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Mathematical study of a model with a density-dependent contact rate

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Abstract

In this manuscript, we consider a parabolic epidemic system modelling the transmission of diseases like Hepatitis B with a general density dependent rate in a closed population and bounded domain. Our model takes into account diffusion of population as well as two classes of infected individuals: the chronic carriers and the acute infected humans. Based on realistic assumptions, we study the existence of a solution for the model and look for steady states using critical points or perturbation arguments.

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1 Introduction

The role of the spatial diffusion is important in the comprehension of the spatial dynamic of the disease (see Bengfort et al[4], Skellam [27], Ducrot et al.[11] and references therein). Another important parameter is the form of contact rate (see discussion in Thieme[30]). Combine these two features in one model with a general density[1] dependent rate is sometime difficult since we need to discuss about the existence of a solution for the model. One disease is concerned by our model: Hepatitis B (for a good review of the disease, see [12, 13, 14, 25, 16, 35]). Its principal route of contamination in Africa is the horizontal one[23, 33, 34], especially in Cameroon[12, 20]. Moreover Hepatitis B virus genotypes show a characteristic geographic distribution with a proposed association with human migration[16]. We construct a model by adding a continuous space structure to the ordinary differential equation proposed by Bonzi et al.[5, 15]. We neglect vertical transmission[3]. We assume that the strength of diffusion is homogeneous in space. Then quoting Bengfort et al[4], Fick's and Fokker-Planck's laws of diffusion are identical. Our aim is to explore the existence of a solution for the model with a focus in the bilinear contact case sketched as "[transmission rate] × [infectious]^m × [susceptible individuals]$^\alpha$" with powers m and α in $(0; 1)$. The note is organized as follow. We present the model in the next section and analyse the existence of a solution for the parabolic and elliptic systems linked to our model in the section before the discussion.

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[1] Considering densities is sometime biologically more relevant[18].

2 Preliminary

In this model the spatial variable x belongs to a bounded set $\Omega \subset R^N$ (with $N \in N - \{0\}$) with a regular boundary and a general density dependent "immediate" contact rate[2] is defined as $\lambda_m(t,x)S(t,x)^\alpha$ with $\alpha, m \geq 0$, $\lambda_m(t,x) = \beta_i i^m(t,x) + \beta_e e^m(t,x)$ and

$$\begin{cases} \partial_t S(t,x) - d_S \Delta S(t,x) = & CS - \lambda_m(t,x)S^\alpha - \mu(x)S \\ \partial_t i(t,x) - d_i \Delta i(t,x) = & -\left(d_1(x) + \mu(x)\right) i(t,x) + p\lambda_m(t,x)S^\alpha \\ \partial_t e(t,x) - d_e \Delta e(t,x) = & -\left(d_2(x) + \mu(x)\right) e(t,x) + q\lambda_m(t,x)S^\alpha \\ S(t,x') = & 0 \text{ with } t \geq 0 \text{ and } x' \in \partial\Omega \\ i(t,x') = & 0 \text{ with } t \geq 0 \text{ and } x' \in \partial\Omega \\ e(t,x') = & 0 \text{ with } t \geq 0 \text{ and } x' \in \partial\Omega \\ S(0,x) = & S_0(x) \geq 0 \text{ with } x \in \Omega \\ i(0,x) = & i_0(x) \geq 0 \text{ with } x \in \Omega \\ e(0,x) = & e_0(x) \geq 0 \text{ with } x \in \Omega \end{cases} \tag{1}$$

with $p + q = 1$ (p and q are two positive constants), the laplacian $\Delta := \Delta_x$ and C the strictly positive influx rate through births of the susceptible individuals. Here $S(t,x)$ denotes the space-specific density of the susceptible individuals, $e(t,x)$ and $i(t,x)$ respectively denotes the space-specific density of chronic carriers and acute infected individuals (that can be symptomatic or asymptomatic). We consider for the long term behaviour of the system that the density of recovered individuals is decoupled from the model (1) since we are working with a class of immunizing diseases[5]. Moreover to perform our analysis we shall assume that the contacts between individuals are homogeneous among the different cohorts. We assume that (positive transmission rates) β_i, β_e and (death rates sufficiently smooth) μ, $\mu + d_1$, $\mu + d_2$ are positive and bounded (that means

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[2] In fact a more accurate force of infection[30] is $\lambda_m(t,x,i,e) = \int_\Omega p_1(t,x,y)(i(t,y) + e(t,y))dy$ whose second order expansion may lead to simple diffusion through e.g. the kernel $p_1(t,x,y) \equiv k(x-y)$ [21, Li et al (2009)].

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values in a positive and bounded interval with a finite supremum norm). S_0, i_0 and e_0 are also sufficiently smooth, positive and also bounded. We will consider Dirichlet conditions with a C^1 regular domain Ω. Roughly speaking, Dirichlet one induces (with enough regularity on $\partial\Omega$) that the normal vector is inward the domain: in some sense boundary $\partial\Omega$ repels individuals inside Ω [28, Corollary 2.3, page 124].

The model (1) re-writes in the following abstract form (E_{q,β_i,β_e})

$$\begin{cases} \partial_t X(t,x) = D\Delta X - MX - F(X,q,\beta_i,\beta_e) \\ X(t,x') = 0 \text{ with } t \geq 0 \text{ and } x' \in \partial\Omega \\ X(0,x) = X_0(x) \geq 0 \text{ with } x \in \Omega \end{cases} \tag{2}$$

with two real numbers p and q in $[0;1]$ such that $p+q=1$,

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} := \begin{pmatrix} S \\ i \\ e \end{pmatrix},$$

$$X_0 = \begin{pmatrix} S_0 \\ i_0 \\ e_0 \end{pmatrix},$$

$$D = \begin{pmatrix} d_S & 0 & 0 \\ 0 & d_i & 0 \\ 0 & 0 & d_e \end{pmatrix} \text{ and } M = \begin{pmatrix} \mu & 0 & 0 \\ 0 & (\mu + d_1) & 0 \\ 0 & 0 & (\mu + d_2) \end{pmatrix}.$$

Moreover we set

$$B = \begin{pmatrix} C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $F(X) \equiv F_{q,\beta_i,\beta_e}(X) = F(X,q,\beta_i,\beta_e) = BX + X_1^{\alpha}\lambda_m(t,x) \begin{pmatrix} 1 \\ q-1 \\ -q \end{pmatrix}.$

Received: October 30, 2025

3 Existence of a solution for the system (2) and study of the steady state(s)

3.1 Existence of a solution for the system (2)

The problem (1) (with appropriate boundary conditions) with $m, \alpha \geq 1$ is wellposed by using (analytic) positive semigroup generated by the 3-dimensional Laplacian to construct a classical positive solution through a variation of constants formula and maximum principles[8, 28] and assuming enough regularity on initial conditions (at $t = 0$, on boundary, on linear and non-linear operators of the reaction diffusion system over a bounded domain $\Omega \subset R^N$)[28, (Theorem 3.1, page 127) and (Corollary 3.2, page 129)]. We will pay attention in most part of this work on the case $m, \alpha < 1$ with Dirichlet conditions where methods of Carrero and Lizana[8] and Hal Smith[28] should be adapted since $F(X)$ is not twice continuously differentiable in classic sense. Hal Smith[28] presents some results on the stability of equilibria when F is quasimonotone and irreducible. Moreover the case $m, \alpha \geq 1$ will be discussed in some Sobolev spaces like $H^1(\Omega)$.

We study the existence and unicity of a solution for the system (2) with $x \in \Omega = R^N$ using ideas of Theorem 5.1 in page 476 of J. L. Lions[22]. Boundary conditions are then understood in the limit sense. We define new variables U and V in the corresponding systems because of the iterative process used later in Proposition 3.1 for the proof of the existence of respective weak solution of (1). V is a version of U for the iterative process described in Proposition 3.1. We set then

$$U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \text{ and } V = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \text{ with } \Phi(U) = \begin{pmatrix} \Phi_1(U_1) \\ \Phi_2(U_2) \\ \Phi_3(U_3) \end{pmatrix} := \begin{pmatrix} \mu(x)U_1 - C\Psi(U_1) \\ (d_1 + \mu)U_2 \\ (d_1 + \mu)U_3 \end{pmatrix}.$$

Received: October 30, 2025

Moreover we consider

$$W(V) = \begin{pmatrix} W_1(V) \\ W_2(V) \\ W_3(V) \end{pmatrix}^T := \begin{pmatrix} -(\beta_i {V_2}^m + \beta_e {V_3}^m) \\ p\beta_i {V_1}^\alpha \\ q\beta_e {V_1}^\alpha \end{pmatrix}^T$$

$$\text{and } \; G(V) = \begin{pmatrix} G_1(V) \\ G_2(V) \\ G_3(V) \end{pmatrix} := \begin{pmatrix} 0 \\ p\beta_e {V_1}^\alpha {V_3}^m \\ q\beta_i {V_1}^\alpha {V_2}^m \end{pmatrix}.$$

X, Φ, V and G are column vectors and W is a row vector. We work on the linear case $\Psi(U_1) = U_1$ considered in $W_1(V)$, first component of $W(V)$. We denote $diag(W) := \begin{pmatrix} W_1 & 0 & 0 \\ 0 & W_2 & 0 \\ 0 & 0 & W_3 \end{pmatrix}$ and $W * U := diag(W).U$ as matrix product of matrix $diag(W)$ by column vector U. We define U^ν as

$$U^\nu := \begin{pmatrix} U_1^{\nu(1)} \\ U_2^{\nu(2)} \\ U_3^{\nu(3)} \end{pmatrix} := \begin{pmatrix} U_1^\alpha \\ U_2^m \\ U_3^m \end{pmatrix}.$$

Wherein the equation (2) inspire (if $U = V$ over $\Omega = R^N$, with $N \in N$ natural number > 0) the following system for our fixed point method

$$\begin{cases} \partial_t U - D\Delta U(t,x) + \Phi(U) = W(V) * U^\nu + G(V) \\ \qquad\qquad\qquad\qquad U(0,x) = U_0(x) \geq 0 \text{ with } x \in \Omega = R^N. \end{cases} \tag{3}$$

We will work on each row $k \in \{1;2;3\}$ of (3) and consider (for $u = U_k$ and small values of $u_{0,k}(x) = U_k(0,x)$)[22]. We follow for (3), the methods of J. L. Lions described in Theorem 5.1 in page 476[22]. Since the matrix D is invertible for $d_S, d_i, d_e > 0$, we assume for sake of simplicity that D is the identity matrix instead of using a change of variables in space through the diffusivity coefficients $d_S, d_i, d_e > 0$. We obtain then the following fixed point strategy for our problem (3). We set $T(x,y,t) = \frac{1}{(4\pi t)^{N/2}} exp\left(-\frac{|x-y|}{4t}\right)$ and the operators H and K_V defined respectively for Hölder continuous functions u_0 and U respectively defined on R^N and $[0;+\infty) \times R^N$ with a specific weighted norm[3] and the following definitions for $k \in \{1;2;3\}$:

$$\begin{cases} H(u_{0,k})(t,x) = \displaystyle\int_{R^N} T(x,y,t) u_{0,k}(y) dy \\ K_V(U_k)(t,x) = \displaystyle\int_0^t d\sigma \int_{R^N} T(x,y,t-\sigma)\left(-\Phi_k(U_k)(y) + W_k(U)(y)U^{\nu(k)}(y) + G_k(U(y))\right) dy \end{cases} \tag{4}$$

Then for sufficiently large[4] $N \in N^*$or sufficiently small initial condition u_0, there exists a solution U of (3) satisfying the nonlinear integral equation for each $k \in \{1;2;3\}$:

$$U_k = H(u_{0,k}) + K_V(U_k) \tag{5}$$

Proof. We define (for $n \in N$) the sequence

$$\begin{cases} u_1 = H(u_{0,k}) \\ u_{n+1} = H(u_{0,k}) + K_V(u_n). \end{cases} \tag{6}$$

Following ideas of [22, Theorem 5.1 in page 476] with with $\nu > 0$ and $N \in N^0$, we set $\kappa > 0$ small enough and choose the norm

$$\begin{cases} |||\phi||| = sup_{x \in R^N, t \geq 0} \dfrac{|\phi(x,t)|}{\rho(t,x)} \\ \rho(t,x) = T(x,0,\kappa). \end{cases} \tag{7}$$

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[3] See (7): $|||\phi||| = sup_{x \in R^N, t \geq 0} \frac{|\phi(x,t)|}{\rho(t,x)}$ where $\rho(t,x) = \rho = T(x,0,\kappa)$

[4] We recall that N is the set of nonnegative integers or natural numbers and N^* is the set of positive integers or nonzero natural numbers.

Our strategy is working on each row k separately and let the other U-variables (collected in V) in some nonlinear terms (including $W(V)$ but excluding $U^{\nu(k)}$) and use then Schauder's Fixed point result for operator K_V to obtain the resulting solution to (3) depending on V. As pointed by [22], for small values of the three components of the initial data $u_{0,k}$ (with $0 \leq u_{0,k} \leq \delta T(x,0,\kappa)$, $\delta = (4\pi\kappa)^{N/2}\sigma$) with an arbitrary number $\sigma > 0$, we get (with $c_0, c_1, c_2 \geq$)

$$|||u_{n+1}||| \leq \delta + c_0|||u_n|||^{\nu(k)+m} + c_1(|||u_n||| + |||u_n|||^{\nu(k)})$$

Then from a discrete Gronwall result, there exists a function (independent of n) $M : \delta M(\delta)$ (with $lim_{\delta\to 0}M(\delta) = 0$) such that $|||u_n||| \leq M(\delta)$. Moreover

$$|||u_{n+1} - u_n||| = |||K_V(u_n) - K_V(u_{n-1})|||$$

and

$$|||K_V(u_n) - K_V(u_{n-1})||| \leq c_2.(\delta.(M(\delta)+1)^{\nu(k)-1}) + s_p$$

where

$$s_p := sup_{x\in R^N}(|\mu(x) + d_1(x) + d_2(x) + C|)|||u_n - u_{n-1}|||$$

if $c_2.(\delta.(M(\delta)+1)^{\nu(k)-1}) + sup_{x\in R^N}(|\mu(x) + d_1(x) + d_2(x) + C|) < 1$ (where c_0, c_1, c_2 are constants), then discrete Gronwall property shows that

$$lim_{n\to+\infty}|||u_{n+1} - u_n||| = 0.$$

Thus $\{u_n\}$ is a Cauchy sequence (for small values of $u_{0,k}$ and Φ coefficients -mortalities and death rates-). Then $\{u_n\}$ admits limits u_k^V which is solution of (6).

We consider now a map $g : V \mapsto g(V) = (u_k^V)_{k\in\{1;2;3\}}$ as [22, Theorem 5.1 in page 476]. Using a fixed point theorem as Schauder's one, one can prove the existence of a classic solution $U = u^* = (u_k^*)_{k\in\{1;2;3\}}$ of (3). $\square$

3.2 Some methods for the characterization of equilibrium

An equilibrium X of (2) satisfies

$$-D\Delta X = -MX - F(X, q, \beta_i, \beta_e). \tag{8}$$

3.2.1 Energy and critical points

We consider $[X, Y]$ as the usual scalar product in R^N. The idea is to generate an energy function J of the system (8) and prove that gradient of J vanish at equilibrium which is then a critical point of J. Practically we start by finding an antiderivative (in X) $\tilde{F}$ of $D^{-1}F$. Then through a variational method (with a Dirichlet boundary condition) [11, 31], energy J could take the following form (with $q = \beta_j$):

$$J_{q,\beta_e}(X) = \frac{1}{2}\int_\Omega [\nabla X, \nabla X]dx + \frac{1}{2}\int_\Omega [D^{-1}MX, X]dx + \int_\Omega [\tilde{F}(X, q, \beta_e)]dx \tag{9}$$

We recall the basic "Implicit Function Theorem" generally admitted[2, 24] (see also [Theorem A, page 339 in Appendix A [9]]).

Theorem 1 ([2, 24]). *Let F be a function in $C^k(T \times U, Y)$, $k \geq 1$, where Y is a Banach space and T with U are respectively open subsets of the Banach spaces Z and X. Suppose that $F(\lambda^*, u^*) = 0$, with $(\lambda^*, u^*) \in T \times U$ and $D_uF(\lambda^*, u^*)$, the partial derivative of F with respect to u evaluated at the (λ^*, u^*), is invertible from X to Y. Then there exist neighbourhoods Θ of λ^* in Z and U^* of u^* in X and a map $g \in C^k(\Theta, X)$ such that*

- *i)* $F(\lambda, g(\lambda)) = 0$ *for all* $\lambda \in \Theta$;
- *ii)* $F(\lambda, u) = 0$, $(\lambda, u) \in \Theta \times U^*$, *implies* $u = g(\lambda)$;
- *iii)* $g'(\lambda) = -[D_uF(\lambda, g(\lambda))]^{-1} \circ D_\lambda F(\lambda, g(\lambda))$, $\lambda \in \Theta$.

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Let us now give a basic definition of a critical point. Let H be a Hilbert space whose norm and scalar product will be denoted, respectively, by $||.||$ and $(.|.)$.

Definition 2 ([24]). A critical point of a functional $G \in C^1(H, R)$ is an element $u \in H$ such that $\nabla G(u) = 0$, where ∇G denotes the gradient of G, defined through the relationship

$$\mathbf{d}G(u)[v] = (\nabla G(u)|v) \ , \ \forall v \in H$$

where $\mathbf{d}G(u)[v]$ denotes the Frechet derivative of G with respect to u applied to v.

3.2.2 Another method for small values of β_e: perturbation and bifurcation

We start by presenting the definition of a Bifurcation point.

Definition 3 ([9]). Let W and Y be real Banach spaces, Ω an open subset of W, $\omega : I \to \Omega$, where I an interval, and $G : \Omega \to Y$ be continuous map. Suppose there is a simple arc C_* in Ω given by $C_* = \{\omega(t), t \in I\}$ such that $G(\omega) = 0$. If there is a number $\tau \in I$ such that every neighbourhood of $\omega(\tau)$ contains zeros of G not in C_*, then $\omega(\tau)$ is called a Bifurcation point for the equation $\omega(\tau) = 0$ with respect to the curve C_*.

In many situations W is of the form $R \times X$, where X is a real Banach space, and

$$C_* = \{(\lambda; 0)|\lambda \in R, 0 \in X\}.$$

The basic problem of bifurcation theory is the bifurcation points for $G = 0$ with respect to C_* and $G^{-1}\{0\}$ near such points.

[[9]] A necessary condition for $\lambda^* \in R$ to be a bifurcation point for $G(u, x) = 0$ is that $G_u(\lambda^*; 0)$ is not invertible.

The perturbation methods as Crandall-Rabinowitch method[9] are based on Implicit function Theorem 1. Let us define $\phi(X, q, \beta_i, \beta_e)$:

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$C^{2,\alpha} \to C^{0,\alpha}$ with $C^{\pi,\alpha}$ a Hölder space, $\pi \in \{0;2\}$, $\alpha \in (0;1)$ and

$$\phi(X,q,\beta_i,\beta_e) = D\Delta X - MX - F(X,q,\beta_i,\beta_e).$$

Here, we could study the problem $(E_{0,0}) : \phi(X,q,\beta_i,\beta_e) = 0$ as solution of equation (8), for possible steady state(s) of (2) similarly to Tavares[31][5] and use perturbation arguments to solve the case (E_{q,β_i,β_e}) for small values of q and β_e (see [5]) as biological assumption: the third coordinate, the number/density of latent infectives for hepatitis B virus received a very few part of new infectives (for small q) and are less infective (for small β_e) than the acute infectives seen as the second coordinate.

In fact if for some X^*, $D_X\phi(X^*,0,0)$ is invertible then there exists steady state(s) $X = X(q,\beta_e)$[6] lying in a neighbourhood of X^* for the semiflow of (2). But the converse ($D_X\phi(X^*,0,0)$ not invertible) leads through a bifurcation method to possible different solutions $X = X(q,\beta_e)$ if the kernel of $D_X\phi(X^*,0,0)$ is of dimension 1.

3.2.3 A fixed point method for $N \geq 2$, $\alpha \in (0;1)$ and $m \in (0;+\infty)$

By setting with $\Omega \subset R^2$, $X_1 := S$, $X_2 := i$ and $X_3 := e$, any equilibrium X of problem (1) satisfies:

$$\begin{cases} -d_S\Delta X_1(x) = C\Psi(X_1) - \left(\lambda_m^0(t,x) + \mu(x)\right) {X_1}^{\alpha} \\ -d_i\Delta X_2(x) = -\left(d_1(x) + \mu(x)\right) X_2(x) + p\lambda_m^0(x){X_1}^{\alpha} \\ -d_e\Delta X_3(x) = -\left(d_2(x) + \mu(x)\right) X_3(x) + q\lambda_m^0(x){X_1}^{\alpha} \\ \quad X_1(x') = 0 \text{ with } x' \in \partial\Omega \\ \quad X_2(x') = 0 \text{ with } x' \in \partial\Omega \\ \quad X_3(x') = 0 \text{ with } x' \in \partial\Omega \end{cases} \tag{10}$$

with $\lambda_m^0(x) = \beta_i {X_2}^m(x) + \beta_e {X_3}^m(x)$. We slightly modify equation (10) in to the equation (3.2.3) in order to used a fixed point theorem

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[5] But we lack symmetry in coefficients of nonlinearities contrary to [31].

[6] $X(q,\beta_e)$ is a solution of equation (8).

Received: October 30, 2025

as Schauder through minimizing sequences compactness. We recall the **Rellich - Kondrachov** Theorem 4.

Theorem 4 (Rellich - Kondrachov, Part of Theorem 9.16 p. 290 [6])**.** *Suppose that in R^n, Ω is bounded and of class C^1. Then we have the following compact injections:*

** (a) If $n = 2$ (used in this subsection), then $H^1(\Omega)$ is continuously embedded in $L^q(\Omega)$, $\forall q \in [1; +\infty[$;*

** (b) If $n = 1$, then $H^1(\Omega)$ is continuously embedded in $C\left(\overline{\Omega}\right)$.*

and this important result

Theorem 5 (Poincaré inequality, Corollary 9.19, page 290 in [6])**.** *: Suppose that $1 < s \leq \infty$ and Ω is a bounded open set in R^N. Then there exists a constant θ (depending on Ω and s) such that:*

$$\|u\|_{L^s(\Omega)} \leq \theta \|\nabla u\|_{L^s(\Omega)}, \ \forall u \in W_0^{1,s}(\Omega).$$

We will mostly use this Theorem 5 in the case where $s = 2$ and $W_0^{1,2}(\Omega) = H_0^1(\Omega)$. Poincaré inequality remains true if Ω has finite measure and also if Ω has a bounded projection on some axis. We define both variables X and Y in the corresponding systems because of the iterative process used later in subsection 3.2.3 for the proof of the existence of respective weak solution of (10). Y is a version of X for the iterative process described in the subsection **3.2.3.2**. We set

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \text{ and } Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

We define also

$$\Phi(X) = \begin{pmatrix} \Phi_1(X_1) \\ \Phi_2(X_2) \\ \Phi_3(X_3) \end{pmatrix} = \begin{pmatrix} \mu(x)X_1 - C\Psi(X_1) \\ (d_1 + \mu)X_2 \\ (d_2 + \mu)X_3 \end{pmatrix}$$

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$$\text{and}\ \ W(Y) = \begin{pmatrix} W_1(Y) \\ W_2(Y) \\ W_3(Y) \end{pmatrix}^T = \begin{pmatrix} -(\beta_i {Y_2}^m + \beta_e {Y_3}^m) \\ p\beta_i {Y_1}^\alpha \\ q\beta_e {Y_1}^\alpha \end{pmatrix}^T$$

$$\text{with}\ \ G(Y) = \begin{pmatrix} G_1(Y) \\ G_2(Y) \\ G_3(Y) \end{pmatrix} = \begin{pmatrix} 0 \\ p\beta_e {Y_1}^\alpha {Y_3}^m \\ q\beta_i {Y_1}^\alpha {Y_2}^m \end{pmatrix}$$

X, Φ, Y and G are columns vectors and W is a row vector. In the first step for each $s \in \{1;2;3\}$, consider X_c as parameter in the row $s-th$ of (10) with $c \in \{1;2;3\} - \{s\}$. We work on the case $\Psi(X_1) = X_1$. From the equation (10), we add Dirichlet boundary condition and obtain

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X(x')= 0 with x'∈ $\partial\Omega$, (11)Where X^ν is defined as $X^\nu = \begin{pmatrix} X_1^{\nu(1)} \\ X_2^{\nu(2)} \\ X_3^{\nu(3)} \end{pmatrix}$ with $\begin{pmatrix} X_1^{\nu(1)} \\ X_2^{\nu(2)} \\ X_3^{\nu(3)} \end{pmatrix} := \begin{pmatrix} X_1^\alpha \\ X_2^m \\ X_3^m \end{pmatrix}$ and $X_0 = \begin{pmatrix} S_0 \\ i_0 \\ e_0 \end{pmatrix}$.

3.2.3.1. Critical point theory used for the system (3.2.3)**.** For each row $k \in \{1;2;3\}$ of (3.2.3) we solve

$$\begin{cases} -D_{kk}\Delta X_k + \Phi_k(X_k) = W_k(Y)X_k^{\nu(k)} + G_k(Y) \text{ in } \Omega \\ X_k(x') = 0 \text{ with } x' \in \partial\Omega, \end{cases} \tag{12}$$

re-written into the following compact form

$$\begin{cases} -D\Delta X = L(X,Y) \text{ in } \Omega \\ X(x') = 0 \text{ with } x' \in \partial\Omega, \end{cases} \tag{13}$$

such that for $k \in \{1;2;3\}$, $L = (L_k)_{k\in\{1;2;3\}}$ and $L_k(X,Y) = -\Phi_k(X_k) + W_k(Y)X_k^{\nu(k)} + G_k(Y)$. We obtain from (13) by multipling each row by a C^∞ test function v

$$\int_\Omega D_{kk}\nabla X_k \nabla v dx + \int_\Omega (\Phi_k(X_k) - W_k(Y)X_k^{\nu(k)})v dx - \int_\Omega G_k(Y) v dx = 0 \tag{14}$$

Since we consider the Dirichlet condition $X_k(x') = 0$ where $x' \in \partial\Omega$ with $\Phi_k^*(X_k)$ the antiderivative in X_k of $\Phi_k(X_k)$, then a solution $X_k(Y)$ of (14) is a critical point $(J_k'(X_k) = 0)$ of the energy function

$$\begin{cases} J_k(X_k) = & \frac{D_{kk}}{2}\int_\Omega |\nabla X_k|^2 - \frac{1}{(1+\nu(k))}\int_\Omega W_k(Y(X)){X_k}^{1+\nu(k)}(x)dx \\ & + \int_\Omega \Phi^*(X_k)dx - \int_\Omega G_k(Y)X_k(x)dx. \end{cases} \tag{15}$$

Let us define a minimizing sequence.

Definition 6. Let E be a metric space and $F : E \to R$ a map, $K \subset E$ a non empty set. A sequence $\{y_n\}_n \subset K$ is a minimizing sequence of F in K if

$$lim_{n\to+\infty}F(y_n) = inf_{x\in K}F(x)$$

Practically, solving (15) leads to the following minimizing problem (16) (with existence of a unique minimizing sequence with compactness through Rellich' theorem 4). If we define $J^*(X_k) := \frac{1}{2}\int_\Omega |\nabla X_k|^2 + \int_\Omega \Phi^*(X_k)dx$, then the candidate solution X satisfies through its components X_k:

$$inf_{\left\{\int_\Omega X_k W_k(Y(x))X_k(x)^{\nu(k)+1}dx=1\right\}} J^*(X_k). \tag{16}$$

Since the nonlinearity $X_k(x)^{\nu(k)+1}$ is homogeneous in X_k, the existence of a solution U_k can be found by setting roughly $U_k := \lambda^{\frac{1}{\nu(k)-1}}u_k^*$ for some Lagrange multiplier λ and u_k^* satisfying

$$-D_{kk}\Delta u_k^* + \Phi_k(u_k^*) = \lambda W_k(V){u_k^*}^{\nu(k)}$$

Finally:

$$-D_{kk}\Delta U_k + \Phi_k(U_k) = W_k(V){U_k}^{\nu(k)}$$

Then if the Palais-Smale condition (definitions 1.2.5, 1.2.6, theorem 1.2.8)[24, 1] is satisfied then

$$-D_{kk}\Delta U_k + \Phi_k(U_k) = W_k(V){U_k}^{\nu(k)}$$

Received: October 30, 2025

has a (nontrivial) solution.

Since $\Omega \subset R^N$:

1- **If we add transitions from acute infectious X_2 to carriers X_3 and the reverse transition, it would not significantly modified the schemes and ideas developed in this note.**

2- If we assume that the limits for $|x| \to +\infty$ of functions $d_u(x)$ and $\mu(x)$ for $u \in \{S, i, e\}$ are finite with appropriate conditions[1], then each functional J^* has a Mountain Pass critical point by a compactness condition obtained through the use of Palais-Small sequences.

3- The existence of a solution for each subsystem (12) of (13) of the model could be also studied using "Mountain Pass" methods[7, 17] with $N \geq 3$ in L^2 with appropriate antiderivative (in respect to space variable) of the right hand-side of each row of (13): there exists then a nontrivial and positive solution $X_k(Y)$ for each row $k \in \{1; 2; 3\}$ of (13).

3.2.3.2. A Solution of the system (3.2.3) **using an iterative strategy.**

We need the following Proposition 3.2.3 useful later to seek a solution $X = X(Y)$ of the system (3.2.3).

Assume that Ω is a bounded and smooth domain in R^N, $\frac{1}{1-\alpha} > 1$, $m = 1 - \alpha$, $q_0 = \frac{2}{1-\alpha} > 1$ for $\alpha \in (0; 1)$ (then $2 < q_0 < \frac{2N}{N-2}$ if $N > 3$; or for $2 < q_0 < +\infty$ if $N \in \{1; 2\}$) and $\mu, d_1, d_2 \in L^{q_0}(\Omega)$.

Then there exist three Ω-dependent constants N_1, N_2 and N_3 such that (for well suited parameters and eventually reducing the "size" of Ω or increasing/decreasing values of $d_S, d_i, d_e > 0$) these a priori estimates for the solution $X = X(Y)$ hold (for positive Ω-dependent constants $c_1, c_2, c_3, N_1, N_2, N_3$):

- $||X_1||_{H_0^1(\Omega)} \leq c_1^{\frac{1}{1-\alpha}} ||\nabla X_1||_{H_0^1(\Omega)} \leq N_1^{\frac{1}{1-\alpha}} \left[||Y_2||^m_{H_0^1(\Omega)} + ||Y_3||^m_{H_0^1(\Omega)} + ||\mu||_{L^{q_0}(\Omega)} \right]^{\frac{1}{1-\alpha}}$

- $||X_2||_{H_0^1(\Omega)} \leq c_2^{\frac{1}{1-\alpha}} ||\nabla X_2||_{H_0^1(\Omega)} \leq N_2^{\frac{1}{1-\alpha}} \left[||Y_1||^m_{H_0^1(\Omega)} + ||Y_3||^m_{H_0^1(\Omega)} + ||\mu + d_1||_{L^{q_0}(\Omega)}\right]^{\frac{1}{1-\alpha}}$
- $||X_3||_{H_0^1(\Omega)} \leq c_3^{\frac{1}{1-\alpha}} ||\nabla X_3||_{H_0^1(\Omega)} \leq N_3^{\frac{1}{1-\alpha}} \left[||Y_1||^m_{H_0^1(\Omega)} + ||Y_2||^m_{H_0^1(\Omega)} + ||\mu + d_2||_{L^{q_0}(\Omega)}\right]^{\frac{1}{1-\alpha}}$.

Sketch of proof. Using Young's $\left(\text{for } 1 < s < \infty, a \geq 0, b \geq 0 : ab \leq \frac{1}{s}a^s + \frac{s-1}{s}b^{\frac{s}{s-1}}\right)$ with Hölder's inequalities[6, Theorem 4.6, page 92] jointly with Rellich - Kondrachov Theorem 4 and Poincaré inequality Theorem 5, it can be shown using relation (14) with $v = X_k$, for each row $k \in \{1; 2; 3\}$ and $\alpha \in (0; 1)$ with Sobolev embeddings[6], the following result:

- $||X_1||^{1-\alpha}_{H_0^1(\Omega)} \leq c_1(\Omega) ||\nabla X_1||^{1-\alpha}_{H_0^1(\Omega)} \leq N_1(\Omega, \frac{1}{d_S}) \left[||Y_2||^m_{H_0^1(\Omega)} + ||Y_3||^m_{H_0^1(\Omega)} + ||\mu||_{L^{q_0}(\Omega)}\right]$
- $||X_2||^{1-\alpha}_{H_0^1(\Omega)} \leq c_2(\Omega) ||\nabla X_2||^{1-\alpha}_{H_0^1(\Omega)} \leq N_2(\Omega, \frac{1}{d_i}) \left[||Y_1||^m_{H_0^1(\Omega)} + ||Y_3||^m_{H_0^1(\Omega)} + ||\mu + d_1||_{L^{q_0}(\Omega)}\right]$
- $||X_3||^{1-\alpha}_{H_0^1(\Omega)} \leq c_3(\Omega) ||\nabla X_3||^{1-\alpha}_{H_0^1(\Omega)} \leq N_3(\Omega, \frac{1}{d_e}) \left[||Y_1||^m_{H_0^1(\Omega)} + ||Y_2||^m_{H_0^1(\Omega)} + ||\mu + d_2||_{L^{q_0}(\Omega)}\right]$.

□

The purpose of the iterative strategy is the following claim: We construct a map $\Gamma : Y \mapsto X(Y)$ from $(H_0^1(\Omega))^3$ to $(H_0^1(\Omega))^3$ such that $X(Y)$ is a solution of

$$\begin{cases} -D\Delta X(Y) = L(X(Y), Y) \text{ in } \Omega \\ X(Y)(x') = 0 \text{ with } x' \in \partial\Omega. \end{cases} \tag{17}$$

The idea is to assume that the Lebesgue measure of Ω is small enough and then use inequalities from Proposition 3.2.3 in order to construct a ball $B(0; R)$ (of finite radius R and centred at 0) globally invariant in $H_0^1(\Omega)$ by Γ since there is a continuous and compact injection from $H_0^1(\Omega)$ into $L^s(\Omega)$, for all $1 < s < +\infty$. Then using a fixed point theorem (as Shauder theorem) to obtain a function $Y_0 = X(Y_0)$ (with X continuously dependent of X_0), one gets

$$\begin{cases} -D\Delta Y_0 = L(Y_0, Y_0) \text{ in } \Omega \\ Y_0(x') = 0 \text{ with } x' \in \partial\Omega, \end{cases} \tag{18}$$

And Y_0 is called a mild solution of (3.2.3) if each of its k-correspondent coordinate is solution of (14).

Construction of an iterative sequence: With this method, we solve iteratively the problem (3.2.3) by setting iterates $(X_{[n]})_n$ as follow.

Step 0: we solve (even row by row similarly to the Lax-Milgram theorem (Corollary 5.8, page 140[6]), using the Schauder theorem (Theorem 9.33, page 317[6]) in the framework Hölder spaces, the (De Giorgi, Nash, Stampacchia, Theorem 9.34) result or the Duality method, described in page 318[6])

$$\begin{cases} -D\Delta X_{[0]} = -\Phi(X_{[0]}) \text{ in } \Omega \\ X_{[0]}(x') = 0 \text{ with } x' \in \partial\Omega, \end{cases} \tag{19}$$

Step n: For $n \in N^*$ and knowing $X_{[n-1]}$, we solve similarly to the Step [0]:

$$\begin{cases} -D\Delta X_{[n]} = W(X_{[n-1]})X_{[n]}^{\nu} - \Phi(X_{[n]}) + G(X_{[n-1]}) \text{ in } \Omega \\ X_{[n]}(x') = 0 \text{ with } x' \in \partial\Omega. \end{cases} \tag{20}$$

The main idea here is to prove that this sequence $(X_{[n]})_n$ is of Cauchy and bounded in the complete normed linear space $(H_0^1(\Omega))^3$ and use a compactness argument to prove that the limit X^* of $(X_{[n]})_n$ exists and satisfies (17). If we assume enough regularity on W and G, [6, Corollary 9.37, page 320] can be adapted to prove the positivity of the solution $X_{[n]}$ of (20) at each step n.

4 Discussion

We study existence of a solution for the system (1) and, under some suitable assumptions, we show how to seek equilibrium using critical points or perturbation arguments. Pao[26, Example (6.2) page 27] shows that a system similar to (2) with an Hölder continuous function F could have multiple solutions even in the case $\Omega = [a; b]$ in R. It would be interesting to extend this work by considering Neumann or Robin type boundary conditions[28, 26]. The main difficulty (and perspective) is to study travelling-wave solutions and diffusion driven instability for this system, under appropriate conditions on parameters[19, 8, 32]. Another perspective could be to introduce age structure, a vaccination strategy and migration. But the main problem which could arise then in mathematical analysis will be the control of the parameters in order to avoid the blow-up of interesting solution(s). Bengfort et al[4] state that Fick's and Fokker-Planck's laws of diffusion yield very different results if there are spatial heterogeneities. It could be challenging to study mathematically (in the general cases with spatial heterogeneities even with singularities or variable exponents), the impacts of these laws on wellposedness and dynamical properties of our model (2). Another guess will be to study the following general system of equations with spatial and ages structures

$$\begin{cases} (\partial_t + \partial_a)s(t,a,x) = d\Delta s(t,a,x) - \mu s(t,a,x) - \lambda(t)s(t,a,x), \quad t > 0,\ a > 0, \\ s(t,0,x) = \Lambda(x), \\ (\partial_t + \partial_\tau)i_k(t,\tau,x) = D_{kk}\Delta i_k(t,\tau,x) - (\mu + \gamma_k)i_k(t,\tau,x),\ t > 0,\ \tau > 0,\ k = 1, \cdots, N_0, \\ i(t,0,x) = \lambda(t,x)\int_0^A p_k(a)s(t,a,x)da, \\ \lambda(t,x) = \sum_{k=1}^{N_0}\int_0^A \beta_k(\tau)i(t,\tau,x)d\tau, \\ N_0 \in N - \{0\} \end{cases}$$

posed on some spatial bounded domain Ω and supplemented together with no flux boundary conditions on $\partial\Omega$, using the results of Ducrot et al.[10] for one compartment of infectious ($k = 1$). For individuals of the population studied, A denotes the maximal age,

a the chronological age and τ the age of infection.

References

[1] A. Ambrosetti and Andrea Malchiodi, Perturbation methods and semilinear elliptic problems on r^n, *Springer Science & Business Media*, **240** (2006).

[2] A. Ambrosetti and G. Prodi, A Primer of Nonlinear Analysis, *Cambridge studies in advanced mathematics*, **34** (1993).

[3] Anfumbom K.W. Kfutwah, M. C. Tejiokem and R. Njouom A low proportion of HBeAg among HBsAg-positive pregnant women with known HIV status could suggest low perinatal transmission of HBV in Cameroon, *Virology Journal*, **9:62** (2012).

[4] M. Bengfort, H. Malchow, F. M. Hilker, The Fokker-Planck law of diffusion and pattern formation in heterogeneous environments, *J. Math. Biol.*, **DOI 10.1007/s00285-016-0966-8** (2016).

[5] B. Bonzi, A. A. Fall, A. Iggidr, and G. Sallet, Stability of differential susceptibility and infectivity epidemic models, *J. Math. Biol.*, **81** (1981), 20-47.

[6] H. Brézis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, *Universitext coll.*, (2010).

[7] H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Universitext coll.*, **XXXV** (1983), 437-477.

[8] G. Carrero and M. Lizana, Pattern formation in a SIS epidemiological mode, *Canadian Applied Mathematics Quaterly*, **11, Number 1** (2003).

Received: October 30, 2025

[9] M. G. Crandall and P. H. Rabinowitz, Bifurcation from simple eigenvalues, *Journal of Functional Analysis*, **8, no. 2** (1971), 321-340.

[10] A. Ducrot and P. Magal, Travelling wave solutions for an infection-age structured epidemic model with external supplies, *Nonlinearity*, **24** (2011), 2891-2911.

[11] A. Ducrot and M. Langlais, Global weak solution for a singular two component reaction-diffusion system, *Bull. London Math. Soc.*, **46 (1)** (2014), 1-12.

[12] W.J. Edmunds, G. F. Medley, D. J. Nokes, A. J. Hall, H. C. Whittle, The influence of age on the development of the hepatitis B virus (HBV) carrier state, *Proc. Roy. Soc. London Series B*, **253** (1993), 197-201.

[13] W. J. Edmunds, G. F. Medley, D. J. Nokes, C. J. O'Callaghan, H. C. Whittle and A. J. Hall, Epidemiological patterns of hepatitis B virus (HBV) in highly endemic areas, *Epidemiol Infect.*, **117** (1996), 313-325.

[14] W. J. Edmunds, G. F. Medley, D. J. Nokes, The transmission dynamics and control of hepatitis B virus in the Gambia, *Stat. Med.*, **15** (1996), 2215-2233.

[15] A. A. Fall, G. Sallet, A. Iggidr, Modélisation de la transmission verticale de l'Hépatite B, *CARI*, **81** (1981), 20-47.

[16] S. M. Jazayeri, S. M. Alavian and W. F. Carman, Hepatitis B virus: origin and evolution, *Journal of Viral Hepatitis*, **17** (2010), 229-235.

[17] I. Ghaffour, Régularité des Solutions d'EDPs Elliptiques: Inégalité de Caccioppoli (Regularity of elliptic PDE' solutions: Caccioppoli inequality), *Msc thesis, University Abou Bekr Belkaid - Tlemcenn, Algeria.*, (2011).

[18] A. L. Greer, Cheryl J. Briggs and James P. Collins, Testing a key assumption of host-pathogen theory: density and disease transmission, *Oikos*, **Volume 117, Issue 11** (2008), 1667-1673.

[19] W. Huang, Uniqueness of traveling wave solutions for a biological reaction-diffusion equation, *J. Math. Anal. Appl.*, **316** (2006), 42-59.

[20] Y. Kouakep T., D. E. Houpa D., Optimal test strategies for hepatitis B vaccination with no vertical transmission, *Gen. Math. Notes*, **20** (2014).

[21] T. Li, Y. Li, H.W. Hethcote, Periodic travelling waves in SIRS endemic models, *Mathematical and Computer Modelling*, **49** (2009), 393-401.

[22] J. L. Lions, Some methods solving nonlinear boundary problems. *In french*, Quelques méthodes de résolution des problèmes aux limites non linéaires, *Dunod*, (1969).

[23] A. Majid, J. Muhammad, K. Tamana1, M. N. Riaz, M. Naveed, T. A. Khan1, A. Ghafoor, A. A. Sheikh, M. N. Zahid, Real Time PCR Based Detection Of Hepatitis B Infection In Suspected Liver Disease Patients Of District MARDAN-KPK, *International Journal of Advanced Research*, **Volume 1, Issue 7** (2013), 145-148.

[24] C. O. Ndaw, Perturbation methods and Applications, *African Institute for Mathematical Sciences (AIMS) [https://sites.google.com/a/aims-senegal.org/archives/2014-2015/Cheikhou%20Oumar%20NDAW.pdf?attredirects=0&d=1], Senegal, 13 June 2015.,* **Volume 1, Issue 7** *(2015).*

[25] D. J. Nokes, A. J. Hall, W. J. Edmunds, G. F. Medley and H. C. Whittle, The influence of age on the development of the hepatitis B carrier state, Proc. Biol. Sci., **23** *(1993),197-201.*

Received: October 30, 2025

[26] C. V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum Press, New York, (1992).

[27] J. G. Skellam, andom dispersal in theoretical populations, Biometrika, **38** *(1951),196-218.*

[28] H. L. Smith, (Chapter 7: Quasimonotone systems of parabolic equations) in Monotone dynamical systems: An introduction to the theory of competitive and cooperative systems, Coll. Mathematical surveys and monographs, **41, AMS** *(1995).*

[29] H. L. Smith, P. Waltman, Perturbation of a globally stable steady state, Proc. AMS, **127** *(1999), 447-453.*

[30] H. Thieme, Global stability of the endemic equilibrium in infinite dimension: Lyapunov functions and positive operators, J. Diff. Eq., **250** *(2011), 3772-3801.*

[31] H. Tavares, S. Terracini, G. Verzini and T. Weth, Existence and Nonexistence of Entire Solutions for Non-Cooperative Cubic Elliptic Systems, Comm. in Partial Differential Equations, **Volume 36, Issue 11** *(2011).*

[32] A. M. Turing, The chemical basis of morphogenesis, Philos. Trans. R. Soc. B, **237** *(1952), 32-72.*

[33] WHO, Hepatitis B and breastfeeding, WHO., **November, No. 22** *(1996).*

[34] L. Zou, S. Ruan, and W. Zhang, Modelling the transmission dynamics and control of Hepatitis B virus in China, SIAM J. Appl. Math., **70** *(2010), 3121-3139.*

[35] L. Zou, W. Zhang, and S. Ruan, Modelling the transmission dynamics and control of Hepatitis B virus in China, J. Theoret. Biol., **262** *(2010), 330-338.*

[36] D.S. Moak, The q-analogue of the Laguerre polynomials, J. Math. Anal. Appl., **81** *(1981), 20-47.*

Received: October 30, 2025