} q^{\binom{\theta}{2}} \frac{x^{\theta}}{[\theta]_{p,q}!}. \tag{4}$$

The (p,q)-derivative of an expression f alongside regards to x, expressed as $D_{p,q,x}f(x)$ is stated as [2]:

$$D_{p,q,x}f(x) = \frac{f(px)-f(qx)}{(p-q)x}, \qquad x \neq 0,$$

that satisfy the next rules:

$$D_{p,q,x}(f(x)g(x)) = g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x). \tag{5}$$

In particular, we gain [2]

$$D_{p,q,x}x^{\theta} = [\theta]_{p,q}x^{\theta-1}, \tag{6}$$

$$D_{p,q,x}e_{p,q}(\mu x) = \mu e_{p,q}(\mu px) \tag{7}$$

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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

and

$$D_{p,q,x}E_{p,q}(\mu x) = \mu E_{p,q}(\mu q x).$$

Considering a function f, the value for the (p,q)-definite integral is established through

$$\int_0^a f(x)\, d_{p,q}x = (p-q)\, a \sum_{\theta=0}^{\infty} \frac{p^\theta}{q^{\theta+1}} f\left(\frac{p^\theta}{q^{\theta+1}} a\right). \tag{8}$$

We extended the concept shift operator $L_{\mu,x}$ [11] for each (p,q)-function of the three variables $f(x,y,z)$ is stated below:

$$L_{\mu,x} f(x,y,z) = f(\mu x, y, z). \tag{9}$$

$$L_{\mu^\tau,x} f(x,y,z) = L^{\tau}_{\mu,x} f(x,y,z) = f(\mu^\tau x, y, z), \qquad \tau \in \mathcal{Z},$$

$$L^{-1}_{\mu,x} f(\mu x, y, z) = f(x,y,z), \tag{10}$$

wherein the operator $L_{\mu,x}$ is reversed by $L^{-1}_{\mu,x}$. $L_{\mu,x} = I$, wherein I denotes the outcome for the identity operator $If(x,y,z) = f(x,y,z)$.

Hermite polynomials offer s traightforward a nd a daptable answers to boundary value problems, which have many uses in applied sciences including probability, numerical analysis and combinatorics, umbral calculus, the quantum harmonic oscillator as well as explore the statistical characteristics for chaotic light. Dattoli and his colleagues [7, 5] also discovered more uses for Hermite polynomials in quantum mechanics problem theory and optical wave transmission. Furthermore, Dattoli and his colleagues [8] examined and explored the features for the 3VHP $H_\theta(x,y,z)$.

The q-Hermite polynomials and their generalized, multi-dimensional and Higher forms have many uses in modeling quantum computing and mathematical physics [11, 12, 14, 15, 20]. Raza *et al.* [15] used a

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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

particular generating relation to create and categorize the 2-variable q and (p, q)-Hermite polynomials $H_{\theta,q}(x, y)$ and $H_{\theta,p,q}(x, y)$.

$$e_q(x\xi)e_q(y\xi^2) = \sum_{\theta=0}^{\infty} H_{\theta,q}(x, y)\frac{\xi^\theta}{[\theta]_q!}$$

and

$$e_{p,q}(x\xi)e_{p,q}(y\xi^2) = \sum_{\theta=0}^{\infty} H_{\theta,p,q}(x, y)\frac{\xi^\theta}{[\theta]_{p,q}!}, \tag{11}$$

with a series expression for $H_{\theta,p,q}(x, y)$ [15]:

$$H_{\theta,p,q}(x, y) = [\theta]_{p,q}!\sum_{\omega=0}^{[\theta/2]} \frac{p^{\binom{\theta-2\omega}{2}} x^{\theta-2k} p^{\binom{\omega}{2}} y^\omega}{[\theta-2\omega]_{p,q}![\omega]_{p,q}!}. \tag{12}$$

The operational identity of $H_{\theta,p,q}(x, y)$ is stated as [15]:

$$H_{\theta,p,q}(x, y) = e_{p,q}(yX^2_{p,q,x})p^{\binom{\theta}{2}}x^\theta, \tag{13}$$

where

$$X^\tau_{p,q,x} = p^{-\binom{\tau}{2}}\Big(L^{-1}_{p,x}\Big)^\tau\Big(D_{p,q,x}\Big)^\tau, \qquad \tau \geq 1 \tag{14}$$

and $L^{-1}_{p,x}$ is the inverse shift operator provided by equation (10) and the (p, q)-operator $X^2_{p,q,x} = p^{-1}L^{-2}_{p,x}D^2_{p,q,x}$.

The (p, q)-partial derivative for the (p, q)-exponential function $e_{p,q}(y\xi^2)$ alongside relation to ξ is specified via [15]:

$$D_{p,q,\xi}\ e_{p,q}(y\xi^2) = py\xi\ e_{p,q}(p^2y\xi^2) + qy\xi\ e_{p,q}(pqy\xi^2). \tag{15}$$

In 2024, Fadel *et al.* [11] defined the subsequent generating formula for the 3VqHP $H_{\theta,q}(x, y, z)$ [11]:

$$e_q(x\xi)\ e_q(y\xi^2)e_q(z\xi^3) = \sum_{\theta=0}^{\infty} H_{\theta,q}(x, y, z)\frac{\xi^\theta}{[\theta]_q!}. \tag{16}$$

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This led us to construct (p,q)-Hermite polynomials in three variables and study the corresponding formalism. In many branches of mathematics and science, such as mathematical physics, quantum-phase-space mechanics, wave propagation, and classical mechanics, these polynomials are essential. Researchers will focus especially on the 3-variable (p,q)-Hermite polynomials, which are the result of their (p,q)-analogue study. They have discovered new and fascinating possibilities that could expand the theory of (p,q)-special functions, which have attracted more attention. Based by Fadel's *et al.* investigation [11] into 3VqHP $H_{\theta,q}(x,y,z)$ and knowing that the 3VHP $H_{\theta}(x,y,z)$ have applications in numerous disciplines of mathematics and science. The written work is structured into four sections: Section 2 contains the 3V(p,q)HP $H_{\theta,p,q}(x,y,z)$, which are discussed at the current study based on our findings. A handful of its properties are investigated, including their generating function, series series expression, (p,q)-partial derivatives, (p,q)-partial differential equations and (p,q)-integral representations. Section 3 contains the operational identity, pure recurrence relation and addition formulas for the 3V(p,q)HP $H_{\theta,p,q}(x,y,z)$. Section 4 contains the conclusion. To summarize, many of the ideas and conclusions reported in this work are novel and distinct from commonly recognized findings in the literature.

2 Certain features for (p,q)-Hermite polynomials of three-variable

Throughout this section, we cover (p,q)-Hermite polynomials of three variables and explore their properties as generating function, series expression, (p,q)-partial derivatives, (p,q)-partial differential equations and (p,q)-integral representations.

We generate the (p,q)-Hermite polynomials for three-variable $H_{\theta,p,q}(x,y,z)$ having a specific generating formula employing formulas (11) and

International Journal of Applied Mathematics
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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

(16):

$$e_{p,q}(x\xi)\ e_{p,q}(y\xi^2)e_{p,q}(z\xi^3)=\sum_{\theta=0}^{\infty}H_{\theta,p,q}(x,y,z)\frac{\xi^\theta}{[\theta]_{p,q}!}. \tag{17}$$

Stretching the left part for formula (17) via formula (3), that we gain

$$\sum_{\theta=0}^{\infty}\ \sum_{\omega=0}^{\infty}\sum_{\tau=0}^{\infty}\frac{p^{\binom{\theta}{2}}x^\theta p^{\binom{\omega}{2}}y^\tau p^{\binom{\tau}{2}}z^\tau\ \xi^{\theta+2\omega+3\tau}}{[\theta]_{p,q}![\omega]_{p,q}![\tau]_{p,q}!}=\sum_{\theta=0}^{\infty}H_{\theta,p,q}(x,y,z)\frac{\xi^\theta}{[\theta]_{p,q}!},$$

whereby, afterwards, executing the next series of reorganization approaches ([4] see page 782, Eq. 1.6):

$$\sum_{\theta=0}^{\infty}\ \sum_{\omega=0}^{\infty}A(\omega,\theta)=\sum_{\theta=0}^{\infty}\ \sum_{\omega=0}^{[\theta/2]}A(\omega,\theta-2\omega), \tag{18}$$

we gain

$$\sum_{\theta=0}^{\infty}\ \sum_{\omega=0}^{[\theta/2]}\frac{p^{\binom{\theta-2\omega}{2}}x^{\theta-2\omega}p^{\binom{\omega}{2}}y^\omega\ \xi^\theta}{[\theta-2\omega]_{p,q}![\omega]_{p,q}!}\sum_{\tau=0}^{\infty}\frac{p^{\binom{\tau}{2}}z^\tau\ \xi^{3\tau}}{[\tau]_{p,q}!}$$
$$=\sum_{\theta=0}^{\infty}H_{\theta,p,q}(x,y,z)\frac{\xi^\theta}{[\theta]_{p,q}!}.$$

By employing the formula (12), that we acquire

$$\sum_{\theta=0}^{\infty}\ \sum_{\tau=0}^{\infty}\frac{p^{\binom{\tau}{2}}z^\tau H_{\theta,p,q}(x,y)\ \xi^{\theta+3\tau}}{[\tau]_{p,q}![\theta]_{p,q}!}=\sum_{\theta=0}^{\infty}H_{\theta,p,q}(x,y,z)\frac{\xi^\theta}{[\theta]_{p,q}!}.$$

This employs the next series rearrange approach (see [4] page 782, Eq. 1.1, for $m=3$):

$$\sum_{\theta=0}^{\infty}\ \sum_{\omega=0}^{\infty}A(\omega,\theta)=\sum_{\theta=0}^{\infty}\ \sum_{\omega=0}^{[\theta/3]}A(\omega,\theta-3\omega), \tag{19}$$

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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

gives

$$\sum_{\theta=0}^{\infty}\sum_{\tau=0}^{[\theta/3]}\frac{p^{\binom{\tau}{2}}z^{r}H_{\theta-3\tau,p,q}(x,y)\xi^{\theta}}{[\tau]_{p,q}![\theta-3\tau]_{p,q}!}=\sum_{\theta=0}^{\infty}H_{\theta,p,q}(x,y,z)\frac{\xi^{\theta}}{[\theta]_{p,q}!}.$$

Applying formula (12), we gain

$$\sum_{\theta=0}^{\infty}\sum_{\tau=0}^{[\theta/3]}\sum_{\omega=0}^{[(\theta-3\tau)/2]}\frac{p^{\binom{\tau}{2}}z^{\tau}p^{\binom{\theta-3\tau-2\omega}{2}}x^{\theta-3\tau-2\omega}p^{\binom{\omega}{2}}y^{\omega}\xi^{\theta}}{[\tau]_{p,q}![\omega]_{p,q}![\theta-3\tau]_{p,q}![\theta-3\tau-2\omega]_{p,q}!}$$
$$=\sum_{\theta=0}^{\infty}H_{\theta,p,q}(x,y,z)\frac{\xi^{\theta}}{[\theta]_{p,q}!}.$$

Through matching the associated values of ξ for each part, we can obtain the series that describes 3V(p,q)HP $H_{\theta,p,q}(x,y,z)$ as:

$$H_{\theta,p,q}(x,y,z)=[\theta]_{p,q}!\sum_{\tau=0}^{[\theta/3]}\frac{p^{\binom{\tau}{2}}z^{\tau}H_{\theta-3\tau,p,q}(x,y)}{[\tau]_{p,q}![\theta-3\tau]_{p,q}!},\tag{20}$$

or, correspondingly

$$H_{\theta,p,q}(x,y,z)=$$
$$[\theta]_{p,q}!\sum_{\tau=0}^{[\theta/3]}\sum_{\omega=0}^{[(\theta-3\tau)/2]}\frac{p^{\binom{\tau}{2}}z^{\tau}p^{\binom{\theta-3\tau-2\omega}{2}}x^{\theta-3\tau-2\omega}p^{\binom{\omega}{2}}y^{\omega}}{[\tau]_{p,q}![\omega]_{p,q}![\theta-3\tau]_{p,q}![\theta-3\tau-2\omega]_{p,q}!}.\tag{21}$$

The first few 3V(p,q)HP $H_{\theta,p,q}(x,y,z)$ are listed as:

$$\begin{aligned}
&H_{0,p,q}(x,y,z)=1\\
&H_{1,p,q}(x,y,z)=x\\
&H_{2,p,q}(x,y,z)=x^2+[2]_{p,q}\;y\\
&H_{3,p,q}(x,y,z)=x^3+[3]_{p,q}[2]_{p,q}\;xy+[3]_{p,q}[2]_{p,q}z\\
&H_{4,p,q}(x,y,z)=x^4+[4]_{p,q}[3]_{p,q}x^2y+[4]_{p,q}[3]_{p,q}y^2+[4]_{p,q}[3]_{p,q}[2]_{p,q}xz.
\end{aligned}$$

Received: November 06, 2025

International Journal of Applied Mathematics
Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Plus, formulae (20) or (21) make it clear that

$$H_{\theta,p,q}(ax, a^2y, a^3z) = a^\theta H_{\theta,p,q}(x, y, z). \tag{22}$$

Considering $x = z = 0$ and $x = y = 0$ into formula (17), it appears clear that

$$H_{\theta,p,q}(0, y, 0) = \frac{[\theta]_{p,q}!\left(p^{\binom{\theta}{2}}y^\theta\right)^{[\theta/2]}}{[\theta/2]_{p,q}!} \ and$$

$$H_{\theta,p,q}(0, 0, z) = \frac{[\theta]_{p,q}!\left(p^{\binom{\theta}{2}}z^\theta\right)^{[\theta/3]}}{[\theta/3]_{p,q}!}, \quad \theta = 0, 1, 2 \ldots$$

Furthermore, the next boundary conditions are produced when we enter $y = 0$, $z = 0$, and $z = 0$ through expression (17) once by once:

$$H_{\theta,p,q}(x, 0, 0) = p^{\binom{\theta}{2}}x^\theta \quad \text{and} \quad H_{\theta,p,q}(x, y, 0) = H_{\theta,p,q}(x, y). \tag{23}$$

Assuming $y = 0$ in formula (17), then we may gain the subsequent series:

$$H_{\theta,p,q}(x, z) = [\theta]_{p,q}! \sum_{\omega=0}^{[\theta/3]} \frac{p^{\binom{\omega}{2}}z^\omega p^{\binom{\theta-3\omega}{2}}x^{\theta-3\omega}}{[\theta-3\omega]_{p,q}![\omega]_{p,q}!}, \quad \theta = 0, 1, 2 \ldots. \tag{24}$$

Considering $x = [2]_{p,q}x$, $y = -1$, along with $z = 0$ within formula (17) yields [10]:

$$H_{\theta,p,q}\left([2]_{p,q}x, -1, 0\right) = H_{\theta,p,q}(x). \tag{25}$$

The next theorem is employed to show the (p, q)-partial derivatives for the 3V(p, q)HP $H_{\theta,p,q}(x, y, z)$:

Theorem 1. *The next (p, q)-partial derivatives with $H_{\theta,p,q}(x, y, z)$ remain valid:*

$$D_{p,q,x}H_{\theta,p,q}(x, y, z) = [\theta]_{p,q}H_{\theta-1,p,q}(px, y, z), \qquad \theta \geq 1, \tag{26}$$

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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$D_{p,q,y}H_{\theta,p,q}(x,y,z) = [\theta]_{p,q}[\theta-1]_{p,q}H_{\theta-2,p,q}(x,py,z),\ \theta \geq 2, \quad (27)$$

$$D_{p,q,z}H_{\theta,p,q}(x,y,z) = [\theta]_{p,q}[\theta-1]_{p,q}[\theta-2]_{p,q}H_{\theta-3,p,q}(x,y,pz),\ \ \theta \geq 3, \quad (28)$$

$$D^m_{p,q,x}H_{\theta,p,q}(x,y,z) = \frac{[\theta]_{p,q}!p^{\binom{m}{2}}}{[\theta-m]_{p,q}!}H_{\theta-m,p,q}(p^m x,y,z),\ 0 \leq m \leq \theta, \quad (29)$$

$$D^m_{p,q,y}H_{\theta,p,q}(x,y,z) = \frac{[\theta]_{p,q}!p^{\binom{m}{2}}}{[\theta-2m]_{p,q}!}H_{\theta-2m,p,q}(x,p^m y,z),\ 0 \leq m \leq \frac{\theta}{2} \quad (30)$$

and

$$D^m_{p,q,z}H_{\theta,p,q}(x,y,z) = \frac{[\theta]_{p,q}!p^{\binom{m}{2}}}{[\theta-3m]_{p,q}!}H_{\theta-3m,p,q}(x,y,p^m z),\ 0 \leq m \leq \frac{\theta}{3}. \quad (31)$$

Proof. Implementing the (p,q)-partial derivative for each component of (17) regard to x employing formula (7), we derive

$$\xi e_{p,q}(px\xi)e_{p,q}(y\xi^2)e_{p,q}(z\xi^3) = \sum_{\theta=0}^{\infty} D_{p,q,x}H_{\theta,p,q}(x,y,z)\frac{\xi^\theta}{[\theta]_{p,q}!}. \quad (32)$$

Utilizing formula (17) into the left part of the formula (32), gains us

$$\sum_{\theta=1}^{\infty} H_{\theta-1,p,q}(px,y,z)\frac{\xi^\theta}{[\theta-1]_{p,q}!} = \sum_{\theta=0}^{\infty} D_{p,q,x}H_{\theta,p,q}(x,y,z)\frac{\xi^\theta}{[\theta]_{p,q}!}.$$

When the values of ξ from every component of the aforementioned formula are matched, we reach assertion (26).

We determine the (p,q)-partial derivative at every component of the formula (17) according to the values of y and z, subsequently duplicate the processes employed in the formula's proof (26) to yield statements (27) and (28).

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Volume 38 No. 10s, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Applying the methodology from getting equation (26), we as a species determine the 2^{nd}-order (p,q)-partial derivative over each component of expression (17) with regard to x. Subsequently, we use expression (7) to calculate

$$D^2_{p,q,x}H_{\theta,p,q}(x,y,z) = [\theta]_{p,q}[\theta-1]_{p,q}pH_{\theta-2,p,q}(p^2x,y,z),\ \theta \geq 2. \quad (33)$$

In a comparable vein, considering the m^{th} degree (p,q)-partial derivative for every component of equation (17) in regard for x, via equation (7) and performing the previous stages of demonstrating equation (26), we gain assertion (29).

In a comparable manner, if we apply the m^{th} order (p,q)-partial derivatives for every component of the equation (17) alongside regard to y and z, and subsequently perform precisely the same process, we have the assertions (30) and (31). Theorem 1 has been fully proved. □

As before, employing equations (27), (28), (33) and $m = 2,3$ in equation (29), we derive the below (p,q)-partial differential equations of $H_{\theta,p,q}(x,y,z)$:

Corollary 2. *The statement below summarizes the (p,q)-partial differential equations of $H_{\theta,p,q}(x,y,z)$:*

$$pD^2_{p,q,x}H_{\theta,p,q}(x,p^3y,p^4z) = D_{p,q,y}H_{\theta,p,q}(p^2x,p^2y,p^4z) \quad (34)$$

and

$$pD^3_{p,q,x}H_{\theta,p,q}(x,p^4y,p^5z) = D_{p,q,z}H_{\theta,p,q}(p^3x,p^4y,p^4z). \quad (35)$$

Example 2.1. Applying Formula (34) and (35), we have the following:

$$1/3D^2_{1/3,1/5,x}H_{2,1/3,1/5}\Big(x,1/27y,1/81z\Big) - D_{1/3,1/5,y}H_{2,1/3,1/5}\Big(1/9x,1/27y,1/81z\Big) = 0$$

Received: November 06, 2025

International Journal of Applied Mathematics
Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

and

$$1/2D^3_{1/2,1/4,x}H_{7,1/2,1/4}\Big(x,1/16y,1/32z\Big) - D_{1/2,1/4,z}H_{7,1/2,1/4}\Big(1/16x,1/16y,1/16z\Big)=0.$$

We research the (p,q)-integral expressions of $H_{\theta,p,q}(x,y,z)$ as follows:

Theorem 3. *The (p,q)-integral representations for $H_{\theta,p,q}(x,y,z)$*

$$\int_a^b H_{\theta,p,q}(x,y,z)\,d_{p,q}x = p\frac{H_{\theta+1,p,q}\left(\frac{b}{p},y,z\right) - H_{\theta+1,p,q}\left(\frac{a}{p},y,z\right)}{[\theta+1]_{p,q}}, \tag{36}$$

$$\int_a^b H_{\theta,p,q}(x,y,z)\,d_{p,q}y = p\frac{H_{\theta+2,p,q}\left(x,\frac{b}{p},z\right) - H_{\theta+2,p,q}\left(x,\frac{a}{p},z\right)}{[\theta+1]_{p,q}[\theta+2]_{p,q}} \tag{37}$$

and

$$\int_a^b H_{\theta,p,q}(x,y,z)\,d_{p,q}z = p\frac{H_{\theta+3,p,q}\left(x,y,\frac{b}{p}\right) - H_{\theta+3,p,q}\left(x,y,\frac{a}{p}\right)}{[\theta+1]_{p,q}[\theta+2]_{p,q}[\theta+3]_{p,q}}, \tag{38}$$

hold for θ being a positive integer.

Proof. Applying (26), we gain

$$\begin{aligned}\int_a^b H_{\theta,p,q}(x,y,z)\,d_{p,q}x &= \frac{p}{[\theta+1]_{p,q}} \int_a^b D_{p,q;x}\left[H_{\theta+1,p,q}\left(\frac{x}{p},y,z\right)\right]d_{p,q}x \\ &= p\frac{H_{\theta+1,p,q}\left(\frac{b}{p},y,z\right) - H_{\theta+1,p,q}\left(\frac{a}{p},y,z\right)}{[\theta+1]_{p,q}},\end{aligned}$$

which completes the proof (36).

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Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Moreover, applying (27), we obtain

$$\begin{aligned}\int_a^b H_{\theta,p,q}(x,y,z)\, d_{p,q}y &= \frac{p}{[\theta+1]_{p,q}[\theta+2]_{p,q}} \\ &\cdot \int_a^b D_{p,q;y}\left[H_{\theta+2,p,q}\left(x,\frac{y}{p},z\right)\right] d_{p,q}y \\ &= p\frac{H_{\theta+2,p,q}\left(x,\frac{b}{p},z\right) - H_{\theta+1,p,q}\left(x,\frac{a}{p},z\right)}{[\theta+1]_{p,q}[\theta+2]_{p,q}},\end{aligned}$$

which completes the proofs (37).

Furthermore, applying (28), we gain

$$\begin{aligned}\int_a^b H_{\theta,p,q}(x,y,z)\, d_{p,q}x &= \frac{p}{[\theta+1]_{p,q}[\theta+2]_{p,q}[\theta+3]_{p,q}} \\ &\cdot \int_a^b D_{p,q;z}\left[H_{\theta+3,p,q}\left(x,y,\frac{z}{p}\right)\right] d_{p,q}z \\ &= p\frac{H_{\theta+3,p,q}\left(x,y,\frac{b}{p}\right) - H_{\theta+3,p,q}\left(x,y,\frac{a}{p}\right)}{[\theta+1]_{p,q}[\theta+2]_{p,q}[\theta+3]_{p,q}},\end{aligned}$$

which completes the proof (38). □

Corollary 4. *The triple (p,q)-integration of $3V(p,q)HP$ $H_{n,p,q}(x,y,z)$ is as follows:*

$$\begin{aligned}&\int_{x=a}^b \int_{y=c}^d \int_{z=e}^f H_{\theta,q}(x,y,z)\ d_qx\ d_qy\ d_qz = P^3 \frac{[\theta]_q!}{[\theta+6]_q!} \\ &\Bigg(H_{\theta+6,p,q}\Big(\frac{b}{p},\frac{d}{p},\frac{f}{p}\Big) + H_{\theta+6,p,q}\Big(\frac{a}{p},\frac{c}{p},\frac{f}{p}\Big) + H_{\theta+6,p,q}\Big(\frac{b}{p},\frac{c}{p},\frac{e}{p}\Big) \\ &+H_{\theta+6,p,q}\Big(\frac{a}{p},\frac{d}{p},\frac{e}{p}\Big) - H_{\theta+6,p,q}\Big(\frac{b}{p},\frac{c}{p},\frac{f}{p}\Big) - H_{\theta+6,p,q}\Big(\frac{a}{p},\frac{d}{p},\frac{f}{p}\Big) \\ &-H_{n+6,p,q}\Big(\frac{b}{p},\frac{d}{p},\frac{e}{p}\Big) - H_{\theta+6,p,q}\Big(\frac{a}{p},\frac{c}{p},\frac{e}{p}\Big)\Bigg). \quad (39)\end{aligned}$$

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Proof. Integrating (26), then applying equation (36), we gain

$$\int_e^f \int_c^d \left(\int_a^b H_{\theta,p,q}(x,y,z) d_{p,q}x \right) d_{p,q}y d_{p,q}z = \frac{p}{[\theta+1]_{p,q}} \int_e^f \left(\int_c^d H_{\theta+1,q}\Big(\frac{b}{p}, y, z\Big) d_{p,q}y - \int_c^d H_{\theta+1,q}\Big(\frac{a}{p}, y, z\Big) d_{p,q}y \right) d_{p,q}z.$$

Applying equation (37), we obtain

$$\int_e^f \left(\int_c^d \left(\int_a^b H_{\theta,p,q}(x,y,z) d_{p,q}x \right) d_{p,q}y \right) d_{p,q}z$$
$$= \frac{P^2}{[\theta+1]_{p,q}[\theta+2]_{p,q}[\theta+3]_{p,q}} \int_e^f \left(H_{\theta+3,p,q}\Big(\frac{b}{p}, \frac{d}{p}, z\Big) + H_{\theta+3,p,q}\Big(\frac{a}{p}, \frac{c}{p}, z\Big) - H_{\theta+3,p,q}\Big(\frac{b}{p}, \frac{c}{p}, z\Big) - H_{\theta+3,p,q}\Big(\frac{a}{p}, \frac{d}{p}, z\Big) \right) d_{p,q}z.$$

Similarly, integrating equation (38) with respect to z, we gain assertion (39). $\square$

3 Pure recurrence relation, operational identity and addition formulas

In this section, we realize that the use of operational identities has simplified the study of special polynomials, that we can gain the operational rule for the 3V(p,q)HP $H_{\theta,p,q}(x,y,z)$. We also constructed the pure recurrence relation and addition formulas for $H_{\theta,p,q}(x,y,z)$.

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We recall the shift-differential operator $X_{p,q,x}$ satisfied the below property [15]:

$$X^{\tau}_{p,q,x} p^{\binom{\theta}{2}} x^{\theta} = \frac{[\theta]_{p,q}!}{[\theta-\tau]_{p,q}!} p^{\binom{\theta-\tau}{2}} x^{\theta-\tau}. \tag{40}$$

Theorem 5. *The subsequent operational identity is achieved through the 3V(p, q)HP $H_{\theta,p,q}(x, y, z)$:*

$$H_{\theta,p,q}(x, y, z) = e_{p,q}(zX^{3}_{p,q,x}) e_{p,q}(yX^{2}_{p,q,x}) p^{\binom{\theta}{2}} x^{\theta}, \tag{41}$$

wherein the 2^{nd} (p, q)-operator $X^{2}_{p,q,x} = p^{-1} L^{-2}_{p,x} D^{2}_{p,q,x}$ and 3^{th} (p, q)-operator $X^{3}_{p,q,x} = p^{-3} L^{-3}_{p,x} D^{3}_{p,q,x}$.

Proof. From formula (40), we gain

$$D^{3\tau}_{p,q,x}\ H_{\theta,p,q}(x, y) = p^{3} \frac{[\theta]_{p,q}!}{[\theta-3\tau]_{p,q}!} H_{\theta-3\tau,p,q}(p^{3}x, y), \tag{42}$$

or correspondingly

$$D^{3\tau}_{p,q,x}\ H_{\theta,p,q}(x, y) = p^{3} L^{3}_{p} \frac{[\theta]_{p,q}!}{[\theta-3\tau]_{p,q}!} H_{\theta-3\tau,p,q}(x, y). \tag{43}$$

Applying equation (43) in the right portion of formula (20), we gain

$$H_{\theta,p,q}(x, y, z) = \sum_{\tau=0}^{[\theta/3]} \frac{p^{-3} L^{-3}_{p} p^{\binom{\tau}{2}} z^{\tau} D^{3\tau}_{p,q,x} H_{\theta,p,q}(x, y)}{[\tau]_{p,q}!}, \tag{44}$$

or, correspondingly

$$H_{\theta,p,q}(x, y, z) = \sum_{\tau=0}^{[\theta/3]} \frac{p^{\binom{\tau}{2}} \left(zX^{3}_{p,q,x}\right)^{\tau} H_{\theta,p,q}(x, y)}{[\tau]_{p,q}!}.$$

Using formula (3) in the right portion of the equation mentioned above yields

$$H_{\theta,p,q}(x, y, z) = e_{p,q}(zX^{3}_{p,q,x}) H_{\theta,p,q}(x, y).$$

We obtain the assertion (41) by employing the equation (13). □

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Example 3.1. For $\theta = 5, 7$ into equation (41), we have the next operational identities of $H_{5,p,q}(x, y, z)$ and $H_{7,p,q}(x, y, z)$:

$$H_{5,p,q}(x,y,z) = e_{p,q}(zX^3_{p,q,x})e_{p,q}(yX^2_{p,q,x})p^{10}x^5,$$

and

$$H_{7,p,q}(x,y,z) = e_{p,q}(zX^3_{p,q,x})e_{p,q}(yX^2_{p,q,x})p^{21}x^7,$$

To accomplish the fundamental recurrence relation for the 3V(p, q)HP $H_{\theta,p,q}(x, y, z)$, it is necessary to demonstrate the next specific lemma:

Lemma 3.1. If $D_{p,q,\xi}$ corresponds to the (p, q)-partial derivative relating to ξ, subsequently

$$D_{p,q,\xi}\, e_{p,q}(z\xi^3) = p^2z\xi^2\, e_{p,q}(p^3z\xi^3)+pqz\xi^2\, e_{p,q}(p^2qz\xi^3)+q^2z\xi^2\, e_{p,q}(pq^2z\xi^3). \tag{45}$$

Proof. Applying formulas (3) and (6), we gain

$$D_{p,q,\xi}\ e_{p,q}(z\xi^3) = \sum_{\theta=0}^{\infty} \frac{p^{\binom{\theta}{2}} z^\theta [3\theta]_{p,q}\xi^{3-1}}{[\theta]_{p,q}!}.$$

Since, from equation (1), we have $\frac{[3\theta]_{p,q}}{[\theta]_{p,q}}$ $=p^{2\theta} + p^\theta q^\theta + q^{2\theta}$, therefore the right component of the preceding expression generates

$$D_{p,q,\xi}\ e_{p,q}(z\xi^3) = \sum_{\theta=1}^{\infty} \frac{p^{\binom{\theta}{2}} z^\theta (p^{2\theta} + p^\theta q^\theta + q^{2n})\xi^{3\theta-1}}{[\theta-1]_{p,q}!}.$$

Streamlining the right part yields

$$D_{p,q,\xi}\ e_{p,q}(z\xi^3) = \sum_{\theta=0}^{\infty} \frac{p^{\binom{\theta+1}{2}} (p^2z)^{\theta+1}\xi^{3\theta+2}}{[\theta]_{p,q}!} + \sum_{\theta=0}^{\infty} \frac{p^{\binom{\theta+1}{2}} (pqz)^{\theta+1}\xi^{3\theta+2}}{[\theta]_{p,q}!} + \sum_{\theta=0}^{\infty} \frac{p^{\binom{\theta+1}{2}} (q^2z)^{\theta+1}\xi^{3\theta+2}}{[\theta]_{p,q}!}.$$

Received: November 06, 2025

International Journal of Applied Mathematics
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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Utilizing the comprehension that $\binom{\theta}{2} + \theta = \binom{\theta+1}{2}$ within the right portion in the previous expression, we produce

$$D_{q,\xi}\ e_q(z\xi^3) = p^2 z\xi^2 \sum_{\theta=0}^{\infty} \frac{p^{\binom{\theta}{2}}(p^3 z\xi^3)^\theta}{[\theta]_q!} + pqz\xi^2 \sum_{\theta=0}^{\infty} \frac{p^{\binom{\theta}{2}}(p^2 qz\xi^3)^\theta}{[\theta]_q!} + q^2 z\xi^2 \sum_{\theta=0}^{\infty} \frac{p^{\binom{\theta}{2}}(pq^2 z\xi^3)^\theta}{[\theta]_q!}. \quad (46)$$

Employing formula (3) over the right component of the previously mentioned formula produces the statement (45). □

Listed below theorem demonstrates a pure recurrence relation of the $H_{\theta,p,q}(x,y,z)$:

Theorem 6. *The subsequent pure recurrence connection to 3V(p,q)HP $H_{\theta,p,q}(x,y,z)$ over $\theta \geq 2$ might be expressed as:*

$$\begin{aligned} H_{\theta+1,p,q}(x,y,z) &= xH_{\theta,p,q}(px,py,pz) + py[\theta]_{p,q}\Big(pH_{\theta-1,p,q}(qx,p^2y,pz) \\ &\quad + qH_{\theta-1,p,q}(qx,pqy,pz)\Big) + pz[\theta]_{p,q}[\theta-1]_{p,q} \\ &\quad \Big(pH_{\theta-2,p,q}(qx,qy,p^3z) + qH_{\theta-2,p,q}(qx,qy,p^2qz) \\ &\quad + q^2p^{-1}H_{\theta-2,p,q}(qx,qy,pq^2z)\Big)\ (\theta \geq 2). \end{aligned} \quad (47)$$

Proof. Considering the q-derivative of each of the part of formula (17) via regard to ξ and the formula (5) for (p,q)-differentiation, we as a species can easily calculate

$$\begin{aligned} \sum_{\theta=0}^{\infty} D_{p,q,\xi} H_{\theta,p,q}(x,y,z)\frac{\xi^\theta}{[\theta]_{p,q}\ \ !} &= \Big[D_{p,q,\xi}e_{p,q}(x\xi)e_{p,q}(py\xi^2) \\ &\quad + e_{p,q}(qx\xi)D_{p,q,\xi}(\ e_{p,q}(y\xi^2)\Big]e_{p,q}(pz\xi^3) \\ &\quad + \Big[e_{p,q}(qx\xi)e_{p,q}(qy\xi^2)\Big]D_{p,q,\xi}e_{p,q}(z\xi^3). \end{aligned}$$

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Employing formula (6) into the left component of the aforementioned formula and formulas (15) and (45) in the right part, correspondingly, we as a species acquire

$$\begin{aligned}\sum_{\theta=1}^{\infty} H_{\theta,p,q}(x,y,z)\frac{\xi^{\theta-1}}{[\theta-1]_{p,q}\ !} &= xe_{p,q}(px\xi)e_{p,q}(py\xi^2)e_{p,q}(pz\xi^3)\\ &+ py\xi e_{p,q}(qx\xi)\ e_{p,q}(p^2y\xi^2)e_q(pz\xi^3)\\ &+ qy\xi e_{p,q}(qx\xi)e_{p,q}(pqy\xi^2)e_{p,q}(pz\xi^3)\\ &+ p^2z\xi^2 e_{p,q}(qx\xi)e_{p,q}(qy\xi^2)e_{p,q}(p^3z\xi^3)\\ &+ pqz\xi^2\ e_{p,q}(qx\xi)e_{p,q}(qy\xi^2)e_{p,q}(p^2qz\xi^3)\\ &+ q^2z\xi^2\ e_{p,q}(qx\xi)e_{p,q}(qy\xi^2)e_{p,q}(pq^2z\xi^3).\end{aligned} \tag{48}$$

Finally, we may deduce our claim (47) by applying the formula (17) to the right component of (48) and matching the two equivalent powers of ξ in the two portions of the final formula. Theorem 6 is currently fully proved. □

Example 3.2. For $\theta = 5, 7$, equation (47), we have the next operational representations of $H_{5,p,q}(x,y,z)$ and $H_{7,p,q}(x,y,z)$:

$$\begin{aligned}&H_{6,p,q}(x,y,z) - xH_{5,p,q}(px,py,pz) - py[5]_{p,q}\Big(pH_{4,p,q}(qx,p^2y,pz)\\ &\quad + qH_{4,p,q}(qx,pqy,pz)\Big) - pz[5]_{p,q}[4]_{p,q}\Big(pH_{3,p,q}(qx,qy,p^3z)\\ &\quad\quad + qH_{3,p,q}(qx,qy,p^2qz) + q^2p^{-1}H_{3,p,q}(qx,qy,pq^2z)\Big) = 0\end{aligned}$$

and

$$\begin{aligned}&H_{8,p,q}(x,y,z) - xH_{7,p,q}(px,py,pz) - py[7]_{p,q}\Big(pH_{6,p,q}(qx,p^2y,pz)\\ &\quad + qH_{6,p,q}(qx,pqy,pz)\Big) - pz[7]_{p,q}[6]_{p,q}\Big(pH_{5,p,q}(qx,qy,p^3z)\\ &\quad\quad + qH_{5,p,q}(qx,qy,p^2qz) + q^2p^{-1}H_{5,p,q}(qx,qy,pq^2z)\Big) = 0.\end{aligned}$$

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Definition 3.1. We construct the next addition formulas through replacing x, y and z by $x_1 \oplus x_2$, $y_1 \oplus_{p,q} y_2$ and $z_1 \oplus_{p,q} z_2$ respectively in the generating functions of (p,q)-Hermite polynomials of three variables $H_{n,p,q}(x,y,z)$ at equation (17), we gain

$$e_{p,q}(x_1 t)\ e_{p,q}(x_2 t) e_{p,q}(y t^2) e_{p,q}(z t^3) = \sum_{n=0}^{\infty} H_{n,p,q}(x_1 \oplus_{p,q} x_2, y, z) \frac{t^n}{[n]_{p,q}!}, \tag{49}$$

$$e_{p,q}(xt)\ e_{p,q}(y_1 t) e_{p,q}(y_2 t^2) e_{p,q}(z t^3) = \sum_{n=0}^{\infty} H_{n,p,q}(x, y_1 \oplus_{p,q} y_2, z) \frac{t^n}{[n]_{p,q}\ !} \tag{50}$$

and

$$e_{p,q}(xt)\ e_{p,q}(y t^2) e_{p,q}(z_1 t^3) e_{p,q}(z_2 t^3) = \sum_{n=0}^{\infty} H_{n,p,q}(x, y, z_1 \oplus_{p,q} z_2) \frac{t^n}{[n]_{p,q}!}, \tag{51}$$

where (cf. [19])

$$e_{p,q}\left(x_1 \oplus_{p,q} x_2\right) = e_{p,q}\left(x_1\right) e_{p,q}\left(x_2\right). \tag{52}$$

We now establish the series expressions for $H_{n,p,q}(x_1 \oplus_{p,q} x_2, y, z)$, $H_{n,p,q}(x, y_1 \oplus_{p,q} y_2, z)$ and $H_{n,p,q}(x, y, z_1 \oplus_{p,q} z_2)$ as follows:

Theorem 7. *The below series expressions for $H_{n,p,q}(x_1 \oplus_{p,q} x_2, y, z)$, $H_{n,p,q}(x, y_1 \oplus_{p,q} y_2, z)$ and $H_{n,p,q}(x, y, z_1 \oplus_{p,q} z_2)$ hold true*

$$H_{\theta,p,q}\left(x_1 \oplus_{p,q} x_2, y, z\right) = \sum_{\omega=0}^{\theta}\ H_{\theta-\omega,p,q}\left(x_1, y, z\right) p^{\binom{\omega}{2}} \left(x_2\right)^{\omega} \begin{bmatrix} \theta \\ \omega \end{bmatrix}_{p,q}, \tag{53}$$

$$H_{\theta,p,q}\left(x, y_1 \oplus_{p,q} y_2, z\right) = [\theta]_{p,q}! \sum_{\omega=0}^{[\theta/2]}\ \frac{H_{\theta-2\omega,p,q}\left(x, y_1, z\right)}{[\omega]_{p,q}! [\theta-2\omega]_{p,q}!} p^{\binom{\omega}{2}} \left(y_2\right)^{\omega} \tag{54}$$

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and

$$H_{\theta,p,q}\left(x,y,z_{1}\oplus_{p,q}z_{2}\right)=[\theta]_{p,q}!\sum_{\omega=0}^{[\theta/3]}\frac{H_{\theta-3\omega,p,q}\left(x,y,z_{1}\right)}{[\omega]_{p,q}![\theta-3\omega]_{p,q}!}p^{\binom{\omega}{2}}\left(z_{2}\right)^{\omega}. \tag{55}$$

Proof. It is readily seen from (17) that

$$\sum_{\theta=0}^{\infty}H_{\theta,p,q}\left(x_{1}\oplus_{p,q}x_{2},y,z\right)\frac{\zeta^{\theta}}{[\theta]_{p,q}!}$$

$$=e_{p,q}\left(\left(x_{1}+x_{2}\right)\zeta\right)\ e_{p,q}(y\xi^{2})e_{p,q}(z\xi^{3})$$

$$=e_{p,q}\left(x_{1}\zeta\right)e_{p,q}\left(x_{2}\zeta\right)\ e_{p,q}(y\xi^{2})e_{p,q}(z\xi^{3})$$

$$=\sum_{\theta=0}^{\infty}p^{\binom{\theta}{2}}\left(x_{2}\right)^{\theta}\frac{\zeta^{\theta}}{[\theta]_{p,q}!}\sum_{\theta=0}^{\infty}\ H_{\theta,p,q}\left(x_{1},y,z\right)\frac{\zeta^{\theta}}{[\theta]_{p,q}!}$$

$$=\sum_{\theta=0}^{\infty}\sum_{\omega=0}^{\theta}\ H_{\theta-\omega,p,q}\left(x_{1},y,z\right)p^{\binom{\omega}{2}}\begin{bmatrix}\theta\\ \omega\end{bmatrix}_{p,q}\left(x_{2}\right)^{\omega}\frac{\zeta^{\theta}}{[\theta]_{p,q}!},$$

which gives the alleged result (53).

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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Moreover, it is readily seen from (17) that

$$\sum_{\theta=0}^{\infty} H_{\theta,p,q}\left(x, y_1 \oplus_{p,q} y_2, z\right) \frac{\zeta^{\theta}}{[\theta]_{p,q}!}$$

$$= e_{p,q}(x\xi)\ e_{p,q}\left(\left(y_1 + y_2\right)\zeta^2\right) e_{p,q}(z\xi^3)$$

$$= e_{p,q}\left(x\zeta\right) e_{p,q}\left(y_1\zeta^2\right)\ e_{p,q}(y_2\xi^2) e_{p,q}(z\xi^3)$$

$$= \sum_{\theta=0}^{\infty} p^{\binom{\theta}{2}}\left(y_2\right)^{\theta} \frac{\zeta^{2\theta}}{[\theta]_{p,q}!} \sum_{\theta=0}^{\infty}\ H_{\theta,p,q}\left(x, y_1, z\right) \frac{\zeta^{\theta}}{[\theta]_{p,q}!}$$

$$= \sum_{\theta=0}^{\infty}\left([\theta]_{p,q}! \sum_{\omega=0}^{[\theta/2]}\ \frac{H_{\theta-2\omega,p,q}\left(x, y_1, z\right)}{[\omega]_{p,q}![\theta-2\omega]_{p,q}!} p^{\binom{\omega}{2}}\left(y_2\right)^{\omega}\right) \frac{\zeta^{\theta}}{[\theta]_{p,q}!},$$

which gives the alleged result (54).

Furthermore, it is readily seen from (17) that

$$\sum_{\theta=0}^{\infty} H_{\theta,p,q}\left(x, y, z_1 \oplus_{p,q} z_2\right) \frac{\zeta^{\theta}}{[\theta]_{p,q}!}$$

$$= e_{p,q}(x\xi)\ e_{p,q}\left(y\zeta^2\right) e_{p,q}(\left(z_1 + z_2\right)\xi^3)$$

$$= e_{p,q}\left(x\zeta\right) e_{p,q}\left(y\zeta^2\right)\ e_{p,q}(z_1\xi^3) e_{p,q}(z_2\xi^3)$$

$$= \sum_{\theta=0}^{\infty} p^{\binom{\theta}{2}}\left(z_2\right)^{\theta} \frac{\zeta^{3\theta}}{[\theta]_{p,q}!} \sum_{\theta=0}^{\infty}\ H_{\theta,p,q}\left(x, y, z_1\right) \frac{\zeta^{\theta}}{[\theta]_{p,q}!}$$

$$= \sum_{\theta=0}^{\infty}\left([\theta]_{p,q}! \sum_{\omega=0}^{[\theta/3]}\ \frac{H_{\theta-3\omega,p,q}\left(x, y, z_1\right)}{[\omega]_{p,q}![\theta-3\omega]_{p,q}!} p^{\binom{\omega}{2}}\left(z_2\right)^{\omega}\right) \frac{\zeta^{\theta}}{[\theta]_{p,q}!},$$

Received: November 06, 2025

Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

which gives the alleged result (55). □

Following the same procedure for proof as Theorem 7 and using formulas (11), (17) and (52), we deduce the following results:

Corollary 8. *For any positive integer θ, we gain the following relations:*

$$H_{\theta,p,q}\left(x_1 \oplus_{p,q} x_2, y_1 \oplus_{p,q} y_2, z\right) = \sum_{\omega=0}^{\theta} \begin{bmatrix} \theta \\ \omega \end{bmatrix}_{p,q} H_{\omega,p,q}(x_1, y_1, z_1) H_{\theta-\omega,p,q}(x_2, y_2) \tag{56}$$

and

$$H_{\theta,p,q}\left(x_1 \oplus_{p,q} x_2, y_1 \oplus_{p,q} y_2, z_1 \oplus_{p,q} z_2\right) = \sum_{\omega=0}^{\theta} \begin{bmatrix} \theta \\ \omega \end{bmatrix}_{p,q} H_{\omega,p,q}(x_1, y_1, z_1) H_{\theta-\omega,p,q}(x_2, y_2, z_2). \tag{57}$$

4 Conclusions

Properties associated with conventional three-variable Hermite polynomials are extensively employed in investigating charged-beam transfer in standard mechanics, as well as for estimating quantum-phase-space mechanics and investigating their properties using umbral techniques. This explanation presents additional aspects of the 3V(p,q)HP $H_{\theta,p,q}(x,y,z)$ and their associated formalism. We thoroughly examine generating functions, series formulas, (p,q)-partial derivatives and (p,q)-partial differential equations and (p,q)-integral representations. We also investigate their pure recurrence relation, operational identity and addition formulas. The mathematical examination of these polynomials has opened up new opportunities for expanding the theory of (p,q)-special functions, their families, as well as hybrid forms from diverse classes, opening up opportunities for future research.

Received: November 06, 2025

International Journal of Applied Mathematics
Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

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