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Some New Existence and Stability Results for Nonlocal Generalized Hilfer Fractional Volterra-Fredholm System

Riyam Bassim Abdulmaged[1], Ihab Hadi Jumaah[2], Alan Jalal Abdulqader[3]

[1,2,3] Department of Mathematics, College of Education
Mustansiriyah University
Bagdad - 10052, IRAQ
e-mail: riyam.basim2023@uomustansiriyah.edu.iq
e-mail: ihabhadi5@uomustansiriyah.edu.iq
e-mail: alanjalal515@uomustansiriyah.edu.iq

Abstract

This study focuses on the Hadamard-Hilfer Volterra-Fredholm system with non-local conditions. The existence of solutions is demonstrated by applying Krasnoselskii's fixed point theorem (FPT). When the Banach contraction mapping concept is applied, the solutions are distinct and well-defined under the correct conditions. The stability of solutions is also examined using the Ulam-Hyers (U-H) and Ulam-Hyers-Rassias (U-H-R) stability concepts.

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1 Introduction

Since fractional differential equations are utilized extensively in many scientific fields, including physics, engineering, and mathematics, they are now considered to be among the most crucial topics for mathematicians (see [1, 2, 3, 4]). At the same time, numerous researchers provided arbitrary order explanations of significant theoretical findings in differential equations (DEs) and fractional calculus; for clarification, one might refer to monographs by Kilbas et al. [5] and Samko et al. [6]. In fact, the searching for the existence of solutions to DEs with various fractional derivatives (FD) using many FPTs has become the focus of interest for many authors in recent time, see([7, 8, 9, 10]). Furthermore, many researchers are attracted to study the stability of solutions because it is check the behavior of solutions over the time. Also, many stability styles such as U-H and U-H-R have been developed and get interesting by audience, see ([11, 12, 13, 14]). On the other hand, the attractive results for various classes of fractional integro-differential equations (FIDEs) are considered in ([15, 16, 17, 18, 19, 20]). The existence of solutions for the nonlinear FIDEs studied by Ahmad et al. [21].

$$^{C}D^{q}E(\vartheta) = \phi(\vartheta, E(\vartheta), (\theta E)(\vartheta), (\psi E)(\vartheta)), \quad \vartheta \in [0,1], \quad q \in (1,2],$$

with the boundary condition

$$\begin{aligned} E'(0) + aE(M_1) = 0 \\ bE'(1) + E(M_2) = 0 \end{aligned}$$

where $0 < M_1 \leq M_2 < 1$, and $^{C}D^{q}_{1+}$, is known as Caputo FD of order q and $a, b \in (0,1)$, with the property

$$(\theta E)(\vartheta) = \int_0^{\vartheta} y(\vartheta, s)E(\nabla)ds, \quad (\psi E)(\vartheta) = \int_0^{\vartheta} \mho(\vartheta, s)E(\nabla)ds$$

In [22], authors apply Schaefer's FPT to seek for the existence of solutions of fractional Hadamard-Hilfer IDEs and using U-H to

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analyze the stability. Additionally, the authors in [23] establish the existence of solutions for Hadamard-Hilfer (GH) fractional boundary value problems (BVP) by employing FPTs, specifically those of Krasnoselskii and Schaefer, along with the Leray-Schauder nonlinear alternative. Regarding recent findings related to $GH-$fractional derivative, see ([24, 25, 26]).
Inspired by the aforementioned research, we investigate the existence and stability properties of solutions to the following (GH)-fractional Volterra-Fredholm system:

$$^{GH}D_{1^+}^{\grave{\mathrm{Q}},\omega}u(\vartheta) = \phi(\vartheta, u(\vartheta), (\theta u)(\vartheta), (\psi u)(\vartheta), (\Re u)(\vartheta)),\ \vartheta \in \grave{\mathrm{Q}} = [1, e], \omega \in [0, 1], \tag{1}$$

$$\begin{aligned} ^{H}D^{\mho-2}u(1) &= 0 \\ ^{H}D^{\mho-1}u(1) &= {}^{H}I^{\mho-1}u(\mu), \end{aligned} \tag{2}$$

where $\mho = \grave{\mathrm{Q}} + n\omega - \grave{\mathrm{Q}}\omega$, $^{H}D^{\mho-2}$ is Hadamard FD of order $\mho - 2$, $^{H}I^{\mho-1}$ is Hadamard fractional integral of order $\mho-1$ and $^{GH}D_{1^+}^{\grave{\mathrm{Q}},\omega}$ is Hilfer-Hadamard FD with $\phi : \grave{\mathrm{Q}} \times \mathcal{R}^4 \to \mathcal{R}$ is continuous with the property

$$\begin{aligned} (\theta u)(\vartheta) &= \int_1^{\vartheta} y(\vartheta, s)u(\nabla)ds < \infty, \\ (\psi u)(\vartheta) &= \int_1^{\vartheta} z(\vartheta, s)u(\nabla)ds < \infty, \\ (\Re u)(\vartheta) &= \int_1^{e} m(\vartheta, s)u(\nabla)ds < \infty, \end{aligned} \tag{3}$$

where y, z, m are continuous functions on $\grave{\mathrm{Q}} \times \grave{\mathrm{Q}}$.

This paper is organized as follows. Section 2 provides preliminary results that will be utilized in subsequent sections. In Section 3, we establish the existence and uniqueness of a viable integral solution to the system (1)-(2) by apply standard FPTs. Section 4 discusses the U-H stability and U-H-R stability of the solutions.

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2 Mathematical Background

Definition 2.1. [5] Let $\gamma \in \mathcal{R}$, then the Hadamard fractional integral:

$$I_{a+}^{\gamma}\phi(C) := \frac{1}{\Gamma(\grave{Q})}\int_{a+}^{C}\left(\ln\left(\frac{C}{Q}\right)\right)^{\grave{Q}-1}\frac{\phi(Q)}{Q}dQ. \tag{4}$$

Definition 2.2. [5] Let $\gamma \in \mathcal{R}$, with $n = [\Box] + 1$ and $\delta = \left(x\frac{d}{dx}\right)$; then the Hadamard derivative of fractional of a continuous function ϕ is defined as:

$$D_{a+}^{\gamma}\phi(C) := \frac{1}{\Gamma(n-\grave{Q})}\delta^{n}\int_{a+}^{C}\left(\ln\left(\frac{C}{Q}\right)\right)^{n-\grave{Q}-1}\frac{\phi(Q)}{Q}dQ. \tag{5}$$

where Γ is the gamma function and $[\gamma]$ denotes the integer part of the real number γ.

Definition 2.3. [1] Let $\phi : [0, +\infty) \to R$, is differentiable function then the Hilfer-Hadamard FD order $n-1 < \gamma < n$, and type $0 \le \omega \le 1$, is defined by

$$\begin{aligned}
&\left({}^{GH}D^{\grave{Q},\omega}\phi\right)(\vartheta) \\
&\quad = \left({}^{H}I^{\omega(n-\grave{Q})}\delta^{n H}I^{(n-\grave{Q})(1-\omega)}\phi\right)(\vartheta) \\
&\quad = \left({}^{H}I^{\omega(n-\grave{Q})}\delta^{n H}I^{(n-\mho)}\phi\right)(\vartheta) \\
&\quad = \left({}^{H}I^{\omega(n-\grave{Q}) H}D^{\mho}\phi\right)(\vartheta), \quad \mho = \grave{Q} + n\omega - \grave{Q}\omega,
\end{aligned}$$

where $\delta = \frac{d}{dt}$, and ${}^{H}I^{(\cdot)}, {}^{H}D^{(\cdot)}$ are defined as Hadamard fractional integral and Hadamard FD respectively.

Definition 2.4. [15] The initial value problem (1) is known as $U-H$ stable if d any positive constant such that for each $\epsilon > 0$, when $w \in C(\grave{Q}, \mathcal{R})$, is a solution of the below inequality

$$\left|{}^{GH}D_{1+}^{\grave{Q},\omega}u(\vartheta) - \phi(\vartheta, u(\vartheta), (\theta u)(\vartheta), (\psi u)(\vartheta), (\Re u)(\vartheta))\right| \le \epsilon, \quad \vartheta \in \grave{Q}, \tag{6}$$

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and $\exists$ an additional solution $y \in C(\grave{\mathrm{Q}}, \mathcal{R})$, to equation (1) that satisfies the following inequality

$$|u(\vartheta) - y(\vartheta)| \leq d\epsilon, \quad \vartheta \in \grave{\mathrm{Q}}, \tag{7}$$

Definition 2.5. [15] The system (1) is said to possess $\mathcal{U} - \mathcal{H} - \mathcal{R}$ stability with respect to a function $\beta \in C(\grave{\mathrm{Q}}, \mathcal{R})$ and let V be a positive. Specifically, for any $\epsilon > 0$ and any function $u_1 \in C(\grave{\mathrm{Q}}, \mathcal{R})$ that satisfies the inequality

$$\left| {}^{GH}D_{1+}^{\grave{\mathrm{Q}},\omega} u(\vartheta) - \phi(\vartheta, u(\vartheta), (\theta u)(\vartheta), (\psi u)(\vartheta), (\Re u)(\vartheta)) \right| \leq \epsilon\beta(\vartheta), \quad \vartheta \in \grave{\mathrm{Q}}, \tag{8}$$

thus, there exist a solution $S \in C(\grave{\mathrm{Q}}, \mathcal{R})$ of the problem (1) satisfied

$$|u(\vartheta) - S(\vartheta)| \leq V\epsilon\beta(\vartheta), \quad \vartheta \in \grave{\mathrm{Q}}. \tag{9}$$

Theorem 2.6. *Let* $\grave{\mathrm{Q}} > 0, 0 \leq \omega \leq 1, \mho = \grave{\mathrm{Q}} + n\omega - \grave{\mathrm{Q}}\omega, n = [\grave{\mathrm{Q}}] + 1$. *If* $f \in L^1(a,b)$ *and* $\left({}^{H}I^{n-\mho}\phi\right)(\vartheta) \in AC_\delta^n[a,b]$, *then*

$$\begin{aligned} {}^{H}I_a^{\grave{\mathrm{Q}}} \left({}^{GH}D^{\grave{\mathrm{Q}},\omega}\phi\right)(\vartheta) &= {}^{H}I_a^{\mho}\left({}^{GH}D^{\mho}\phi\right)(\vartheta) \\ &= \phi(\vartheta) - \sum_{k=0}^{n-1} \frac{\left(\delta^{n-k-1}\left({}^{H}I_a^{n-\mho}\phi\right)\right)(a)}{\Gamma(\mho - k)} \left(\ln\left(\frac{\vartheta}{a}\right)\right)^{\mho-k-1} \end{aligned}$$

clearly that $\Gamma(\mho - k)$ *exists for all* $k = 1, 2, \ldots, n-1$ *for* $\mho \in [\grave{\mathrm{Q}}, n]$.

Theorem 2.7. *[15] (Krasnoselskii FPT). Consider a Banach space* E *and a subset* $z \subseteq E$ *be a closed, convex and nonempty. Suppose that* F_1 *and* F_2 *are two operators satisfying the following conditions:*

1. $F_1 z_1 + F_2 z_2 \in Z$, *whenever* $z_1, z_2 \in Z$,
2. F_1 *is compact and continuous;*
3. F_2 *is a contraction.*

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Then there exists $x \in Z$ such that $x = F_1 x + F_2 x$.

Lemma 2.8. *For any $u(\vartheta) \in C(\grave{\mathrm{Q}}, \mathcal{R})$, $\quad 1 < \grave{\mathrm{Q}} \leq 2$, then the IVP (1)-(3) has a solution*

$$u(\vartheta) = \frac{\ln(\grave{\mathrm{d}}))^{\mho-1}}{\Gamma(\mho) - b} \int_1^{\mu} \frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{\mathrm{Q}}-2}}{\Gamma(\mho + \grave{\mathrm{Q}} - 1)} h(\nabla)\frac{d\nabla}{\nabla} + \int_1^{\frac{\vartheta}{(\mho)}} \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{\mathrm{Q}}-1}}{\Gamma(\grave{\mathrm{Q}})} h(\nabla)\frac{d\nabla}{\nabla}.$$

Proof. Applying the Hadamard fractional integral on (1) and apply Theorem 2.6, we obtain

$$u(\vartheta) - \frac{\delta\left({}^{H}I^{2-\mho}u\right)(1)}{\Gamma(\mho)}(\ln(\vartheta))^{\mho-1} - \frac{\left({}^{H}I^{2-\mho}u\right)(1)}{\Gamma(\mho-1)}(\ln(\vartheta))^{\mho-2} = {}^{H}I_1^{\grave{\mathrm{Q}}}h(\vartheta),$$

$$u(\vartheta) = \frac{\delta\left({}^{H}I^{2-\mho}u\right)(1)}{\Gamma(\mho)}(\ln(\vartheta))^{\mho-1} + \frac{\left({}^{H}I^{2-\mho}u\right)(1)}{\Gamma(\mho-1)}(\ln(\vartheta))^{\mho-2} + {}^{H}I_1^{\grave{\mathrm{Q}}}h(\vartheta),$$

which can be rewritten as follows:

$$u(\vartheta) = c_0(\ln(\vartheta))^{\mho-1} + c_1(\ln(\vartheta))^{\mho-2} + \int_1^{\vartheta} \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{\mathrm{Q}}-1}}{\Gamma(\grave{\mathrm{Q}})} h(\nabla)\frac{d\nabla}{\nabla}.$$

Next, we will use the boundary conditions specified in (2) to determine the values of the constants c_0 and c_1

$$D^{\mho-2}u(\vartheta) = \Gamma(\mho)c_0(\ln(\vartheta)) + c_1 + \int_1^{\vartheta} \frac{\left(\ln\left(\left(\frac{\vartheta}{\nabla}\right)\right)\right)^{n-\mho-3}}{\Gamma(\mho-2)} \left(\int_1^{\vartheta} \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{\mathrm{Q}}-1}}{\Gamma(\grave{\mathrm{Q}})} h(\nabla)\frac{d\nabla}{\nabla}\right)\frac{d\nabla}{\nabla},$$

$$D^{\mho-2}u(1) = \Gamma(\mho)c_0(\ln(1)) + c_1 + \int_1^{1} \frac{\left(\ln\left(\left(\frac{\vartheta}{\nabla}\right)\right)\right)^{n-\mho-3}}{\Gamma(\mho-2)} \left(\int_1^{\mathring{\eth}} \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{\mathrm{Q}}-1}}{\Gamma(\grave{\mathrm{Q}})} h(\nabla)\frac{d\nabla}{\nabla}\right)\frac{d\nabla}{\nabla},$$

we get $c_1 = 0$,
which can be rewritten as follows:

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$$u(\vartheta)=c_0(\ln(\vartheta))^{\mho-1}+\int_1^{\vartheta}\frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})}h(\nabla)\frac{d\nabla}{\nabla}.$$

Now, take the derivative $D^{\mho-1}$, to both sides

$$D^{\mho-1}u(\vartheta)=D^{\mho-1}c_0(\ln(\vartheta))^{\mho-1}+D^{\mho-1}\int_1^{\vartheta}\frac{\left(\ln\left(\frac{\dot{\phi}}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})}h(\nabla)\frac{d\nabla}{\nabla}$$

$$D^{\mho-1}u(1)=\Gamma(\mho)c_0+\int_1^{1}\frac{\left(\ln\left(\left(\frac{\vartheta}{\nabla}\right)\right)\right)^{n-\mho-2}}{\Gamma(\mho-1)}\left(\int_1^{\vartheta}\frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})}h(\nabla)\frac{d\nabla}{\nabla}\right)\frac{d\nabla}{\nabla}$$

$$D^{\mho-1}u(1)=\Gamma(\mho)c_0$$

But, from the condition $D^{\mho-1}u(1)=I^{\mho-1}u(\mu)$, implies that

$$D^{\mho-1}u(\vartheta)={}_1^{\mu}I^{\mho-1}c_0(\ln(\vartheta))^{\mho-1}+I^{\mho-1}\int_1^{\vartheta}\frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})}h(\nabla)\frac{d\nabla}{\nabla}$$

$$\Gamma(\mho)c_0-c_0\frac{\Gamma(\mho)(\ln(\vartheta))^{2\mho-2}}{\Gamma(2\mho-1)}=\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho-2}}{\Gamma(\mho-1)}\left(\int_1^{\nabla}\frac{\left(\ln\frac{\nabla}{r}\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})}h(r)\frac{dr}{r}\right)\frac{d\nabla}{\nabla}.$$

Let $b=\frac{\Gamma(\mho)(\ln(\grave{\partial}))^{2\mho-2}}{\Gamma(2\mho-1)}$, then

$$c_0=\frac{1}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho-2}}{\Gamma(\mho-1)}\left(\int_1^{\nabla}\frac{\left(\ln\frac{\nabla}{r}\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})}h(r)\frac{dr}{r}\right)\frac{d\nabla}{\nabla}.$$

Let

$$\Upsilon=\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho-2}}{\Gamma(\mho-1)}\left(\int_1^{\nabla}\frac{\left(\ln\frac{\nabla}{r}\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})}h(r)\frac{dr}{r}\right)\frac{d\nabla}{\nabla}$$

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by using Fubini's theorem, we get

$$\Upsilon = \int_1^{\mu}\int_r^{\mu} \frac{(\ln(\mu)-\ln(\nabla))^{\mho-2}}{\Gamma(\mho-1)} \frac{(\ln(\nabla)-\ln(r))^{\grave{Q}-1}}{\Gamma(\Im)} \frac{d\nabla}{\nabla} h(r)\frac{dr}{r}. \tag{10}$$

Suppose that $\ln(\nabla) = \ln(r) + u(\ln(\mu) - \ln(r))$, when $s = r \to u = 0$ and $s = \mu \to u = 1$, $\frac{d\nabla}{\nabla} = \left(\ln\left(\frac{\mu}{r}\right)\right) du$. From (10), we obtain

$$\Upsilon = \int_1^{\mu} \frac{\left(\ln\left(\frac{\mu}{r}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho-1)\Gamma(\grave{Q})} \left(\int_0^1 (1-u)^{\mho-2} u^{\grave{Q}-1} du\right) h(r)\frac{dr}{r}$$

By using beta function we have

$$u(\grave{\boldsymbol{\partial}}) = \frac{1}{\Gamma(\mho)-b}\int_1^{\mu} \frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)} h(\nabla)\frac{d\nabla}{\nabla}(\ln(\grave{\boldsymbol{J}}))^{\mho-1} + \int_1^{\vartheta} \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})} h(\nabla)\frac{d\nabla}{\nabla}.$$

3 Existence and Uniqueness Results

In this section, we investigate the existence of solutions for the system (1)-(2) then we prove uniqueness.

Let ϕ be a continuous function, and $C(\grave{Q}, \mathcal{R})$ be the Banach space which consisting of all continuous functions of real-valued defined on

$$\|u\| = \sup\{|u(\vartheta)|; \vartheta \in \grave{Q}\}.$$

In order to establish our findings, we use the below assumptions:
(**F1**) Let $L > 0$, be a constant such that

$$|\phi(\vartheta, u(\vartheta), (\theta u)(\vartheta), (\psi u)(\vartheta), (\Re u)(\vartheta))| \leq L.$$

(**F2**) There exists positive functions $k_1(\vartheta), k_2(\vartheta), k_3(\vartheta)$, and $k_4(\vartheta)$, such that

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$|\phi(\vartheta, u(\vartheta), (\theta u)(\vartheta), (\psi u)(\vartheta), (\Re u)(\vartheta)) - \phi(\vartheta, w(\vartheta), (\theta w)(\vartheta), (\psi w)(\vartheta), (\Re w)(\vartheta))|$
$\leq k_1(\vartheta)|u - w| + k_2(\vartheta)|\theta u - \theta w| + k_3(\vartheta)|\psi u - \psi w| + k_4(\vartheta)|\Re u - \Re w|$.

For all $\vartheta \in \grave{\mathrm{Q}}$.

$$y_0 = \sup_{\vartheta \in \grave{\mathrm{Q}}} \int_1^{\vartheta} y(\vartheta, s) u(\nabla) ds, \quad z_0 = \sup_{\vartheta \in \grave{\mathrm{Q}}} \int_1^{\vartheta} z(\vartheta, s) u(\nabla) ds, \quad m_0 = \sup_{\vartheta \in \grave{\mathrm{Q}}} \int_1^{e} m(\vartheta, s) u(\nabla) ds$$

$$d_2 = \sup_{\vartheta \in [1,e]} \left\{ I^{\grave{\mathrm{Q}}} k_1(\vartheta), I^{\grave{\mathrm{Q}}} k_2(\vartheta), I^{\grave{\mathrm{Q}}} k_3(\vartheta), I^{\grave{\mathrm{Q}}} k_4(\vartheta) \right\}$$

$$d_1 = \max \left\{ I^{\mho + \grave{\mathrm{Q}} - 1} k_1(\mu), I^{\mho + \grave{\mathrm{Q}} - 1} k_2(\mu), I^{\mho + \grave{\mathrm{Q}} - 1} k_3(\mu), I^{\mho + \grave{\mathrm{Q}} - 1} k_4(\mu) \right\}.$$

Theorem 3.1. *The system (1)-(2) has at least one solution if $\phi : \grave{\mathrm{Q}} \times \mathcal{R}^4 \to \mathcal{R}$ is continuous and satisfies $(\boldsymbol{F1}) - (\boldsymbol{F2})$.*

Proof. Let the set $G_r = \{u \in \grave{\mathrm{Q}} : \|u\| < r\}$, then G_r is a closed, bounded, and convex set of $\grave{\mathrm{Q}}$. Now, let the operators A and B on $\grave{\mathrm{Q}}$ defined as:

$$(Au)(\vartheta) = \int_1^{\vartheta} \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{\mathrm{Q}} - 1}}{\Gamma(\grave{\mathrm{Q}})} \phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla), \Re(u)(\nabla)) \frac{d\nabla}{\nabla}$$

$$(Bu)(\vartheta) = \frac{(\ln(\vartheta))^{\mho - 1}}{\Gamma(\mho) - b} \int_1^{\mu} \frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho + \grave{\mathrm{Q}} - 2}}{\Gamma(\mho + \grave{\mathrm{Q}} - 1)} \phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla), \Re(u)(\nabla)) \frac{d\nabla}{\nabla}.$$

For any $u \in G_r$ and $\vartheta \in \grave{\mathrm{Q}}$, we acquire

$$\begin{aligned} |(Au)(\vartheta)| &= \left| \int_1^{\vartheta} \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{\mathrm{Q}} - 1}}{\Gamma(\grave{\mathrm{Q}})} \phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla), \Re(u)(\nabla)) \frac{d\nabla}{\nabla} \right| \\ &\leq \int_1^{\vartheta} \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{\mathrm{Q}} - 1}}{\Gamma(\grave{\mathrm{Q}})} |\phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla), \Re(u)(\nabla)) - \phi(\nabla, 0, 0, 0, 0)| \frac{d\nabla}{\nabla} \\ &\quad + \int_1^{\vartheta} \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{\mathrm{Q}} - 1}}{\Gamma(\grave{\mathrm{Q}})} |\phi(\nabla, 0, 0, 0, 0)| \frac{d\nabla}{\nabla} \end{aligned}$$

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$$
\begin{aligned}
&\leq \left(I^{\grave{Q}}k_1(\grave{\dot{Q}}) + y_0 I^{\grave{Q}}k_2(\vartheta) + z_0 I^{\grave{Q}}k_3(\vartheta) + m_0 I^{\grave{Q}}k_4(\vartheta)\right) + \frac{q}{\Gamma(\grave{Q}+1)}(\ln(\vartheta))^{\grave{Q}} \\
&\leq d_2\left(1 + y_0 + z_0 + m_0\right)\|u\| + \frac{q}{\Gamma(\grave{Q}+1)}.
\end{aligned}
$$

Hence, we obtain

$$
\|Au\| \leq d_2\left(1 + y_0 + z_0 + m_0\right)\|u\| + \frac{q}{\Gamma(\grave{Q}+1)}. \tag{11}
$$

By the same way, we estimate $\|Bu_1\|$, let $u_1 \in G_r$, and $\dot{\boldsymbol{\delta}} \in \grave{Q}$, then

$$
\begin{aligned}
&\left|\left(Bu_1\right)(\vartheta)\right| \\
&= \left|\frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)}\phi\left(\nabla, u_1(\nabla), \left(\theta u_1\right)(\nabla), \left(\psi u_1\right)(\nabla), \left(\Re u_1\right)(\nabla)\right)\frac{d\nabla}{\nabla}\right| \\
&\leq \frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)} \\
&\quad \times\left|\phi\left(\nabla, u_1(\nabla), \left(\theta u_1\right)(\nabla), \left(\psi u_1\right)(\nabla), \left(\Re u_1\right)(\nabla)\right) - \phi(\nabla,0,0,0,0)\right|\frac{d\nabla}{\nabla} \\
&\quad + \frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)}|\phi(\nabla,0,0,0,0)|\frac{d\nabla}{\nabla} \\
&\leq \frac{(\ln(\vartheta))^{\mho-1}||u_1||}{\Gamma(\mho)-b} \\
&\quad \times\left(I^{\mho+\grave{Q}-1}k_1(\mu) + y_0 I^{\mho+\grave{Q}-1}k_2(\mu) + z_0 I^{\mho+\grave{Q}-1}k_3(\mu) + m_0 I^{\mho+\grave{Q}-1}k_4(\mu)\right) \\
&\quad + \frac{q(\ln(\mu))^{\mho+\grave{Q}-1}}{\Gamma(\mho+\grave{Q})}\frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}
\end{aligned}
$$

$$
\|Bu_1\| \leq \frac{d_1}{\Gamma(\mho)-b}\left(\left(1 + y_0 + z_0 + m_0\right)\|u_1\| + \frac{q}{\Gamma(\mho+\grave{Q})}\right). \tag{12}
$$

Now, from the result (11) and (12), we obtain that for any $u, u_1 \in G_r$, and $\dot{\delta} \in \grave{Q}$

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$$
\begin{aligned}
\|Au+Bu_1\| &\leq \|Au\|+\|Bu_1\| \\
\|Au+Bu_1\| &\leq d_2\left(1+y_0+z_0+m_0\right)\|u\|+\frac{q}{\Gamma(\grave{Q}+1)} \\
&\quad +\frac{d_1}{\Gamma(\mho)-b}\left(\left(1+y_0+z_0+m_0\right)\|u_1\|+\frac{q}{\Gamma(\mho+\grave{Q})}\right),
\end{aligned}
$$

$$
d_2\left(1+y_0+z_0+m_0\right)\|u\|+\frac{q}{\Gamma(\grave{Q}+1)}+\frac{d_1}{\Gamma(\mho)-b}\left(\left(1+y_0+z_0+m_0\right)\|u_1\|+\frac{q}{\Gamma(\mho+\grave{Q})}\right)\leq R,
$$

then $\|Au+Bu_1\|\leq R$.
Now, we must show that B is a contraction, by assume that $u_1,u_2\in G_r$, and $\vartheta\in\grave{Q}$. Therefor,

$$
\begin{aligned}
&\left|\left(Bu_1\right)(\vartheta)-\left(Bu_2\right)(\vartheta)\right| \\
=&\ \Bigg|\frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)}\phi\left(\nabla,u_1(\nabla),\left(\theta u_1\right)(\nabla),\left(\psi u_1\right)(\nabla),\left(\Re u_1\right)(\nabla)\right)\frac{d\nabla}{\nabla} \\
&\quad -\frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)}\phi\left(\nabla,u_2(\nabla),\left(\theta u_2\right)(\nabla),\left(\psi u_2\right)(\nabla),\left(\Re u_2\right)(\nabla)\right)\frac{d\nabla}{\nabla}\Bigg| \\
\leq&\ \frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)} \\
&\quad \times\left|\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla))-\phi\left(\nabla,u_1(\nabla),\left(\theta u_1\right)(\nabla),\left(\psi u_1\right)(\nabla),\left(\Re u_1\right)(\nabla)\right)\right|\frac{d\nabla}{\nabla} \\
\leq&\ \frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)}\left(k_1(\nabla)|u|+k_2(\nabla)|\theta u|+k_3(\vartheta)|\psi u|+k_4(\vartheta)|\Re u|\right)\frac{d\nabla}{\nabla} \\
\leq&\ \frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\left(I^{\mho+\grave{Q}-1}k_1(\mu)+y_0I^{\mho+\grave{Q}-1}k_2(\mu)+z_0I^{\mho+\grave{Q}-1}k_3(\mu)+m_0I^{\mho+\grave{Q}-1}k_4(\mu)\right)
\end{aligned}
$$

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$$\therefore \quad \left\|(Bu_1)(\vartheta)-(Bu_2)(\vartheta)\right\| \leq \frac{d_1}{\Gamma(\mho)-b}\left(1+y_0+z_0+m_0\right)\left\|u_1-u_2\right\|,$$

we have shown that B is a contraction mapping.

To establish that A is compact and continuous, we must demonstrate that A is both continuous and compact on the region G_r. Furthermore, A is uniformly bounded within G_r, as evidenced by equation (11) we have

$$\begin{aligned}\|Au\| &\leq d_2\left(1+y_0+z_0+m_0\right)\|u\|+\frac{q}{\Gamma(\grave{Q}+1)}\\ &\leq d_2\left(1+y_0+z_0+m_0\right)R+\frac{q}{\Gamma(\grave{Q}+1)}.\end{aligned}$$

Let $u \in G_r$ and $t_1 < t_2$,

$$\begin{aligned}&\left|(Au)(\vartheta_2)-(Au)(\vartheta_1)\right| = \left|\int_1^{t_2}\frac{\left(\ln\frac{t_2}{\nabla}\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})}\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla),(\Re u)(\nabla))\frac{d\nabla}{\nabla}\right.\\
&\left.-\int_1^{t_1}\frac{\left(\ln\frac{t_1}{\nabla}\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})}\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla),(\Re u)(\nabla))\frac{d\nabla}{\nabla}\right|,\\
&\leq\left|\frac{1}{\Gamma(\grave{Q})}\int_1^{t_1}\left(\left(\ln\frac{t_2}{\nabla}\right)^{\grave{Q}-1}-\left(\ln\frac{t_1}{\nabla}\right)^{\grave{Q}-1}\right)\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla),(\Re u)(\nabla))\frac{d\nabla}{\nabla}\right.\\
&\left.+\int_{t_1}^{t_2}\frac{\left(\ln\frac{t_2}{\nabla}\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})}\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla),(\Re u)(\nabla))\frac{d\nabla}{\nabla}\right|,\\
&\leq\frac{L}{\Gamma(\grave{Q}+1)}\left[\left(\ln\frac{t_2}{\nabla}\right)^{\grave{Q}}-\left(\ln\frac{t_1}{\nabla}\right)^{\grave{Q}}\right]_1^{t_1}+\frac{L}{\Gamma(\grave{Q}+1)}\left[\left(\ln\frac{t_2}{\nabla}\right)^{\grave{Q}}\right]_{t_1}^{t_2},\\
&=\frac{L}{\Gamma(\grave{Q}+1)}\left[\left(\ln\frac{t_2}{t_1}\right)^{\grave{Q}}\right]=\frac{L}{\Gamma(\grave{Q}+1)}\left[\left(\ln(\vartheta_2)-\ln(\vartheta_1)\right)^{\grave{Q}}\right]\\
&\to 0 \qquad \text{as} \qquad t_2\to t_1.\end{aligned}$$

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Hence, by the profound Arzela-Ascoli theorem, it follows that A is a compact operator on G_r. Consequently, A is completely continuous. Applying the robust framework of Krasnoselskii's FPT, so we obtain that the boundary value problem described by (1)- (2) has at least one solution within the interval $\grave{Q}$. The proof is thus thoroughly complete.

Theorem 3.2. *Then the problem* (1)-(2) *has a unique solution. If* $\phi : \grave{Q} \times \mathcal{R}^4 \to \mathcal{R}$ *is continuous and satisfies* $(\boldsymbol{F1}) - (\boldsymbol{F2})$*, and suppose that* $\sup_{\vartheta\in[1,e]} \phi(\vartheta, 0, 0, 0, 0) = q$*, and*

$$\left(\frac{d_1 + d_2}{\Gamma(\mho) - b}\right)(1 + y_0 + z_0 + m_0) < 1.$$

Proof Let $B : C(\grave{Q}, \mathcal{R}) \to C(\grave{Q}, \mathcal{R})$ be operator defined as the following

$$(Tu)(\vartheta) = \frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho) - b} \int_1^{\mu} \frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho + \grave{Q} - 1)} \phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla), (\Re u)(\nabla)) \frac{d\nabla}{\nabla} + \int_1^{\delta} \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})} \phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla), (\Re u)(\nabla)) \frac{d\nabla}{\nabla} \tag{13}$$

Our objective is to demonstrate that the operator T admits a fixed p oint w ithin t he s et G_r, w hich c oincides w ith a s olution to the initial value problem specified by equations (1)-(2).

First, we establish the invariance of G_r under T, namely, that $T(G_r) \subseteq G_r$. To this end, consider an arbitrary $u \in G_r$.where, $G_r = \{u \in C : \|u\| < r\}$. For $u \in G_r$. The operator T is bounded set into the bounded sets in $C(\grave{Q}, \mathcal{R})$.

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$$
\begin{aligned}
&|(Tu)(\vartheta)|\\
=&\Big| \frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)}\\
&\times\left(\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla),(\Re u)(\nabla))-\phi(\nabla,0,0,0,0)+\phi(\nabla,0,0,0,0)\right)\frac{d\nabla}{\nabla}\\
&+\int_1^{\vartheta}\frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})}\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla),(\Re u)(\nabla))\\
&-\phi(\nabla,0,0,0,0)+\phi(\nabla,0,0,0,0))\frac{d\nabla}{\nabla}\Big|,\\
&\leq\frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)}\\
&\times\left((k_1(\nabla)|u|+k_2(\nabla)|\theta u|+k_3(\vartheta)|\psi u|+k_4(\vartheta)|\Re u|)+q\right)\frac{d\nabla}{\nabla}\\
&+\int_1^{\vartheta}\frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})}\left((k_1(\nabla)|u|+k_2(\vartheta)|\theta u|+k_3(\vartheta)|\psi u|+k_4(\vartheta)|\Re u|)+q\right)\frac{d\nabla}{\nabla},\\
&\leq\frac{1}{\Gamma(\mho)-b}\left(d_1\left(1+y_0+z_0+m_0\right)+\frac{q}{\Gamma(\mho+\grave{Q})}\right)+d_2\left(1+y_0+z_0+m_0\right)+\frac{q}{\Gamma(\grave{Q}+1)},\\
&\leq\left(\frac{1}{\Gamma(\mho)-b}(d_1+d_2)\right)(1+y_0+z_0+m_0)+q\left(\frac{1}{(\Gamma(\mho)-b)\Gamma(\mho+\grave{Q})}+\frac{1}{\Gamma(\grave{Q}+1)}\right),
\end{aligned}
$$

To prove that B is a contraction mapping, we need to show that for any $u,u_1\in G_r$ and for each $\vartheta\in\grave{Q}$, the following inequality holds

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$$
\begin{aligned}
&\left|(Tu)(\vartheta)-(Tu_1)(\vartheta)\right| \\
&=\left|\frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)}\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla),(\Re u)(\nabla))\frac{d\nabla}{\nabla}\right. \\
&+\int_1^{\vartheta}\frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})}\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla),(\Re u)(\nabla))\frac{d\nabla}{\nabla} \\
&-\frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)}\phi\left(\nabla,u_1(\nabla),(\theta u_1)(\nabla),(\psi u_1)(\nabla),(\Re u_1)(\nabla)\right)\frac{d\nabla}{\nabla} \\
&\left.+\int_1^{\vartheta}\frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})}\phi\left(\nabla,u_1(\nabla),(\theta u_1)(\nabla),(\psi u_1)(\nabla),(\Re u_1)(\nabla)\right)\frac{d\nabla}{\nabla}\right| \\
&=\Big|\frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)} \\
&\times\left(\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla),(\Re u)(\nabla))-\phi\left(\nabla,u_1(\nabla),(\theta u_1)(\nabla),(\psi u_1)(\nabla),(\Re u_1)(\nabla)\right)\right)\frac{d\nabla}{\nabla} \\
&+\int_1^{\vartheta}\frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})} \\
&\times\left(\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla),(\Re u)(\nabla))-\phi\left(\nabla,u_1(\nabla),(\theta u_1)(\nabla),(\psi u_1,(\Re u_1)(\nabla)\right)\right)\right)\frac{d\nabla}{\nabla}\Big| \\
&\le\frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)}\left(k_1(\nabla)|u|+y_0k_2(\nabla)|u|+z_0k_3(\nabla)|u|+m_0k_4(\nabla)|u|\right)\frac{d\nabla}{\nabla} \\
&+\int_1^{\dot{\delta}}\frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})}\left(k_1(\nabla)|u|+y_0k_2(\nabla)|u|+z_0k_3(\nabla)|u|+m_0k_4(\nabla)|u|\right)\frac{d\nabla}{\nabla} \\
&\le\frac{1}{\Gamma(\mho)-b}d_1\left(1+y_0+z_0+m_0\right)+d_2\left(1+y_0+z_0+m_0\right) \\
&\le\left(\frac{d_1+d_2}{\Gamma(\mho)-b}\right)\left(1+y_0+z_0+m_0\right)
\end{aligned}
$$

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Where $\left(\frac{d_1+d_2}{\Gamma(\mho)-b}\right)(1+y_0+z_0+m_0)<1$. Hence, the operator T is a contraction mapping. Consequently, we deduce that T is the unique solution to the problem (1)-(2).

4 Stability Results

In this section, we analyze the stability of our result, focusing on U-H stability and U-H-R stability.

Theorem 4.1. *Assuming condition (***F2***) holds, the problem* (1)-(2) *is U-H stable.*

Proof. Assume that the inequality (6) has a solution known as $w \in C(\grave{\mathrm{Q}}, \mathcal{R})$ i.e,

$$\left|{}^{GH}D_{1+}^{\grave{\mathrm{Q}},\omega} w(\vartheta) - \phi(\vartheta, w(\vartheta), (\theta w)(\vartheta), (\psi w)(\vartheta), (\Re w)(\vartheta))\right| \leq \epsilon, \quad \vartheta \in \grave{\mathrm{Q}}. \tag{14}$$

If $u \in C(\grave{\mathrm{Q}}, \mathcal{R})$ is defined as the unique solution of the problem (1)-(2). Since $u\&w$ are continuous functions on $C(\grave{\mathrm{Q}}, \mathcal{R})$, and from lemma 2.8 we get

$$\begin{aligned} w(\vartheta) = & \frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b} \int_1^{\mu} \frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{\mathrm{Q}}-2}}{\Gamma(\mho+\grave{\mathrm{Q}}-1)} \phi(\nabla, w(\nabla), (\theta w)(\nabla), (\psi w)(\nabla), (\Re w)(\nabla)) \frac{d\nabla}{\nabla} \\ & + \int_1^{\vartheta} \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{\mathrm{Q}}-1}}{\Gamma(\grave{\mathrm{Q}})} \phi(\nabla, w(\nabla), (\theta w)(\nabla), (\psi w)(\nabla), (\Re w)(\nabla)) \frac{d\nabla}{\nabla} \end{aligned}$$

Integrate (14) and we obtain

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$$\begin{aligned}
&\Bigg| w(\vartheta) - \frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{\mathrm{Q}}-2}}{\Gamma(\mho+\grave{\mathrm{Q}}-1)}\phi(\nabla, w(\nabla), (\theta w)(\nabla), (\psi w)(\nabla), (\Re w)(\nabla))\frac{d\nabla}{\nabla} \\
&-\int_1^{\vartheta}\frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{\mathrm{Q}}-1}}{\Gamma(\grave{\mathrm{Q}})}\phi(\nabla, w(\nabla), (\theta w)(\nabla), (\psi w)(\nabla), (\Re w)(\nabla))\frac{d\nabla}{\nabla}\Bigg| \\
&\le \frac{\epsilon(\ln(\vartheta))^{\grave{\mathrm{Q}}}}{\Gamma(\grave{\mathrm{Q}}+1)}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\Big| w(\vartheta) - u(\vartheta)\Big| \\
=&\Bigg| w(\vartheta) - \frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{\mathrm{Q}}-2}}{\Gamma(\mho+\grave{\mathrm{Q}}-1)}\phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla), (\Re u)(\nabla))\frac{d\nabla}{\nabla} \\
&-\int_1^{\vartheta}\frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{\mathrm{Q}}-1}}{\Gamma(\grave{\mathrm{Q}})}\phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla), (\Re u)(\nabla))\frac{d\nabla}{\nabla}\Bigg| \\
&\Big| w(\vartheta) - u(\vartheta)\Big| \\
&=\Bigg| w(\vartheta) - \frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{\mathrm{Q}}-2}}{\Gamma(\mho+\grave{\mathrm{Q}}-1)}\phi(\nabla, w(\nabla), (\theta w)(\nabla), (\psi w)(\nabla), (\Re w)(\nabla))\frac{d\nabla}{\nabla} \\
&+\frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{\mathrm{Q}}-2}}{\Gamma(\mho+\grave{\mathrm{Q}}-1)}\phi(\nabla, w(\nabla), (\theta w)(\nabla), (\psi w)(\nabla), (\Re w)(\nabla))\frac{d\nabla}{\nabla} \\
&-\frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{\mathrm{Q}}-2}}{\Gamma(\mho+\grave{\mathrm{Q}}-1)}\phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla), (\Re u)(\nabla))\frac{d\nabla}{\nabla} \\
&-\int_1^{\vartheta}\frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{\mathrm{Q}}-1}}{\Gamma(\grave{\mathrm{Q}})}\phi(\nabla, w(\nabla), (\theta w)(\nabla), (\psi u)(\nabla), (\Re u)(\nabla))\frac{d\nabla}{\nabla}
\end{aligned}$$

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$$
\begin{aligned}
&+\int_1^{\vartheta} \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})} \phi(\nabla, w(\nabla), (\theta w)(\nabla), (\psi u)(\nabla), (\Re u)(\nabla)) \frac{d\nabla}{\nabla} \\
&\left.-\int_1^{\vartheta} \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})} \phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla), (\Re u)(\nabla)) \frac{d\nabla}{\nabla}\right|, \\
&\leq \frac{\epsilon(\ln(\vartheta))^{\grave{Q}}}{\Gamma(\grave{Q}+1)} + \int_1^{\vartheta} \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})} \left(k_1(\nabla) + y_0 k_2(\nabla) + z_0 k_3(\nabla) + m_0 k_4(\nabla)\right) |w-u| \frac{d\nabla}{\nabla} \\
&+ \frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b} \left(I^{\mho+\grave{Q}-1} k_1(\mu) + y_0 I^{\mho+\grave{Q}-1} k_2(\mu) + z_0 I^{\mho+\grave{Q}-1} k_3(\mu) + m_0 I^{\mho+\grave{Q}-1} k_4(\mu)\right) \|w-u\|,
\end{aligned}
$$

then,

$$
\begin{aligned}
&||w(\vartheta) - u(\vartheta)|| \\
&\leq \frac{\epsilon(\ln(\vartheta))^{\grave{Q}}}{\Gamma(\grave{Q}+1)} + \int_1^{\vartheta} \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})} \left(k_1(\nabla) + y_0 k_2(\nabla) + z_0 k_3(\nabla) + m_0 k_4(\nabla)\right) |w-u| \frac{d\nabla}{\nabla} \\
&+ \frac{v d_1 \left(1 + y_0 + z_0 + m_0\right)}{\Gamma(\mho)-b} \|w(\vartheta) - u(\vartheta)\|,
\end{aligned}
$$

Let $\xi = \frac{v d_1 (1+y_0+z_0+m_0)}{\Gamma(\mho)-b}$, then we get

$$
\begin{aligned}
\|w(\vartheta) - u(\vartheta)\| &\leq \frac{\epsilon(\ln(\vartheta))^{\grave{Q}}}{\Gamma(\grave{Q}+1)(1-\xi)} e^{\left(I^{\grave{Q}} k_1(\vartheta) + y_0 I^{\grave{Q}} k_2(\vartheta) + z_0 I^{\grave{Q}} k_3(\vartheta) + m_0 I^{\grave{Q}} k_4(\vartheta)\right)} \\
&\leq \frac{\epsilon(\ln(\vartheta))^{\grave{Q}}}{\Gamma(\grave{Q}+1)(1-\xi)} e^{d_2(1+y_0+z_0+m_0)}.
\end{aligned}
$$

Hence, the initial value problem (1)-(2) is U-H stable

Theorem 4.2. *Suppose that the assumptions $(\boldsymbol{F1})-(\boldsymbol{F2})$ are satisfied, and we have another condition,*
$(\mathbf{F3})$ *The function $\beta \in C\left(\grave{Q}, R_+\right)$ is increasing and there exist $\mho_\beta > 0$, such that, for each $\dot{\boldsymbol{\delta}} \in \grave{Q}$, we have*

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$$I^{\grave{Q}}\beta(\vartheta) < \mho_\beta \beta(\vartheta). \tag{15}$$

Then, the problem (1)-(2) is U-H-R stable with respect to β.

Proof. Assume that $w \in C(\grave{Q}, \mathcal{R})$ be a solution of the inequality (6) i.e,

$$\left| {}^{GH}D_{1+}^{\grave{Q},\omega} w(\vartheta) - \phi(\vartheta, w(\vartheta), (\theta w)(\vartheta), (\psi w)(\vartheta), (\Re w)(\vartheta)) \right| \leq \epsilon\beta(\vartheta), \quad \vartheta \in \grave{Q}, \tag{16}$$

when u and w being continuous functions on $\grave{Q} \times \grave{Q}$. Nowfrom lemma 2.8

$$w(\vartheta) = \frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho) - b} \int_1^\mu \frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)} \phi(\nabla, w(\nabla), (\theta w)(\nabla), (\psi w)(\nabla), (\Re w)(\nabla)) \frac{d\nabla}{\nabla}$$
$$+ \int_1^\delta \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})} \phi(\nabla, w(\nabla), (\theta w)(\nabla), (\psi w)(\nabla), (\Re w)(\nabla)) \frac{d\nabla}{\nabla}$$

integrate (16) and the result become

$$\left| w(\vartheta) - \frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho) - b} \int_1^\mu \frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)} \phi(\nabla, w(\nabla), (\theta w)(\nabla), (\psi w)(\nabla), (\Re w)(\nabla)) \frac{d\nabla}{\nabla} \right.$$
$$\left. - \int_1^\vartheta \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})} \phi(\nabla, w(\nabla), (\theta w)(\nabla), (\psi w)(\nabla), (\Re w)(\nabla)) \frac{d\nabla}{\nabla} \right|$$

$$\leq \epsilon \int_1^\vartheta \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})} \beta(\vartheta) \frac{d\nabla}{\nabla} = \epsilon I^{\grave{Q}}\beta(\vartheta),$$

$$\left| w(\vartheta) - \frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho) - b} \int_1^\mu \frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)} \phi(\nabla, w(\nabla), (\theta w)(\nabla), (\psi w)(\nabla), (\Re w)(\nabla)) \frac{d\nabla}{\nabla} \right.$$
$$\left. - \int_1^\vartheta \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})} \phi(\nabla, w(\nabla), (\theta w)(\nabla), (\psi w)(\nabla), (\Re w)(\nabla)) \frac{d\nabla}{\nabla} \right| \leq \epsilon \mho_\beta \beta(\vartheta)$$

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On the other hand we have

$$
\begin{aligned}
&\left|w(\vartheta)-u(\vartheta)\right| \\
&=\left|w(\vartheta)-\frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)}\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla),(\Re u)(\nabla))\frac{d\nabla}{\nabla}\right. \\
&\left.-\int_1^{\vartheta}\frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})}\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla),(\Re u)(\nabla))\frac{d\nabla}{\nabla}\right|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|w(\vartheta)-u(\vartheta)\right| \\
&\leq \epsilon\mho_{\beta}\beta(\vartheta)+\left|\frac{(\ln(\vartheta))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{\left(\ln\left(\frac{\mu}{\nabla}\right)\right)^{\mho+\grave{Q}-2}}{\Gamma(\mho+\grave{Q}-1)}(\phi(\nabla,w(\nabla),(\theta w)(\nabla),(\psi w)(\nabla),(\Re w)(\nabla))\right. \\
&\quad-\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla)))\frac{d\nabla}{\nabla}+\int_1^{\vartheta}\frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})} \\
&\quad\left.\times\,(\phi(\nabla,w(\nabla),(\theta w)(\nabla),(\psi w)(\nabla),(\Re w)(\nabla))-\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla),(\Re u)(\nabla)))\frac{d\nabla}{\nabla}\right|,
\end{aligned}
$$

then,

$$
\begin{aligned}
&\|w(\vartheta)-u(\vartheta)\| \\
&\leq \epsilon\mho_{\beta}\beta(\vartheta)+\frac{vd_1\left(1+y_0+z_0+m_0\right)}{\Gamma(\mho)-b}\|w(\vartheta)-u(\vartheta)\| \\
&+\int_1^{\vartheta}\frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})}\left(k_1(\nabla)+y_0k_2(\nabla)+z_0k_3(\nabla)+m_0k_4(\nabla)\right)|w-u|\frac{d\nabla}{\nabla}
\end{aligned}
$$

Let $\xi=\frac{vd_1(1+y_0+z_0+m_0)}{\Gamma(\mho)-b}$ then we obtain

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$$\|w(\vartheta) - u(\vartheta)\| \leq \frac{\epsilon \mho_\beta \beta(\vartheta)}{(1-\xi)} + \frac{1}{(1-\xi)} \int_1^\vartheta \frac{\left(\ln\left(\frac{\vartheta}{\nabla}\right)\right)^{\grave{Q}-1}}{\Gamma(\grave{Q})}$$

$$\times \left(k_1(\nabla) + y_0 k_2(\nabla) + z_0 k_3(\nabla) + m_0 k_4(\nabla)\right) |w - u| \frac{d\nabla}{\nabla}$$

$$\therefore \quad \|w(\vartheta) - u(\vartheta)\| \leq \frac{\epsilon \mho_\beta \beta(\vartheta)}{(1-\xi)} e^{d_2(1+y_0+z_0+m_0)}.$$

Hence the initial value problem (1)-(2) is U-H-R stable

References

[1] Hilfer, R. Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000).

[2] Oldham, K. B. Fractional differential equations in electrochemistry," Advances in Engineering Software, vol. 41, no. 1, pp. 9-12, (2010).

[3] Debnath, L. Recent applications of fractional calculus to science and engineering. Int. J. Math. Math. Sci. 54, 3413-3442 (2003).

[4] Hamoud, A. A., Kechar, C., Ardjouni, A., Emadifar, H., Kumar, A., Laith Abualigah, L. On Hadamard-Caputo implicit fractional integro-differential equations with boundary fractional conditions, Kragujevac Journal of Mathematics 50, no. 3 (2026), 491-504.

[5] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Boston (2006)

[6] Hamoud, A. A., Mohammed, N. M., Emadifar, H., Parvaneh, F., Faraidum K. Hamasalh, F. K., Sahoo, S. K., Khademi,

International Journal of Applied Mathematics

Volume 38 No. 10s, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

M. Existence, uniqueness and HU-stability results for nonlinear fuzzy fractional Volterra-Fredholm integro-differential equations, International Journal of Fuzzy Logic and Intelligent Systems 22, no. 4, 391-400, (2022).

[7] Anwar, A. K., and Murad, S. A. Existence and Ulam stability of solutions for Caputo Hadamard fractional differential equations, General Letters in Mathematics (GLM) 12, no. 2 (2022).

[8] Jameel, S. M., Hamoud, A. A. On nonlinear generalized Caputo fractional implicit Volterra-Fredholm model, Discontinuity, Nonlinearity, and Complexity 14, no. 02, 439-450, (2025).

[9] Jameel, S. A. M., Rahman, S. A., Hamoud, A. A. Analysis of Hilfer fractional Volterra-Fredholm system, Nonlinear Functional Analysis and Applications, 259-273, (2024).

[10] Sharif, A. A., Hamoud, A. A., Maha M. Hamood, M. M., Ghadle, K. P. ON new uniqueness results for Riemann-Liouville fractional Volterra-Fredholm integro-differential equations with deviating arguments, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis, 32, 25-40, (2025).

[11] Ibrahim, R.W. Ulam stability of boundary value problem. Kragujev. J. Math., 37, 287–297, (2013).

[12] Benchohra, M., Lazreg, J.E. Existence and Ulam-stability for nonlinear implicit fractional differential equations with Hadamard derivative. Stud. Univ. Babes-Bolyai Math, 62(1), 27-38, (2017).

[13] Murad, Shayma Adil. ”Certain analysis of solution for the nonlinear two-point boundary value problem with Caputo fractional derivative.” Journal of Function Spaces 2022 (2022).

[14] Dai, Q., Gao, R. M., Li, Z., Wang, C. J. Stability of Ulam-Hyers and Ulam-Hyers-Rassias for a class of fractional differential equations, Adv. Differ. Equ., 2020, 95-103, (2020).

Received: October 10, 2025

International Journal of Applied Mathematics

Volume 38 No. 10s, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

[15] Abdulmaged, R. B., Abdulqader, A. J. Some new results on controllability for nonlocal impulsive fractional Volterra-Fredholm model. Journal of Mathematics and Computer Science, 40, no. 2, 196–208, (2026).

[16] Sousa, J. V., Da Oliveira, E. C. Ulam-Hyers stability of a nonlinear fractional Volterra integro-differential equation, Appl. Math. Lett., 81, 50-56, (2018).

[17] Qun Dai, Shidong Liu, Stability of the mixed Caputo fractional integro-differential equation by means of weighted space method, AIMS Mathematics, 7(2), 2498-2511, (2022).

[18] Murad, Shayma Adil, Hussein Jebrail Zekri, and Samir Hadid. Existence and uniqueness theorem of fractional mixed Volterra-Fredholm integrodifferential equation with integral boundary conditions. International Journal of Differential Equations 2011, 1-14, (2011).

[19] Mardanov M.J., Sharifov Y.A., Aliyev H.N., Existence and uniqueness of solutions for nonlinear fractional integro-differential equations with nonlocal boundary conditions, European Journal of Pure and Applied Mathematics, Vol.15, No.2, 726-735, (2022).

[20] Hamoud, A. Existence and uniqueness of solutions for fractional neutral volterra-fredholm integro differential equations, Advances in the Theory of Nonlinear Analysis and its Application 4 (4), 321-331 (2024).

[21] Hussain, K.H., Hamoud, A.A. Mohammed, N.M. Some new uniqueness results for fractional integro-differntial equations, Nonlinear Functional Analysis and Applications 24 (4), 827-836 (2019).

[22] Vivek, D., K. Kanagarajan, and E. M. Elsayed. A study of fractional integro-differential equations via Hilfer-Hadamard

Received: October 10, 2025

fractional derivative. General Mathematics 27, no. 1 (2019): 71-84.

[23] Ahmad B, Ntouyas SK. Hilfer-Hadamard Fractional Boundary Value Problems with Nonlocal Mixed Boundary Conditions. Fractal and Fractional. 2021; 5(4):195. https://doi.org/10.3390/fractalfract5040195

[24] Vivek, D.; Shahb, K.; Kanagarajan, K. Dynamical analysis of Hilfer-Hadamard type fractional pantograph equations via successive approximation. J. Taibah Univ. Sci. 2019, 13, 225-230.

[25] Bachira, F.S.; Abbas, S.; Benbachir, M.; Benchohra, M. Hilfer-Hadamard fractional differential equations; Existence and Attractivity, Adv. Theory Nonl. Anal. Appl. 5 (2021), 49-57.

[26] Promsakon, C., Sotiris, K. N., Jessada, T. Hilfer-Hadamard nonlocal integro-multipoint fractional boundary value problems. Journal of Function Spaces 2021 (2021), 1-9.