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AN EXTENSIVE STUDY ON GENERALIZED ROUGH CLOSED SETS IN ROUGH TOPOLOGICAL SPACES

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Abstract

This study explores the properties and characterizations of generalized rough closed sets within rough topological spaces to enhance understanding of their structure and applications. Employing theoretical analysis, the research introduces the concept of generalized rough closed sets and investigates their fundamental attributes, including their relationships with rough open and closed sets. Key findings demonstrate that every rough closed set qualifies as a generalized rough closed set, while the union of such sets remains within the same class. The work further characterizes rough spaces where every generalized rough closed set is also a rough closed set, termed Rough-$T_{1/2}$ space, establishing conditions for their equivalence. These insights contribute to the development of refined models in data analysis and digital topology, offering potential for improved techniques in image processing and machine learning. The findings advance the mathematical framework of rough set theory, fostering its practical application in complex data systems.

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Key Words : Rough closed set, Rough open set, Rough topological space, Rough Closure, g-closed set.

. Introduction

Pawlak was the first to establish rough set theory [1]. This theoretical framework has proven valuable in analyzing data to identify ambiguity and uncertainty, as well as in the study of information systems. The identification of key variables or essential components within a dataset is a significant use of rough set theory. Equivalence relations are crucial in this situation.Pawlak's research was the first to establish the concept of an approximation space (U, R) using a non-empty set U and an equivalence relation R.He presented the concepts of lower and upper approximations for any subset Y of U and formulated rough

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equivalence relations to establish rough sets.

Rough topological space (RTS) is the name given to the topological space by some authors [2] who developed topology on traditional rough sets later in 2015. Rough open sets are the elements that make up the rough topology and their complements are rough closed sets. A generalized closed set (g-closed set) was first presented by N.Levin [7] in1970.These sets are essential to the study of digital topology. Many authors have since presented many kinds of such sets in topological spaces. They examined some characteristics as well as how they relate to other kinds of closed sets. It is possible to apply these ideas to rough topological spaces and establish their properties there.

There are important theoretical and practical benefits for mathematics and computational science research when generalized rough closed sets are used in place of ordinary generalized closed sets. A more accurate description of sets that can handle uncertain, incomplete, or granular data is made possible by generalized rough closed sets, which combine the ideas of approximation and boundary regions. These scenarios are commonly found in real-world applications like pattern recognition, medical diagnosis, and knowledge discovery.

By permitting indeterminate membership and nuanced set borders, this method goes beyond the rigidity of generalized closed sets, increasing the adaptability and reach of topological analysis. As a result, generalized rough closed sets are effective instruments for simulating intricate relationships, enabling more reliable decisions and a deeper understanding of the structure of uncertain situations, so enhancing the potential of contemporary mathematical research.

In our work, we present various key characteristics of generalized rough closed sets (GRC sets) in a RTS. Additionally, we introduced a concept related to these sets for this purpose. These sets are potentially applicable to image processing, machine learning, and many other fields.Many rough set models were introduced in the modern age.These include rough sets like intuitionistic fuzzy [15], covering-based, dominance-based [13,14] and decision-theoretic [16,17] rough sets. Some authors such as [18,19,20] introduced algebraic, topological features of rough sets.

2. Preliminaries

The following fundamental definitions and examples are helpful for establishing some attributes and introducing new kinds of closed sets.

Definition 2.1: In a topological space, the intersection of all closed sets containing A is cl(A), while the union of all open sets contained in A is int(A).

Definition2.2: [7] In a topological space a subset J is a g-closed set if any open set U that contains J also contains cl(J).

Definition 2.3: [1] In an approximation space (U, R) for any Y□□U, the subset defined by $\underline{\mathrm{Y}} = \{y \in \mathrm{U} : [y]_R \subseteq \mathrm{Y}\}$ is the lower approximation and $\overline{\mathrm{Y}} = \{y \in \mathrm{U} : [y]_R \cap \mathrm{Y} \neq \varnothing\}$ is the upper approximation of Y. A subset Y is a rough set if $\underline{Y} \neq \overline{Y}$.

Definition 2.4: For any rough set Y, a rough topology, represented by τ, is established on the universe U. The combination (U, τ) is a Rough topological space (RTS). Within this topology, the constituent sets are classified as Rough open sets (RO sets), while their complements are termed Rough closed sets (RC sets).

Definition 2.5: The intersection of RC super sets of H in a RTS is Rough closure of H denoted with Rcl(H).

Definition 2.6: The union of all RO subsets of H is the Rough interior of H denoted with Rint(H).

3. Generalized Rough Closed Sets

This section introduces key aspects of the sets and examines the idea of a generalized rough closed set in a rough topological space.

Definition 3.1: Generalized Rough Closed set (GRCset).

A subset H in a RTS is a GRC set if $Rcl(H) \subseteq P$ whenever $H \subseteq P$ and P is a RO set in U.

Example 3.2:

Let $U = \{1,2,3,4,5\}$, $U/R = \{\{1,2\}, \{3,4\}, \{5\}\}$ and $Y = \{1,3,4\}$. Then, Y is a Rough set. Define $\tau_R(Y) = \{U, \phi, \{1,2,3,4\}, \{5\}\}$. The RC sets are $\{U, \phi, \{1,2,3,4\}, \{5\}\}$.

Consider the subset $Q = \{1, 2, 3\}$, $Rcl(Q) = \{1,2,3,4\}$. The RO sets containing Q are U, {1, 2, 3, 4} and they contain $Rcl(Q)$. Hence, $Q = \{1, 2, 3\}$ is a GRC set.

Theorem 3.3: Every RC set is a GRC set.

Proof: Suppose M is any RC set. Then, $Rcl(M) = M$. Let P be any RO set su ch that $M \subseteq P$. Then, $Rcl(M) = M \subseteq P$. Hence, any RO set containing M also contains $Rcl(M)$. This shows that M is a GRC set.

Remark 3.4: A GRC set need not be a RC set. This is clear from the following example.

Example 3.5:

Let $U = \{1,2,3,4,5\}$, $U / R = \{\{1,2\},\{3,4\},\{5\}\}$, $Y = \{1,3,4\}$ and $\tau_R(Y) = \{U, \phi, \{1,2,3,4\},\{5\}\}$. Then, $(U, \tau_R(Y))$ is a RTS. The RC sets are $\{U, \phi, \{1, 2, 3, 4\}, \{5\}\}$. Consider the subset $N = \{1, 2, 3\}$ in the space. The RO sets containing N are U and {1, 2, 3, 4}. The RC sets containing N are U and {1, 2, 3, 4} and their intersection is {1, 2, 3, 4}. So, $Rcl(N) = \{1,2,3,4\}$. Each RO set containing N contains $Rcl(N)$. This shows that N is a GRC set but it is not a RC set.

Theorem 3.6: For any subset P, $Rcl(P)$ is a GRC set.

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Proof: Suppose G be any RO set containing $\mathrm{Rcl(P)}$.But $\mathrm{Rcl}(\mathrm{Rcl(P)}) = \mathrm{Rcl}(\mathrm{P})$ which implies $\mathrm{Rcl}(\mathrm{Rcl}(\mathrm{P})) \subseteq \mathrm{G}$.Hence, $\mathrm{Rcl(P)}$ is a GRC set.

Theorem 3.7: The sets $\varnothing, \mathrm{U}$ are GRC sets.

Proof: Since Rcl□□□□□□□□□the set □□is a GRC set. The only RO set containing U is itself and U contains all the sets. Hence, U is also a GRC set.

Theorem 3.8: Union of two GRC sets is a GRC set.

Proof: Suppose A_1 and A_2 be GRC sets and M be a RO set containing $\mathrm{A}_1 \cup \mathrm{A}_2$. Then, $\mathrm{A}_1 \subseteq \mathrm{M}$ and $\mathrm{A}_2 \subseteq \mathrm{M}$.By the definition $\mathrm{Rcl}(\mathrm{A}_1) \subseteq \mathrm{M}$ and $\mathrm{Rcl}(\mathrm{A}_2) \subseteq \mathrm{M}$. But, $(\mathrm{A}_1 \cup \mathrm{A}_2) \subseteq \mathrm{Rcl}(\mathrm{A}_1) \cup \mathrm{Rcl}(\mathrm{A}_2) \subseteq \mathrm{M}$ which shows that $\mathrm{A}_1 \cup \mathrm{A}_2$ is a GRC set.

Remark 3.9: The above theorem need not hold for intersection. This can be seen from the following example.

Example 3.10:

Let $\mathrm{U} = \{1,2,3,4,5\}$, $\mathrm{U/R} = \{\{1,2\},\{3,4\},\{5\}\}$, $\mathrm{Y} = \{1,5\}$ and the topology $\tau_R(\mathrm{Y}) = \{\mathrm{U},\varnothing,\{1,2,5\},\{1,2\},\{5\}\}$.Then, $(\mathrm{U}, \tau_{\mathrm{R}}(\mathrm{Y}))$ is a RTS. The sets $\mathrm{A} = \{2,3,5\}$ and $\mathrm{C} = \{1,2,4,5\}$ are GRC sets but $\mathrm{A} \cap \mathrm{C} = \{2,5\}$ is not a GRC set. The RO set $\{1,2,5\}$ contains $\mathrm{A} \cap \mathrm{C} = \{2,5\}$.The Rough Closure $\mathrm{A} \cap \mathrm{C}$ is the intersection of RC sets containing $\mathrm{A} \cap \mathrm{C}$.In the space, the RC sets are $\{\mathrm{U},\varnothing,\{3,4\},\{3,4,5\},\{1,2,3,4\}\}$.The RC set containing $\mathrm{A} \cap \mathrm{C}$ is only U and hence $\mathrm{Rcl}(\mathrm{A} \cap \mathrm{C}) = \mathrm{U}$.But it is not a subset of $\{1,2,5\}$.This shows that $\mathrm{A} \cap \mathrm{C}$ is not a GRC set.

Theorem 3.11: In a RTS if A is a GRC set, Rcl(B) □ A □ B, then B is a GRC set.

Proof: Suppose M is an RO set containing B. Then, A □ B □ M and M contains A. Since A is a GRC set, Rcl□A□ □ M. But, Rcl□Rcl□B□□□= Rcl□B□□□Rcl□B□ □ Rcl□A□ □ M.

Hence, B is a GRC set.

Theorem 3.12: If S is a GRC set, then $\mathrm{Rcl}\{y\} \cap \mathrm{S} \neq \varnothing$ for each $y \in \mathrm{Rcl}(\mathrm{S})$.

Proof: Suppose $\mathrm{Rcl}\{y\} \cap \mathrm{S} = \varnothing$ for some $y \in \mathrm{Rcl}(\mathrm{S})$.Then, $\mathrm{S} \subseteq (\mathrm{Rcl}\{y\})^c$. Since $\mathrm{Rcl}\{y\}$ is a RC set, $(\mathrm{Rcl}\{y\})^c$ is a RO set containing S.But S is a GRC set and $\mathrm{Rcl}(\mathrm{S}) \subseteq (\mathrm{Rcl}\{y\})^c \Rightarrow y \notin \mathrm{Rcl}\{y\}$ which is a contradiction. Hence, $\mathrm{Rcl}\{y\} \cap \mathrm{S} \neq \varnothing$ for each $y \in \mathrm{Rcl}(\mathrm{S})$.

Theorem 3.13: If W is a GRC set, then $\mathrm{Rcl}(\mathrm{W}) - \mathrm{W}$ doesn't contain any nonempty RC set.

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Proof: If possible, F is a RC set contained in $\text{Rcl}(\text{W})-\text{W}$.Then $\text{U}-\text{F}$ is a RO set containing W. Since, W is a GRC set, $\text{Rcl}(\text{W})\subseteq \text{U}-\text{F}$.This implies that $\text{F}\subseteq \text{U}-\text{Rcl}(\text{W})$. Therefore, $\text{F}\subseteq(\text{U}-\text{Rcl}(\text{W}))\cap(\text{Rcl}(\text{W})-\text{W})=\varnothing$.Hence, $\text{F}=\varnothing$. This proves that $\text{Rcl}(\text{W})-\text{W}$ doesn't contain any nonempty RC set.

Theorem 3.14: A GRC set is the disjoint union of a RC set and RO set.

Proof: Put $\text{M}=\text{Rint}(\text{A})$ and $\text{N}=\text{Rcl}(\text{A})-\text{Rint}(\text{A})$ where A is a GRC set. Then, M and N are RO set, RC set respectively and they are disjoint. Since A is a GRC set and M, N are subsets of $\text{A}, \text{M}\cup\text{N}$ is a subset of A.Suppose $y\in\text{A}$.If $y\in\text{Rint}(\text{A})$, then $y\in\text{M}\cup\text{N}$.If $y\notin\text{Rint}(\text{A})$ and $y\in\text{A}$, then $y\in\text{Rcl}(\text{A})-\text{Rint}(\text{A})$.This shows that $y\in\text{M}\cup\text{N}$.

Theorem 3.15: In a RTS $(\text{U},\tau_{\text{R}}(\text{Y}))$, for each $x\in\text{U}$, $\{x\}$ is either a RC set or $\text{U}-\{x\}$ is a GRC set.

Proof: Let $x\in\text{U}$.If $\{x\}$ is not a RC set, $\text{U}-\{x\}$ is not a RO set. The only RO set containing $\text{U}-\{x\}$ is U only. Hence, $\text{U}-\{x\}$ is a GRC set.

Definition 3.16: Rough-$\text{T}_{1/2}$ space:

A RTS $(\text{U},\tau_{\text{R}}(\text{Y}))$ is called a Rough-$\text{T}_{1/2}$ space if every GRC set is a RC set.

The characterization of Rough-$\text{T}_{1/2}$ space is established by the following finding.

Theorem 3.17: A RTS $(\text{U},\tau_{\text{R}}(\text{Y}))$ is a Rough-$\text{T}_{1/2}$ space iff for each $x\in\text{U}$, either $\{x\}$ is a RO set or a RC set.

Proof: Suppose $(\text{U},\tau_{\text{R}}(\text{Y}))$ is a Rough-$\text{T}_{1/2}$ space. From theorem 3.14, $\{x\}$ is a RC set or $\text{U}-\{x\}$ is a GRC set. Since $(\text{U},\tau_{\text{R}}(\text{Y}))$ is a Rough-$\text{T}_{1/2}$ space, $\text{U}-\{x\}$ is an RC set. Therefore, $\{x\}$ is a RO set or a RC set.

Conversely suppose $\{x\}$ is either a RO set or a RC set for each $x\in\text{U}$. Let A be a GRC set and $x\notin\text{A}$.If $\{x\}$ is a RC set, $\text{U}-\{x\}$ is a RO containing A.Since, A is a GRC set, $\text{Rcl}(\text{A})\subseteq\text{U}-\{x\}$ showing $x\notin\text{Rcl}(\text{A})$.On the other hand if $\{x\}$ is a RO set, $x\notin\text{A}$ then $\{x\}\cap\text{A}$ is an empty set.This implies, $x\notin\text{Rcl}(\text{A})$.In both the cases $x\notin\text{Rcl}(\text{A})$. Hence, $\text{Rcl}(\text{A})\subseteq\text{A}$.Thus, A is a RC set.

Conclusion:

This study significantly advances the understanding of generalized rough closed sets within rough topological spaces, revealing their essential properties and relationships with other set types. The findings not only enhance the theoretical framework of rough set theory but also pave the way for practical applications in data analysis, image processing, and

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machine learning, highlighting the potential for improved methodologies in handling complex data systems. Also an important characterization of a Rough-$T_{1/2}$ space was proposed in the work.

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